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# Bispectrum for non-Gaussian homogeneous and isotropic field on the plane

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*Dedicated to Lajos Tamássy on the occasion of his 90th birthday*

**Abstract.** The object of this paper is to characterize the third order moments (cumulants) and bispectra of a homogeneous isotropic field defined on a plane. We establish a one to one correspondence between the third order cumulants and the bispectra of such a process in terms of Bessel functions.

## 1. Introduction

In many real applications associated with random fields, the assumption of Gaussianity may be sometimes unrealistic. For example, consider the data of cosmic microwave background (CMB) anisotropies provided by NASA which some scientists believe to be non-Gaussian, [MP11]. Although CMB data given are on a surface of the sphere there are problems concerning on the primordial field on the whole space including the investigation of the bispectrum as well, see [VWHK00], [YKW07], [AC12]. In time series analysis the non-Gaussianity has been well studied [SRG80], [SRG84], [Hin82], [TM98], [Ter99]. It is known that for a Gaussian time series the bispectrum and all higher order spectra greater then two are zero, and equally well known is the fact that for a non Gaussian process defined on a real line the higher order cumulant spectra and higher order

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cumulants uniquely determine each other [Bri01]. No such results are available and well known for a homogeneous isotropic field defined on a plane.

In this paper we consider homogeneous and isotropic fields which are not necessarily Gaussian. The second order properties of such a field are well known, both the covariance and the spectrum depend on the distance between the locations and the wave numbers respectively, [Whi54], [Yad83]. The third order covariances (third order cumulants) depends on three locations and because of the invariance under shifting (homogeneity) and invariance under the rotation (isotropy), it depends only on the distances between locations. In other words the rigid body movement keeps the triangle defined by the three locations fixed. We show that the third order covariances define the bispectrum which depends on three wave numbers forming a triangle. The main result of this paper describes the unique relation between the third order covariances and the bispectrum of a homogeneous and isotropic field on the plane.

**1.1. Homogeneous and isotropic field on the plane.** We consider a homogeneous real valued stochastic field $X(\underline{x})$ on $\mathbb{R}^2$ with $EX(\underline{x}) = 0$. Let us suppose that $X(\underline{x})$ is continuous (in mean square sense), its spectral representation is

$$X\left(\underline{x}\right) = \int_{\mathbb{R}^2} e^{i\underline{x}\cdot\underline{\omega}} Z\left(d\underline{\omega}\right), \quad \underline{\omega}, \underline{x} \in \mathbb{R}^2, \tag{1}$$

with a finite spectral measure

$$E\left|Z\left(d\underline{\omega}\right)\right|^2 = F_0\left(d\underline{\omega}\right).$$

By homogeneity we mean (in strict sense ) the distribution of $X(\underline{x})$ is translation invariant, see [Yag87] for details. Rewrite $X(\underline{x})$ in terms of polar coordinates

$$X\left(r,\varphi\right) = \int_0^\infty \int_0^{2\pi} e^{i\rho r\cos(\varphi-\eta)} Z\left(\rho d\rho d\eta\right),$$

where $\underline{x} = (r,\varphi)$, $\underline{\omega} = (\rho,\eta)$ are polar coordinates, $r = |\underline{x}| = \sqrt{x_1^2 + x_2^2}$, and $\rho = |\underline{\omega}|$, $\underline{x}\cdot\underline{\omega} = r\rho\cos(\varphi-\eta)$. Now we use the Jacobi–Anger expansion, see [AW01] Section 11,

$$e^{i\rho r\cos(\varphi-\eta)} = \sum_{\ell=-\infty}^{\infty} i^\ell J_\ell\left(\rho r\right) e^{i\ell(\varphi-\eta)}, \tag{2}$$

and substitute it into the above spectral representation of $X(r,\varphi)$

$$X\left(r,\varphi\right) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell\left(\rho r\right) Z_\ell\left(\rho d\rho\right) \tag{3}$$

where $J_\ell$ denotes the Bessel function of the first kind [AS92], and the series of stochastic spectral measures $Z_\ell(\rho d\rho)$ is connected to the $Z(d\underline{\omega})$ by the integral

$$Z_\ell\left(\rho d\rho\right) = \int_0^{2\pi} i^\ell e^{-i\ell\eta}\, Z\left(\rho d\rho d\eta\right).$$

The representation (3) will be an orthogonal (uncorrelated) representation if we assume that $F_0(d\underline{\omega})$ is isotropic (invariant under rotations) $F_0(d\underline{\omega}) = E|Z(d\underline{\omega})|^2 = E|Z(\rho d\rho d\eta)|^2 = F(\rho d\rho)d\eta$. Note here that the general theorem of Yadrenko ([Yad83] Theorem 1. pp. 5) on the spectral representation of a homogeneous and isotropic field $X(r,\varphi)$ gives the representation (3) with real valued stochastic spectral measures $Z_\ell(\cdot)$ constructed directly form the field $X(r,\varphi)$ itself. The stochastic spectral measures $Z_\ell(\rho d\rho)$ defined above is complex valued and has the following property

$$Z_{-\ell}\left(\rho d\rho\right) = (-1)^\ell\, \overline{Z_\ell\left(\rho d\rho\right)}. \tag{4}$$

Indeed, since the field $X(\underline{x})$ is real valued we have

$$X\left(r,\varphi\right) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho) = \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} \int_0^\infty J_\ell\left(\rho r\right) \overline{Z_\ell\left(\rho d\rho\right)}.$$

Moreover, from the well known formula $J_\ell(\cdot) = (-1)^\ell J_{-\ell}(\cdot)$ we have

$$\begin{aligned}\sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} \int_0^\infty J_\ell\left(\rho r\right) \overline{Z_\ell\left(\rho d\rho\right)} &= \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} \int_0^\infty J_{-\ell}\left(\rho r\right)(-1)^\ell\, \overline{Z_\ell\left(\rho d\rho\right)} \\ &= \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} \int_0^\infty J_{-\ell}\left(\rho r\right) Z_{-\ell}\left(\rho d\rho\right),\end{aligned}$$

hence (4) follows. The identity (4) implies that the $\ell^{th}$ term and the $-\ell^{th}$ terms of the expansion (3) are conjugates of each other. Moreover $Z_\ell(\cdot)$ is orthogonal

$$\begin{aligned}\operatorname{Cov}\left(Z_{\ell_1}\left(\rho_1 d\rho_1\right), Z_{\ell_2}\left(\rho_2 d\rho_2\right)\right) &= \int_0^{2\pi} i^{(\ell_1-\ell_2)} e^{-i(\ell_2-\ell_1)\eta} F_0\left(d\eta\rho d\rho\right) \\ &= \delta_{\ell_1-\ell_2} 2\pi F\left(\rho d\rho\right),\end{aligned} \tag{5}$$

where $\delta_{\ell_1-\ell_2}$ is the Kronecker delta. Note that the spectral measure $F(\rho d\rho)$ of $Z_\ell(\rho d\rho)$ does not depend on $\ell$. We shall assume in particular cases that $F(\rho d\rho)$ is absolutely continuous, i.e. $F(\rho d\rho) = \sigma^2|A(\rho)|^2\rho d\rho$, here $\sigma^2|A(\rho)|^2$ is usually known as the second order spectrum. In view of this observation, we can rewrite $X(r,\varphi)$ in terms of white noise measures $W_\ell(\rho d\rho)$ with constant spectrum

$$\operatorname{Cov}\left(W_{\ell_1}\left(\rho_1 d\rho_1\right), W_{\ell_2}\left(\rho_2 d\rho_2\right)\right) = \delta_{\ell_1-\ell_2}\sigma^2\rho d\rho,$$

hence (3) becomes

$$X\left(r,\varphi\right) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi}\int_0^\infty J_\ell\left(\rho r\right) A\left(\rho\right) W_\ell\left(\rho d\rho\right).$$

## 2. Isotropy on the plane

We consider rotations about the origin of the coordinate system. Under a rotation (passive) $g \in SO(2)$, we mean a rotation when vectors remain fixed, but the point it defines is given by a new set of coordinates. A rotation $g$ is characterized by an angle $\gamma$ and by the rotation matrix

$$g = \begin{bmatrix}\cos\gamma & -\sin\gamma\\ \sin\gamma & \cos\gamma\end{bmatrix}.$$

If $\underline{x} \in \mathbb{R}^2$ is given in polar coordinates $\underline{x} = (r,\varphi)$, then $g\underline{x} = (r,\varphi-\gamma)$, and as usual the operator $\Lambda(g)$ acts on functions $f(r,\varphi)$, such that $\Lambda(g)f(r,\varphi) = f(g^{-1}(r,\varphi)) = f(r,\varphi+\gamma)$.

The isotropy usually is defined through the invariance of the covariance structure. This is satisfactory for Gaussian cases but for non-Gaussian fields we need invariance of higher order cumulants as well. We use a stronger definition to achieve a similar invariance to be able to define third order spectrum, which we will propose below.

*Definition 1.* A homogeneous stochastic field $X(\underline{x})$ is strictly isotropic if all finite dimensional distributions of $X(\underline{x})$ are invariant under rotation.

If the homogeneous field $X(\underline{x})$ is Gaussian, then the isotropy of the spectral measure $F_0(d\underline{\omega})$, i.e. in polar coordinates $F_0(d\underline{\omega}) = F(\rho d\rho)d\eta/(2\pi)$, implies

$$\operatorname{Cov}\left(\Lambda\left(g\right)X\left(\underline{x}_1\right), \Lambda\left(g\right)X\left(\underline{x}_2\right)\right) = \operatorname{Cov}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right)\right),$$

for each $\underline{x}_1$, $\underline{x}_2$ and for every $g \in SO(2)$. That is the distribution of a Gaussian isotropic field is invariant under rotation. The definition of isotropy given above is a generalization of this property for non-Gaussian case. In general, the isotropy follows and followed by that all higher order moments are also invariant under rotation. Let us consider homogeneous and isotropic stochastic field $X(\underline{x}) = X(r,\varphi)$, ($r > 0$, $\varphi \in [0, 2\pi)$) on the plane defined by (3)

$$X(r,\varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho),$$

where $Z_\ell$ is an array of measures, orthogonal to each other satisfying (5). In this way an isotropic random field $X(\underline{x})$ can be decomposed into a countable number of mutually uncorrelated spectral measures defined on the real line instead of on the whole plane, [Adl10].

The rotation $g$ takes effect on the 'spherical harmonics' $e^{i\ell m\varphi}$ ($m = \pm 1$), as $\Lambda(g)e^{i\ell m\varphi} = e^{i\ell m(\varphi+\gamma)} = e^{i\ell m\gamma}e^{i\ell m\varphi}$, since the $e^{i\ell m\varphi}$ can be considered as a function of $\varphi$. The isotropy of $X(r,\varphi)$ implies that the distribution of $X(r,\varphi)$ does not change under rotations $g \in SO(2)$. Consider

$$\begin{aligned}\Lambda(g) X(r,\varphi) &= \sum_{\ell=-\infty}^{\infty} e^{i\ell(\varphi+\gamma)} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho) \\ &= \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) e^{i\ell\gamma} Z_\ell(\rho d\rho) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho),\end{aligned}$$

and because of the above the distribution of $Z_\ell(\rho d\rho)$ and $e^{i\ell\gamma}Z_\ell(\rho d\rho)$ should be the same. Now it is evident that for a Gaussian random field $X(r,\varphi)$, the necessary and sufficient condition of isotropy is that $Z_\ell(\rho d\rho)$ are independent. Indeed under isotropy assumption we have

$$\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)\right) = e^{i(\ell_1+\ell_2)\gamma} \operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)\right),$$

for each $\gamma$, hence either $\ell_1+\ell_2 = 0$, or otherwise $\operatorname{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)) = 0$, and therefore

$$\operatorname{Cov}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)\right) = 0,$$

unless $\ell_1 = \ell_2$.

In general, we have that under assumption of isotropy the $p^{th}$ order cumulants satisfy the following equation

$$\begin{aligned}&\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), \ldots, Z_{\ell_p}(\rho_p d\rho_p)\right) \\ &\qquad = e^{i(\ell_1+\ell_2\cdots+\ell_p)\gamma} \operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), \ldots, Z_{\ell_p}(\rho_p d\rho_p)\right),\end{aligned}$$

that is either $\ell_1+\ell_2\cdots+\ell_p=0$, or $\operatorname{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), \ldots, Z_{\ell_p}(\rho_p d\rho_p))=0$. In turn, if this assumption is satisfied then the cumulants $\operatorname{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), \ldots, Z_{\ell_p}(\rho_p d\rho_p))$ re invariant under rotation and the field is isotropic.

**2.1. Spectrum.** In this section we briefly review results already known for second order spectra of the field $X(r,\varphi)$ before we determine similar results for bispectra. For notational convenience let us denote the integral $\int_0^\infty J_\ell(\rho r)Z_\ell(\rho d\rho)$ by $z_\ell(r)$, and consider the covariance

$$\begin{aligned}\operatorname{Cov}\left(X\left(\underline{x}\right),X\left(\underline{y}\right)\right) &= \operatorname{Cov}\left(\sum_{\ell_1=-\infty}^{\ell_1=\infty} e^{i\ell_1\varphi_1} z_{\ell_1}\left(r_1\right), \sum_{\ell_2=-\infty}^{\ell_2=\infty} e^{i\ell_2\varphi_2} z_{\ell_2}\left(r_2\right)\right)\\ &= 2\pi\int_0^\infty \sum_{\ell=-\infty}^{\infty} e^{i\ell(\varphi_1-\varphi_2)} J_\ell\left(\rho r_1\right) J_\ell\left(\rho r_2\right) F\left(\rho d\rho\right)\\ &= 2\pi\int_0^\infty J_0\left(\rho r\right) F\left(\rho d\rho\right),\end{aligned}$$

where $r_1=|\underline{x}|$, $r_2=|\underline{y}|$ and $r=|\underline{x}-\underline{y}|$. In arriving at the above we used the addition formula

$$J_0\left(\rho r\right) = \sum_{\ell=-\infty}^{\infty} e^{i\ell(\varphi_1-\varphi_2)} J_\ell\left(\rho r_1\right) J_\ell\left(\rho r_2\right),$$

of Bessel functions, see [EMOT54] Tom2, Ch7, 7.6.2.(6), [Yad83]. Now one may derive the same result using the properties of homogeneity and isotropy.

We are going to apply some special cases of the series expansion given by (3), namely if the location $(r,\varphi)$ is on the $y$-axis i.e., it points on the direction of the 'North pole' ($N=(0,1)$),

$$X\left(rN\right) = \sum_{\ell=-\infty}^{\infty} i^\ell \int_0^\infty J_\ell\left(\rho r\right) Z_\ell\left(\rho d\rho\right), \tag{6}$$

and at the origin

$$X\left(\underline{0}\right) = \int_{\mathbb{R}^2} Z\left(d\underline{\omega}\right) = \int_0^\infty Z_0\left(\rho d\rho\right). \tag{7}$$

Let $r=|\underline{x}-\underline{y}|$, $\mathcal{C}_2(r)=\operatorname{Cov}(X(\underline{x}),X(\underline{y}))$, and use the invariance under translation and rotation to obtain

$$\begin{aligned}\operatorname{Cov}(X(\underline{x}),X(\underline{y})) = \operatorname{Cov}(X(\underline{x}-\underline{y}),X(0)) &= \operatorname{Cov}(X(rN),X(0))\\ &= 2\pi\int_0^\infty J_0(\rho r)F(\rho d\rho).\end{aligned}$$

The above shows one to one correspondence between the second order covariance and its spectral density function, see [Yad83]. In particular for absolutely continuos spectral measure $F(\rho d\rho)=\sigma^2|A(\rho)|^2\rho d\rho$ we have

$$\mathcal{C}_2(r) = 2\pi\int_0^\infty J_0(\rho r)\sigma^2|A(\rho)|^2\rho d\rho,$$

in turn

$$\sigma^2|A(\rho)|^2 = \frac{1}{2\pi}\int_0^\infty J_0(\rho r)\mathcal{C}_2(r)rdr,$$

when both integrals exist, see [Bri74], [Yag87]. The above property of Hankel transform used above is based on the following property of Bessel functions

$$\int_0^\infty J_\ell\left(\rho r\right) J_\ell\left(\kappa r\right) rdr = \frac{\delta\left(\rho-\kappa\right)}{\rho}, \tag{8}$$

where $\delta(\rho-\kappa)$ denotes the Dirac 'function', more precisely $\delta(\cdot)$ is a distribution (measure), see [AW01] Section 11.

## 3. Bispectrum

If the field is not Gaussian then the second order properties do not characterize the distribution. The next characteristics are, in a row, the third order moments. The third order structure of a homogeneous and isotropic stochastic field $X(\underline{x})$ is described by either the third order covariances (third order cumulants) in spatial domain or the bispectrum in frequency domain. Using the spectral representation (1) of $X(\underline{x})$, we obtain the third order cumulants (central moments) and it is given by

$$\begin{aligned}\operatorname{Cum}(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3)) &= \iiint\limits_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} e^{i(\Sigma_1^3\underline{x}_k\cdot\underline{\omega}_k)}\operatorname{Cum}(Z(d\underline{\omega}_1), Z(d\underline{\omega}_2), Z(d\underline{\omega}_3))\\ &= \iiint\limits_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} e^{i(\Sigma_1^3\underline{x}_k\cdot\underline{\omega}_k)} S_3(\underline{\omega}_1,\underline{\omega}_2,\underline{\omega}_3)\delta(\Sigma_1^3\underline{\omega}_k)\prod_{k=1}^3 d\underline{\omega}_k,\end{aligned}$$

where $S_3(\underline{\omega}_1,\underline{\omega}_2,\underline{\omega}_3) = S_3(\underline{\omega}_1,\underline{\omega}_2,-\underline{\omega}_1-\underline{\omega}_2)$ denotes the bispectral density. Under isotropy for each $g\in SO(2)$

$$\begin{aligned}\operatorname{Cum}(X(g\underline{x}_1), X(g\underline{x}_2), X(g\underline{x}_3)) &= \iiint\limits_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} e^{i(\Sigma_1^3\underline{x}_k\cdot\underline{\omega}_k)} S_3(g\underline{\omega}_1, g\underline{\omega}_2, g\underline{\omega}_3)\delta(\Sigma_1^3\underline{\omega}_k)\prod_{k=1}^3 d\underline{\omega}_k\\ &= \operatorname{Cum}(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3)),\end{aligned}$$

hence $S_3(g\underline{\omega}_1, g\underline{\omega}_2, g\underline{\omega}_3) = S_3(\underline{\omega}_1,\underline{\omega}_2,\underline{\omega}_3) = S_3(\rho_1,\rho_2,\rho_3)$. Now we apply the invariance of the third order covariance

$$\begin{aligned}\operatorname{Cum}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right), X\left(\underline{x}_3\right)\right) &= \operatorname{Cum}\left(X\left(0\right), X\left(\underline{x}_2-\underline{x}_1\right), X\left(\underline{x}_3-\underline{x}_1\right)\right)\\ &= \operatorname{Cum}\left(X\left(0\right), X\left(\left|\underline{x}_2-\underline{x}_1\right| N\right), X\left(g\left(\underline{x}_3-\underline{x}_1\right)\right)\right),\end{aligned}$$

where $g$ denotes the rotation carrying $\underline{x}_2 - \underline{x}_1$ into the $|\underline{x}_2 - \underline{x}_1|$ times 'North pole' $(N = (0,1))$.

The third order covariance $\operatorname{Cum}(X(0), X(\underline{x}_2), X(\underline{x}_3))$ depends on the length of vectors $\underline{x}_2$, $\underline{x}_3$ and the angle $\varphi$ between them, this way a triangle with vertices 0, $\underline{x}_2$, $\underline{x}_3$ is formed with length of the third side $r_1$, such that $r_1^2 = r_2^2 + r_3^2 - 2r_2r_3\cos\varphi$. According to this definition of $r_1$, we introduce the notation

$$\mathcal{C}(r_1, r_2, r_3) = \operatorname{Cum}(X(0), X(\underline{x}_2), X(\underline{x}_3)).$$

We show that the bispectrum $S_3(\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3)$ for a homogeneous and isotropic stochastic field $X(\underline{x})$ depend on wave numbers $\rho_1, \rho_2, \rho_3$, $(\rho_k = |\underline{\omega}_k|)$ only, such that the wave numbers $\rho_1$, $\rho_2$, and $\rho_3$ satisfy the triangle relation. The angle between sides $\rho_2$ and $\rho_3$, will be denoted by $\eta$.

The following theorem shows that the usual connection between the third order covariances and spectra is valid in a particular form for the third order covariance and the bispectra as well.

**Theorem 2.** *The third order covariance $\mathcal{C}(r_1, r_2, r_3)$ and the corresponding bispectrum $S_3(\rho_1, \rho_2, \rho_3)$ are given by*

$$\mathcal{C}(r_1, r_2, r_3) = 2\pi \iint_0^\infty \int_0^\pi (J_0(w_+) + J_0(w_-)) S_3(\rho_1, \rho_2, \rho_3)\, d\eta \prod_{k=2}^3 \rho_k d\rho_k,$$

*in turn*

$$S_3(\rho_1, \rho_2, \rho_3) = \frac{1}{(2\pi)^3} \iint_0^\infty \int_0^\pi (J_0(w_+) + J_0(w_-)) \mathcal{C}(r_1, r_2, r_3)\, d\varphi e \prod_{k=2}^3 r_k dr_k,$$

*where $r_1^2 = r_2^2 + r_3^2 - 2r_2r_3\cos\varphi$, $\rho_1^2 = \rho_2^2 + \rho_3^2 - 2\rho_2\rho_3\cos\eta$, and*

$$w_+ = \sqrt{(\rho_2 r_2)^2 + (\rho_3 r_3)^2 - 2\rho_2 r_2 \rho_3 r_3 \cos((\eta + \varphi))},$$
$$w_- = \sqrt{(\rho_2 r_2)^2 + (\rho_3 r_3)^2 - 2\rho_2 r_2 \rho_3 r_3 \cos((\eta - \varphi))}.$$

*In the above, we assume that both integrals exist.*

We note that in some cases it is more convenient to use the transformation

$$\mathcal{T}(\eta, \rho_2, \rho_3 \mid \varphi, r_2, r_3) = J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2\sum_{\ell=1}^\infty \cos(\ell\varphi) J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta),$$

between the bispectrum $S_3(\rho_1, \rho_2, \rho_3)$ and the third order covariance $\mathcal{C}_3(r_1, r_2, r_3)$, namely.

$$\begin{aligned} &\mathcal{C}(r_1, r_2, r_3) \\ &\qquad = 4\pi \iint_0^\infty \int_0^\pi \mathcal{T}(\eta, \rho_2, \rho_3 \mid \varphi, r_2, r_3) S_3(\rho_1, \rho_2, \rho_3) d\eta \prod_{k=2}^3 \rho_k d\rho_k, \qquad (9) \\ &S_3(\rho_1, \rho_2, \rho_3) \\ &\qquad = \frac{1}{4\pi^3} \iint_0^\infty \int_0^\pi \mathcal{T}(\eta, \rho_2, \rho_3 \mid \varphi, r_2, r_3) \mathcal{C}(r_1, r_2, r_3) d\varphi \prod_{k=2}^3 r_k dr_k. \qquad (10) \end{aligned}$$

PROOF. We use the particular representations (7), (6) and obtain

$$\begin{aligned} &\operatorname{Cum}(X(0), X(r_2 N), X(\underline{x}_3)) \\ &= \sum_{\ell_2, \ell_3 = -\infty}^{\infty} \iiint_0^\infty i^{\ell_2} e^{i\ell_3\varphi_3} J_{\ell_2}(\rho_2 r_2) J_{\ell_3}(\rho_3 r_3) \operatorname{Cum}(Z_0(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), Z_{\ell_3}(\rho_3 d\rho_3)) \\ &= \sum_{\ell=-\infty}^{\infty} \iiint_0^\infty e^{i\ell\varphi} J_\ell(\rho_2 r_2) J_{-\ell}(\rho_3 r_3) \operatorname{Cum}(Z_0(\rho_1 d\rho_1), Z_\ell(\rho_2 d\rho_2), Z_{-\ell}(\rho_3 d\rho_3)) \end{aligned}$$

where $\varphi = \pi/2 - \varphi_3$, is the angle between $N$ and $\underline{x}_3$. The third order cumulant of the stochastic spectral measure $Z(d\underline{\omega})$ of the homogeneous field $X(\underline{x})$ is given by

$$\begin{aligned} \operatorname{Cum}\left(Z\left(d\underline{\omega}_1\right), Z\left(d\underline{\omega}_2\right), Z\left(d\underline{\omega}_3\right)\right) &= \delta\left(\Sigma_1^3 \underline{\omega}_k\right) S_3\left(\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3\right) d\underline{\omega}_1 d\underline{\omega}_2 d\underline{\omega}_3 \\ &= \delta\left(\Sigma_1^3 \rho_k \widehat{\underline{\omega}}_k\right) S_3\left(\rho_1, \rho_2, \rho_3\right) \prod_{k=1}^3 \Omega\left(d\widehat{\underline{\omega}}_k\right) \rho_k d\rho_k, \end{aligned}$$

where $\widehat{\underline{\omega}}_k = \underline{\omega}_k / |\underline{\omega}_k|$. The stochastic spectral measures $Z_\ell(\rho d\rho)$ are related to $Z(d\underline{\omega})$ by

$$Z_\ell\left(\rho d\rho\right) = \int_0^{2\pi} i^\ell e^{-i\ell\eta}\, Z\left(d\eta \rho d\rho\right),$$

therefore

$$\begin{aligned} &\operatorname{Cum}\left(Z_0\left(\rho_1 d\rho_1\right), Z_\ell\left(\rho_2 d\rho_2\right), Z_{-\ell}\left(\rho_3 d\rho_3\right)\right) \\ &\qquad = S_3\left(\rho_1, \rho_2, \rho_3\right) \iiint_0^{2\pi} e^{-i\ell(\eta_3 - \eta_2)}\, \delta\left(\Sigma_1^3 \rho_k \widehat{\underline{\omega}}_k\right) \prod_{k=1}^3 \rho_k d\rho_k d\eta_k. \qquad (11) \end{aligned}$$

In order to understand the usefulness of the Dirac 'function' in polar coordinates we express it by an integral through the Jacobi–Anger expansion (2). Since the

Dirac 'function' is a measure we apply here the theory of generalized functions to obtain

$$\delta(\Sigma_1^3 \rho_k \underline{\widehat{\omega}}_k) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(\underline{\lambda}\cdot\Sigma_1^3 \underline{\omega}_k)} d\underline{\lambda}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} \prod_{k=1}^3 \sum_{\ell_k=-\infty}^{\infty} i^{\ell_k} J_{\ell_k}(\rho_k\lambda) e^{i\ell_k(\eta_k-\xi)} \lambda d\lambda d\xi$$

$$= \frac{1}{2\pi} \int_0^\infty \sum_{\ell_1,\ell_2=-\infty}^{\infty} e^{i(\ell_1(\eta_1-\eta_3)+\ell_2(\eta_2-\eta_3))} J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{-\ell_1-\ell_2}(\rho_3\lambda)\lambda d\lambda. \quad (12)$$

Now substitute (12) into (11). Because

$$\iiint_0^{2\pi} e^{-i\ell(\eta_3-\eta_2)} \prod_{k=1}^3 e^{i\ell_k(\eta_k-\xi)} d\eta_k = \delta_{\ell_1}\delta_{\ell_2+\ell}\delta_{\ell_3-\ell}(2\pi)^3,$$

we get

$$\iiint_0^{2\pi} e^{-i\ell(\eta_3-\eta_2)} \delta(\Sigma_1^3 \rho_k \underline{\widehat{\omega}}_k) d\eta_1 d\eta_2 d\eta_3 = (2\pi)^2 \int_0^\infty J_0(\rho_1\lambda) J_{-\ell}(\rho_2\lambda) J_\ell(\rho_3\lambda)\lambda d\lambda.$$

This integral can be evaluated, if $|\rho_2-\rho_3| < \rho_1 < \rho_2+\rho_3$, and let us denote $R = (\rho_2^2+\rho_3^2-\rho_1^2)/(2\rho_2\rho_3)$, $\rho_1^2 = \rho_2^2+\rho_3^2-2\rho_2\rho_3\cos(\eta)$, then

$$\int_0^\infty J_0(\rho_1\lambda) J_\ell(\rho_2\lambda) J_\ell(\rho_3\lambda)\lambda d\lambda = \frac{\cos(\ell \arccos(R))}{\pi\rho_2\rho_3\sqrt{1-R^2}},$$

see [PBM86] Tom. II, 2.12.41.16, hence

$$\iiint_0^{2\pi} e^{-i\ell(\eta_1-\eta_2)} \delta\left(\Sigma_1^3 \rho_k \underline{\widehat{\omega}}_k\right) d\eta_1 d\eta_2 d\eta_3 = 4\pi(-1)^\ell \frac{\cos(\ell \arccos(R))}{\rho_2\rho_3\sqrt{1-R^2}}.$$

Since $J_\ell = (-1)^\ell J_{-\ell}$ we have

$$\operatorname{Cum}\left(Z_0(\rho_1 d\rho_1), Z_\ell(\rho_2 d\rho_2), Z_{-\ell}(\rho_3 d\rho_3)\right)$$
$$= 4\pi(-1)^\ell \delta(\rho\triangle) \frac{\cos(\ell \arccos(R))}{\rho_2\rho_3\sqrt{1-R^2}} S_3(\rho_1,\rho_2,\rho_3) \textstyle\prod_{k=1}^3 \rho_k d\rho_k,$$

where $\delta(\rho\triangle) = \delta(\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos\eta-\rho_1^2)$ and therefore the wave numbers $\rho_1$, $\rho_2$, and $\rho_3$ should satisfy the triangle relation. Now can obtain the following expression for the third order covariance

$$\operatorname{Cum}(X(0), X(r_2 N), X(\underline{x}_3)) = 4\pi \sum_{\ell=-\infty}^{\infty} \iiint_0^\infty$$

$$\times\, e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_{-\ell}(\rho_3 r_3)(-1)^\ell \delta(\rho\triangle) \frac{\cos(\ell \arccos(R))}{\rho_2\rho_3\sqrt{1-R^2}} S_3(\rho_1,\rho_2,\rho_3) \prod_{k=1}^{3} \rho_k d\rho_k$$

$$= 4\pi \sum_{\ell=-\infty}^{\infty} \iint_0^\infty \int_{|\rho_2-\rho_3|}^{\rho_2+\rho_3} e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \frac{\cos(\ell \arccos(R))}{\rho_2\rho_3\sqrt{1-R^2}} S_3(\rho_1,\rho_2,\rho_3) \prod_{k=1}^{3} \rho_k d\rho_k$$

$$= 4\pi \sum_{\ell=-\infty}^{\infty} \iint_0^\infty \int_0^\pi e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \frac{\cos(\ell\eta)}{\sqrt{1-\cos^2(\eta)}} S_3(\rho_1,\rho_2,\rho_3) \sin\eta\, d\eta \prod_{k=2}^{3} \rho_k d\rho_k$$

$$= 4\pi \iint_0^\infty \int_0^\pi \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta) S_3(\rho_1,\rho_2,\rho_3) d\eta \prod_{k=2}^{3} \rho_k d\rho_k, \quad (13)$$

where $\rho_1^2 = \rho_2^2 + \rho_3^2 - 2\rho_2\rho_3\cos(\eta)$ and

$$\frac{\rho_1 d\rho_1}{d\eta} = \rho_2\rho_3 \sin\eta.$$

The function in the last row of the expression (13) is

$$\begin{aligned}
\mathcal{T}(\eta,\rho_2,\rho_3 \mid \varphi, r_2, r_3) &= \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta) \\
&= \sum_{\ell=-\infty}^{\infty} \cos(\ell\varphi) J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta) \\
&= J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2\sum_{\ell=1}^{\infty} \cos(\ell\varphi) J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta). \qquad (14)
\end{aligned}$$

The above gives the transformation of the bispectrum $S_3(\rho_1,\rho_2,\rho_3)$ from the third order covariance $\mathcal{C}_3(r_1,r_2,r_3)$. Note that both angles $\varphi$ and $\eta$ together with two sides define the third side $\rho_1$ and $r_1$ of the triangles, given by wave numbers $(\rho_1,\rho_2,\rho_3)$ and distances $(r_1,r_2,r_3)$. The transformation $\mathcal{T}$ can be simplified by using

$$\begin{aligned}
\mathcal{T}(\eta,\rho_2,\rho_3 \mid \varphi, r_2, r_3) &= \frac{1}{2}\left(J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2\sum_{\ell=1}^{\infty} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell(\eta+\varphi))\right) \\
&+ \frac{1}{2}\left(J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2\sum_{\ell=1}^{\infty} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell(\eta-\varphi))\right) \\
&= \frac{1}{2}(J_0(w_+) + J_0(w_-)),
\end{aligned}$$

where $w_+ = \sqrt{(\rho_2 r_2)^2 + (\rho_3 r_3)^2 - 2\rho_2 r_2 \rho_3 r_3 \cos((\eta+\varphi))}$,
$w_- = \sqrt{(\rho_2 r_2)^2 + (\rho_3 r_3)^2 - 2\rho_2 r_2 \rho_3 r_3 \cos((\eta-\varphi))}$, see [EMOT54] Tom.2, Ch7.7.15, for the formula

$$J_0(z)J_0(u) + 2\sum_{\ell=1}^{\infty} J_\ell(z)J_\ell(u)\cos(\ell\vartheta) = J_0(w),$$
$$w = \sqrt{z^2+u^2-2zu\cos(\vartheta)}.$$

We have now established a relationship between the third order covariance and the bispectrum, namely

$$\operatorname{Cum}(X(0), X(r_2 N), X(\underline{x}_3)) = 2\pi \iint_0^\infty \int_0^\pi (J_0(w_+) + J_0(w_-)) S_3(\rho_1,\rho_2,\rho_3)\, d\eta \prod_{k=2}^3 \rho_k d\rho_k.$$

Next we show that the inversion formula

$$S_3(\rho_1,\rho_2,\rho_3) = \frac{1}{(2\pi)^3} \iint_0^\infty \int_0^\pi (J_0(w_+) + J_0(w_-)) \mathcal{C}(r_1,r_2,r_3)\, d\varphi \prod_{k=2}^3 r_k dr_k,$$

is also valid. Consider the integral

$$I(\rho_1,\rho_2,\rho_3 \mid \kappa_1,\kappa_2,\kappa_3)$$
$$= \iint_0^\infty \int_0^\pi \left( J_0(\rho_2 r_2)J_0(\rho_3 r_3) + 2\sum_{\ell=1}^\infty \cos(\ell\varphi) J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3)\cos(\ell\eta)\right)$$
$$\times \left( J_0(\kappa_2 r_2)J_0(\kappa_3 r_3) + 2\sum_{\ell=1}^\infty \cos(\ell\varphi) J_\ell(\kappa_2 r_2) J_\ell(\kappa_3 r_3)\cos(\ell\vartheta)\right) d\varphi \prod_{k=2}^3 r_k dr_k. \quad (15)$$

We notice first that $\cos(\ell\varphi)$ is an orthogonal system on $[0,\pi]$, i.e.

$$\int_0^\pi \cos(\ell_1\varphi)\cos(\ell_2\varphi)\, d\varphi = \delta_{\ell_1=\ell_2}\{\begin{cases} \pi & \text{if } \ell_1 = 0 \\ \frac{\pi}{2} & \text{if } \ell_1 \neq 0 \end{cases},$$

then we integrate (15) with respect to $\varphi$ and obtain

$$I(\rho_1,\rho_2,\rho_3 \mid \kappa_1,\kappa_2,\kappa_3) = \pi \int_0^\infty J_0(\rho_2 r_2) J_0(\kappa_2 r_2) r_2 dr_2 \int_0^\infty J_0(\rho_3 r_3) J_0(\kappa_3 r_3) r_3 dr_3$$
$$+2\pi \sum_{\ell=1}^\infty \cos(\ell\eta)\cos(\ell\vartheta) \int_0^\infty J_\ell(\rho_2 r_2) J_\ell(\kappa_2 r_2) r_2 dr_2 \int_0^\infty J_\ell(\rho_3 r_3) J_\ell(\kappa_3 r_3) r_3 dr_3.$$

Using the integral of Bessel functions (8), we can show

$$\begin{aligned} I\left(\rho_1,\rho_2,\rho_3|\,\kappa_1,\kappa_2,\kappa_3\right) &= \pi^2\delta\left(\rho_2-\kappa_2\right)\delta\left(\rho_3-\kappa_3\right)\left(\frac{1}{\pi}+\frac{2}{\pi}\sum_{\ell=1}^{\infty}\cos\left(\ell\eta\right)\cos\left(\ell\vartheta\right)\right) \\ &= \pi^2\delta\left(\rho_2-\kappa_2\right)\delta\left(\rho_3-\kappa_3\right)\delta\left(\eta-\vartheta\right), \end{aligned}$$

since $1/\pi$ and $\sqrt{\pi/2}\cos(\ell\eta)$ forms an orthonormal system, see [AW01] Section 11. □

We now consider models to define above random precesses.

*Example 3.* The spatial white noise $\partial W(r,\varphi)$, on the plane is given as a generalized field by the series representation

$$\partial W\left(r,\varphi\right)=\sum_{\ell=-\infty}^{\infty}e^{i\ell\varphi}\int_0^{\infty}J_\ell\left(\rho r\right)W_\ell\left(\rho d\rho\right),$$

where $W_\ell(\rho d\rho)$ with $E|W_\ell(\rho d\rho)|^2=\sigma^2\rho d\rho$, see [Yag87], [Yad83]. We define the Laplacian field on the plane by the equation

$$\left(\nabla^2-c^2\right)X\left(r,\varphi\right)=\partial W\left(r,\varphi\right), \tag{16}$$

where

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2},$$

is the Laplacian operator in terms of spherical coordinates. Now we have

$$\begin{aligned} \nabla^2 e^{i\ell\varphi}\int_0^{\infty}J_\ell\left(\rho r\right)Z_\ell\left(\rho d\rho\right) &= e^{i\ell\varphi}\int_0^{\infty}\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{\ell^2}{r^2}\right]J_\ell\left(\rho r\right)Z_\ell\left(\rho d\rho\right) \\ &= -e^{i\ell\varphi}\int_0^{\infty}\rho^2 J_\ell\left(\rho r\right)Z_\ell\left(\rho d\rho\right), \end{aligned}$$

hence

$$\left(\nabla^2-c^2\right)e^{i\ell\varphi}\int_0^{\infty}J_\ell\left(\rho r\right)Z_\ell\left(\rho d\rho\right)=-e^{i\ell\varphi}\int_0^{\infty}\left(\rho^2+c^2\right)J_\ell\left(\rho r\right)Z_\ell\left(\rho d\rho\right).$$

Let us compare the terms in the integrands of the integrals of the equation (16), and obtain

$$-\left(\rho^2+c^2\right)Z_\ell\left(\rho d\rho\right)=W_\ell\left(\rho d\rho\right),\quad \left(\rho^2+c^2\right)^2 2\pi F\left(\rho d\rho\right)=\sigma^2\rho d\rho,$$

$$F\left(\rho d\rho\right)=\frac{\sigma^2\rho d\rho}{2\pi\left(\rho^2+c^2\right)^2}.$$

Hence the covariance is obtained by inversion,

$$\begin{aligned}\operatorname{Cov}\left(X\left(\underline{x}\right),X\left(\underline{y}\right)\right)&=2\pi\int_0^\infty J_0\left(\rho r\right)F\left(\rho d\rho\right)=\int_0^\infty J_0\left(\rho r\right)\frac{\sigma^2\rho d\rho}{\left(\rho^2+c^2\right)^2}\\&=\sigma^2\frac{rK_{-1}\left(cr\right)}{2c}.\end{aligned}$$

This covariance belongs to Matérn Class, see [Whi54] ($K_{-1}=K_1$), in terms of modified Bessel (Hankel) function. The bispectrum of the process defined by the Laplacian model is given by

$$S_3\left(\rho_1,\rho_2,\rho_3\right)=\prod_{k=1}^3\frac{\sigma^2}{\rho_k^2+c^2},$$

where $\rho_1^2=\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos(\eta)$. We express the third order covariances according to the Theorem 2, applying the transformation (14) to the bispectrum. If $(a/b)^2<1$, we have

$$\int_0^\pi\frac{\cos\left(\ell\eta\right)}{b+a\cos\left(\eta\right)}d\eta=\frac{\pi}{\sqrt{b^2-a^2}}\left(\frac{\sqrt{b^2-a^2}-b}{a}\right)$$

see [GR00] 3.613.1. Put

$$\begin{aligned}a=-2\rho_2\rho_3,\quad b=\rho_2^2+\rho_3^2+c^2\\b^2-a^2=\left(\left(\rho_2-\rho_3\right)^2+c^2\right)\left(\left(\rho_2+\rho_3\right)^2+c^2\right),\end{aligned}$$

hence

$$\int_0^\pi\frac{\cos\left(\ell\eta\right)}{\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos\left(\eta\right)+c^2}d\eta=\frac{\pi}{\sqrt{b^2-a^2}}\left(\frac{\sqrt{b^2-a^2}-b}{a}\right)^\ell.$$

In this way we arrive at the Fourier expansion of the third order covariances

$$\mathcal{C}\left(r_1,r_2,r_3\right)=f_0+2\sum_{\ell=1}^\infty f_l\cos\left(\ell\varphi\right),$$

with coefficients

$$f_l=4\pi\iint_0^\infty\frac{\pi}{\sqrt{b^2-a^2}}\left(\frac{\sqrt{b^2-a^2}-b}{a}\right)^\ell J_\ell\left(\rho_2r_2\right)J_\ell\left(\rho_3r_3\right)\prod_{k=2}^3\rho_kd\rho_k.$$

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