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# Proportionally modular numerical semigroups with embedding dimension three

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**Abstract.** In this paper we study numerical semigroups, generated by three positive integers, that are the set of solutions of a Diophantine inequality of the form $ax \mod b \leq cx$. As a consequence, we show that, if these numerical semigroups are irreducible (this is, symmetric or pseudo-symmetric), then they are the set of solutions of a Diophantine inequality of the form $\alpha x \mod \beta \leq x$.

## 1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* is a subset $S$ of $\mathbb{N}$ such that it is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. If $A \subseteq \mathbb{N}$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by $A$, this is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\},\ a_1, \dots, a_n \in A,\ \lambda_1, \dots, \lambda_n \in \mathbb{N}\}.$$

It is well known (see for instance [11]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd\{A\} = 1$, where gcd means greatest common divisor.

Let $S$ be a numerical semigroup and let $X$ be a subset of $S$. We say that $X$ is a *system of generators* of $S$ if $S = \langle X \rangle$. In addition, if no proper subset of $X$ generates $S$, then we say that $X$ is a *minimal system of generators* of $S$. Every

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numerical semigroup admits a unique minimal system of generators. Moreover, such system has finitely many elements (see [1], [11]).

If $n_1 < n_2 < \cdots < n_e$ are the elements of the minimal system of generators of a numerical semigroup $S$, then $n_1$, $n_2$ and $e$ are known as the *multiplicity*, the *ratio*, and the *embedding dimension* of $S$, and they are denoted by $\mathrm{m}(S)$, $\mathrm{r}(S)$ and $\mathrm{e}(S)$, respectively.

Let $m$, $n$ be integers such that $n \neq 0$. We denote by $m \mod n$ the remainder of the division of $m$ by $n$. A *proportionally modular Diophantine inequality* (see [12]) is an expression of the form

$$ax \mod b \leq cx \tag{1}$$

where $a$, $b$, $c$ are positive integers. We call $a$, $b$ and $c$ the *factor*, the *modulus*, and the *proportion* of the inequality, respectively. Let $\mathrm{S}(a, b, c)$ be the set of integer solutions of (1). Then $\mathrm{S}(a, b, c)$ is a numerical semigroup. We say that a numerical semigroup is a *proportionally modular numerical semigroup (*PM-*semigroup)* if it is the set of integer solutions of a proportionally modular Diophantine inequality.

A *modular Diophantine inequality* (see [13]) is an expression of the form

$$ax \mod b \leq x, \tag{2}$$

this is, it is a proportionally modular Diophantine inequality with proportion equal to one. A numerical semigroup is a *modular numerical semigroup (*M-*semigroup)* if it is the set of integer solutions of a modular Diophantine inequality. Therefore, every M-semigroup is a PM-semigroup, but the converse is false. In effect, from Example 26 in [12], we have that the numerical semigroup $\langle 3, 8, 10\rangle$ is a PM-semigroup, but is not an M-semigroup.

A numerical semigroup is an *irreducible numerical semigroup* if it can not be expressed as an intersection of two numerical semigroups containing it properly. In [9] it is shown that the class of irreducible numerical semigroups is the union of two widely studied classes of numerical semigroups: the *symmetric numerical semigroups* and the *pseudo-symmetric numerical semigroups* (see [1], [2], [5]).

From Theorem 16 in [12] and Theorem 6 in [3] we deduce that the numerical semigroup $\langle n, n+1, \ldots, 2n-2\rangle$ is a symmetric PM-semigroup if $n$ is an integer such that $n \geq 3$. Therefore, there exist irreducible PM-semigroups with arbitrary embedding dimension. However, this result is false for M-semigroups. In fact, as a consequence of the results in [7], [8], we have that every irreducible M-semigroup has embedding dimension less than or equal to three. More precisely, in this paper we will show that the irreducible M-semigroups are just the irreducible PM-semigroups with embedding dimension less than or equal to three.

We summarize the content of this article in the following way. From Remark 24 in [14], we know that, if $S$ is a PM-semigroup, then $\gcd\{\mathrm{r}(S), \mathrm{m}(S)\} = 1$. For this reason, we will consider $m$, $r$ integers such that $3 \leq m < r$ and $\gcd\{m, r\} = 1$. In Section 2 we will give in explicit form the elements of the set

$$\mathrm{PM}(m, r) = \{S \mid S \text{ is a PM-semigroup, } \mathrm{m}(S) = m,\ \mathrm{r}(S) = r,\ \text{and } \mathrm{e}(S) = 3\}.$$

Moreover, we will see that $\mathrm{PM}(m, r)$ has cardinality equal to $m + r - \left\lceil \frac{2r}{m} \right\rceil - 3$. We will compute positive integers $a$, $b$, $c$ such that $S = S(a, b, c)$ for each $S \in \mathrm{PM}(m, r)$.

In Section 3 we will study the set

$$\mathrm{Sy}(\mathrm{PM}(m, r)) = \{S \in \mathrm{PM}(m, r) \mid S \text{ is symmetric}\}.$$

We will compute the cardinality of this set from the divisors of $m$ and $r$. We will give positive integers $a$, $b$ such that $S = \mathrm{S}(a, b, 1)$ for each $S \in \mathrm{Sy}(\mathrm{PM}(m, r))$.

Finally, Section 4 will be devoted to the study of the set

$$\mathrm{PSy}(\mathrm{PM}(n_1, n_2)) = \{S \in \mathrm{PM}(n_1, n_2) \mid S \text{ is pseudo-symmetric}\}.$$

We will show that, if $m = 3$, then this set has cardinality equal to 1 and, if $m \geq 4$, then the cardinality is equal to the cardinality of $\{x \in \{m, r\} \mid x \text{ is odd}\}$. At the end, we will compute positive integers $a$, $b$ such that $S = \mathrm{S}(a, b, 1)$ for each $S \in \mathrm{PSy}(\mathrm{PM}(m, r))$.

## 2. PM-semigroups

Let $x_1, x_2, \ldots, x_q$ be a sequence of integer numbers. We say that it is arranged in a convex form if one of the following conditions is satisfied,

(1) $x_1 \leq x_2 \leq \cdots \leq x_q$;

(2) $x_1 \geq x_2 \geq \cdots \geq x_q$;

(3) There exists $h \in \{2, \ldots, q-1\}$ such that $x_1 \geq \cdots \geq x_h \leq \cdots \leq x_q$.

As a consequence of Theorem 31 in [14] (see its proof and Corollary 18 of [14]) we have the next lemma.

**Lemma 2.1.** *A numerical semigroup $S$ is a* PM*-semigroup if and only if there exists a convex arrangement $n_1, n_2, \ldots, n_e$ of its set of minimal generators that satisfies the following conditions,*

(1) $\gcd\{n_i, n_{i+1}\} = 1$ *for all* $i \in \{1, \dots, e-1\}$;

(2) $(n_{i-1} + n_{i+1}) \equiv 0 \mod n_i$ *for all* $i \in \{2, \dots, e-1\}$.

In what follows we will suppose that $n_1$, $n_2$ are integer numbers such that $3 \leq n_1 < n_2$ and $\gcd\{n_1, n_2\} = 1$. Moreover, to simplify the notation we will use the following sets,

- $A(n_1) = \{2, \dots, n_1 - 1\}$;
- $A(n_1, n_2) = \left\{\left\lceil \frac{2n_2}{n_1} \right\rceil, \dots, n_2 - 1\right\}$;
- $B(n_1) = \{k \in A(n_1) \text{ such that } k \mid n_1\}$;
- $B(n_1, n_2) = \{t \in A(n_1, n_2) \text{ such that } t \mid n_2\}$.

For the definition of $A(n_1, n_2)$, remember that, if $q \in \mathbb{Q}$ (where $\mathbb{Q}$ is the set of rational numbers), then we denote $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$. Moreover, for the definitions of $B(n_1)$ and $B(n_1, n_2)$, if $a$, $b$ are positive integers, then we denote by $a|b$ that $a$ divides $b$.

**Lemma 2.2.** *If* $k \in A(n_1)$, *then* $S = \langle n_1, n_2, kn_2 - n_1 \rangle$ *is a* PM-*semigroup with* $\mathrm{e}(S) = 3$. *Moreover,* $n_1 < n_2 < kn_2 - n_1$.

Proof. It is clear that $n_1 < n_2 < kn_2 - n_1$ because $k \geq 2$ (by hypothesis) and $n_1 < n_2$ (by assumption). In order to give the proof, it is enough to see that $kn_2 - n_1 \notin \langle n_1, n_2 \rangle$. Otherwise, there will exist $\lambda, \mu \in \mathbb{N}$ such that $kn_2 - n_1 = \lambda n_1 + \mu n_2$. Then $(k - \mu)n_2 = (\lambda + 1)n_1$. Because $\gcd\{n_1, n_2\} = 1$, we deduce that $k - \mu \geq n_1$ which is a contradiction with the condition $k \in A(n_1)$. Finally, if we consider the arrangement $n_1$, $n_2$, $kn_2 - n_1$ of the set of minimal generators for $S$, then it is a PM-semigroup as a consequence of Lemma 2.1. □

**Lemma 2.3.** *If* $t \in A(n_1, n_2)$, *then* $S = \langle n_1, n_2, tn_1 - n_2 \rangle$ *is a* PM-*semigroup with* $\mathrm{e}(S) = 3$. *Moreover,* $n_1 < n_2 < tn_1 - n_2$.

Proof. Let us have $t \in \mathbb{Z}$. Let us observe that $n_2 < tn_1 - n_2$ if and only if $t > \frac{2n_2}{n_1}$, which is equivalent to $t \geq \left\lceil \frac{2n_2}{n_1} \right\rceil$ because $n_1 \geq 3$. Using a similar reasoning as in Lemma 2.2, we prove that $\mathrm{e}(S) = 3$. Finally, $S$ is a PM-semigroup by Lemma 2.1 with the arrangement $tn_1 - n_2$, $n_1$, $n_2$ of the set of its minimal generators. □

Let us denote the set

$$\mathrm{PM}(n_1, n_2) = \{S \mid S \text{ is a PM-semigroup}, \ \mathrm{m}(S) = n_1, \ \mathrm{r}(S) = n_2, \mathrm{e}(S) = 3\}.$$

**Theorem 2.4.** $\mathrm{PM}(n_1, n_2)$ *is equal to the union of*

$$\mathrm{PM}_1(n_1, n_2) = \{\langle n_1, n_2, kn_2 - n_1 \rangle \mid k \in A(n_1)\}$$

*and*

$$\mathrm{PM}_2(n_1,n_2)=\{\langle n_1,n_2,tn_1-n_2\rangle \mid t\in A(n_1,n_2)\}\,.$$

PROOF. By applying Lemmas 2.2 and 2.3, $\mathrm{PM}_1(n_1,n_2)\cup\mathrm{PM}_2(n_1,n_2)\subseteq\mathrm{PM}(n_1,n_2)$. In order to prove the other inclusion, let us have $S\in\mathrm{PM}(n_1,n_2)$. Then $S$ is minimally generated by $\{n_1,n_2,n_3\}$ with $n_1<n_2<n_3$. From Lemma 2.1, we deduce that $(n_1+n_3)\equiv 0 \mod n_2$ or $(n_2+n_3)\equiv 0 \mod n_1$. If $(n_1+n_3)\equiv 0 \mod n_2$, then $n_3=kn_2-n_1$ for an integer $k\geq 2$. Moreover, $k\leq n_1-1$ because $n_3\in\langle n_1,n_2\rangle$ in other case, and this is a contradiction with the fact that $\{n_1,n_2,n_3\}$ is the minimal system of generators of $S$. On the other hand, if $(n_2+n_3)\equiv 0 \bmod n_1$, then $n_3=tn_1-n_2$ for an integer $t\geq\left\lceil\frac{2n_2}{n_1}\right\rceil$. Moreover, $t\leq n_2-1$ because $n_3\notin\langle n_1,n_2\rangle$. □

Now we compute the cardinality of $\mathrm{PM}(n_1,n_2)$. First, we need a lemma that is the key.

**Lemma 2.5.** *Let $k,t$ be positive integers such that $k\leq n_1-1$ and $t\leq n_2-1$. Then $kn_2-n_1=tn_1-n_2$ if and only if $k=n_1-1$ and $t=n_2-1$.*

PROOF. It is obvious that $kn_2-n_1=tn_1-n_2$ if and only if $(k+1)n_2=(t+1)n_1$. Because $\gcd\{n_1,n_2\}=1$, $k\leq n_1-1$, and $t\leq n_2-1$, we deduce that $(k+1)n_2=(t+1)n_1$ if and only if $k=n_1-1$ and $t=n_2-1$. □

Let us have a set $A$. We denote by $\#A$ the cardinality of $A$.

**Corollary 2.6.** $\#\,\mathrm{PM}(n_1,n_2)=n_1+n_2-\left\lceil\frac{2n_2}{n_1}\right\rceil-3$.

PROOF. From Lemma 2.5, we have that $\mathrm{PM}_1(n_1,n_2)\cap\mathrm{PM}_2(n_1,n_2)$ has cardinality equal to one. By applying Theorem 2.4, we have the conclusion. □

To clarify the previous results, we give an example.

*Example 2.1.* We want to compute $\mathrm{PM}(5,7)$. First, from Corollary 2.6, we know that $\#\,\mathrm{PM}(5,7)=5+7-\left\lceil\frac{14}{5}\right\rceil-3=6$. Second, by applying Theorem 2.4, we have that the elements of $\mathrm{PM}(5,7)$ are,

- $S_1=\langle 5,7,2\cdot 7-5\rangle=\langle 5,7,9\rangle$;
- $S_2=\langle 5,7,3\cdot 7-5\rangle=\langle 5,7,16\rangle$;
- $S_3=\langle 5,7,4\cdot 7-5\rangle=\langle 5,7,23\rangle=\langle 5,7,6\cdot 5-7\rangle$;
- $S_4=\langle 5,7,3\cdot 5-7\rangle=\langle 5,7,8\rangle$;
- $S_5=\langle 5,7,4\cdot 5-7\rangle=\langle 5,7,13\rangle$;
- $S_6=\langle 5,7,5\cdot 5-7\rangle=\langle 5,7,18\rangle$.

Let $S$ be a PM-semigroup. By definition, there exist positive integers $a$, $b$ and $c$ such that $S = \mathrm{S}(a,b,c) = \{x \in \mathbb{N} \mid ax \mod b \le cx\}$. We finish this section giving, for each $S \in \mathrm{PM}(n_1, n_2)$, a triplet $(a,b,c)$ such that $S = \mathrm{S}(a,b,c)$. For this purpose, we introduce some concepts and results.

Let $\mathbb{Q}_0^+$ be the set of nonnegative rational numbers. If $A \subseteq \mathbb{Q}_0^+$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{Q}_0^+, +)$ generated by $A$, this is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\},\ a_1, \ldots, a_n \in A,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

It is clear that $\mathrm{S}(A) = \langle A \rangle \cap \mathbb{N}$ is a submonoid of $(\mathbb{N}, +)$. If $\alpha, \beta$ are two rational numbers such that $\alpha < \beta$, then we denote $[\alpha, \beta] = \{x \in \mathbb{Q} \mid \alpha \le x \le \beta\}$. The next result is a consequence of Lemmas 12 and 21 in [12].

**Lemma 2.7.** *Let $S$ be a numerical semigroup. Then $S$ is a* PM*-semigroup if and only if there exist $\alpha$, $\beta$ positive rational numbers such that $S = \mathrm{S}([\alpha, \beta])$.*

And this one is consequence of Lemma 6 and Corollary 9 in [12].

**Lemma 2.8.** *Let $a$, $b$, $c$ be positive integers such that $c < a < b$. Then* $\mathrm{S}(a,b,c) = \mathrm{S}\big(\big[\frac{b}{a}, \frac{b}{a-c}\big]\big)$.

Following the ideas in [14], we say that $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_r}{b_r}$ is a Bézout sequence if $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r$ are positive integers such that $a_{i+1}b_i - a_i b_{i+1} = 1$ for all $i \in \{1, 2, \ldots, r-1\}$. The following result is Theorem 12 in [14].

**Lemma 2.9.** *If $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_r}{b_r}$ is a Bézout sequence, then*

$$\langle a_1, a_2, \ldots, a_r \rangle = \mathrm{S}\left(\left[\frac{a_1}{b_1}, \frac{a_r}{b_r}\right]\right).$$

In the rest of the paper, we denote by $u, v$ the unique positive integers such that $\frac{n_1}{u} < \frac{n_2}{v}$ is a Bézout sequence and $1 < \frac{n_1}{u}$.

**Proposition 2.10.** (1) *If $k \in A(n_1)$, then*

$$\langle n_1, n_2, kn_2 - n_1 \rangle = \{x \in \mathbb{N} \mid u(kn_2 - n_1)x \mod n_1(kn_2 - n_1) \le kx\}.$$

(2) *If $t \in A(n_1, n_2)$, then*

$$\langle n_1, n_2, tn_1 - n_2 \rangle = \{x \in \mathbb{N} \mid n_2(tu - v)x \mod n_2(tn_1 - n_2) \le tx\}.$$

PROOF. (1) It is clear that $\frac{n_1}{u} < \frac{n_2}{v} < \frac{kn_2-n_1}{kv-u}$ is a Bézout sequence. From Lemma 2.9, we have

$$\langle n_1, n_2, kn_2 - n_1\rangle = \mathrm{S}\left(\left[\frac{n_1}{u}, \frac{kn_2-n_1}{kv-u}\right]\right) = \mathrm{S}\left(\left[\frac{n_1(kn_2-n_1)}{u(kn_2-n_1)}, \frac{n_1(kn_2-n_1)}{n_1(kv-u)}\right]\right).$$

By using Lemma 2.8, we have the conclusion.

(2) The proof is similar to the previous one with the Bézout sequence given by $\frac{tn_1-n_2}{tu-v} < \frac{n_1}{u} < \frac{n_2}{v}$. □

*Example 2.2.* Let $S_1$ be the numerical semigroup that appeared in Example 2.1, this is, $S_1 = \langle 5, 7, 2\cdot 7 - 5\rangle$. Let us observe that $\frac{5}{3} < \frac{7}{4} < \frac{2\cdot 7-5}{2\cdot 4-3} = \frac{9}{5}$ is a Bézout sequence. Therefore,

$$S_1 = \langle 5,7,9\rangle = \mathrm{S}\left(\left[\frac{5}{3}, \frac{9}{5}\right]\right) = \mathrm{S}\left(\left[\frac{45}{27}, \frac{45}{25}\right]\right) = \{x \in \mathbb{N} \mid 27x \mod 45 \leq 2x\}.$$

## 3. Symmetric PM-semigroups

Let $S$ be a numerical semigroup. The greatest integer that does not belong to $S$ is called the Frobenius number of $S$ (see [6]) and denoted by $\mathrm{F}(S)$. In [9] it is shown that a numerical semigroup $S$ is irreducible if and only if it is maximal (with respect to the inclusion order) in the set of all numerical semigroups with fixed Frobenius number. Therefore, on applying [2], we have that a numerical semigroup $S$ is symmetric (respectively, pseudo-symmetric) if and only if $S$ is irreducible and $\mathrm{F}(S)$ is odd (respectively, even).

The next result is Proposition 39 in [14].

**Lemma 3.1.** *Let $S$ be a numerical semigroup with minimal system of generators $\{m_1, m_2, m_3\}$. Let us suppose that $\gcd\{m_1, m_2\} = \gcd\{m_2, m_3\} = 1$ and $dm_2 = m_1 + m_3$. Then $S$ is symmetric if and only if $d = \gcd\{m_1, m_3\}$.*

In the following result it is shown what elements of $\mathrm{PM}(n_1, n_2)$ are symmetric. We denote $\mathrm{Sy}(\mathrm{PM}(n_1, n_2)) = \{S \in \mathrm{PM}(n_1, n_2) \mid S \text{ is symmetric}\}$.

**Proposition 3.2.** $\mathrm{Sy}(\mathrm{PM}(n_1, n_2))$ *is equal to the union of*

$$\mathrm{Sy}(\mathrm{PM}_1(n_1, n_2)) = \{\langle n_1, n_2, kn_2 - n_1\rangle \mid k \in B(n_1)\}$$

*and*

$$\mathrm{Sy}(\mathrm{PM}_2(n_1, n_2)) = \{\langle n_1, n_2, tn_1 - n_2\rangle \mid t \in B(n_1, n_2)\}.$$

Proof. Let us take $k \in A(n_1)$. From Lemmas 2.2 and 3.1, the numerical semigroup $\langle n_1, n_2, kn_2 - n_1\rangle$ is symmetric if and only if $k = \gcd\{n_1, kn_2 - n_1\}$, but this equality is equivalent to the condition $k|n_1$.

If $t \in A(n_1, n_2)$, we use Lemmas 2.3 and 3.1, and a similar reasoning to the previous one.

By using Theorem 2.4 we finish the proof. □

As a consequence of Proposition 3.2 and Lemma 2.5, we have the following result.

**Corollary 3.3.** $\#\,\mathrm{Sy}(\mathrm{PM}(n_1, n_2)) = \#B(n_1) + \#B(n_1, n_2)$.

Again, we give an example to clarify the previous results.

*Example 3.1.* Let us show that there do not exist symmetric PM-semigroups such that the embedding dimension is 3, the multiplicity is 5, and the ratio is 49. In fact, if $k \in \{2, 3, 4\}$ and $t \in \left\{\left\lceil \frac{98}{5}\right\rceil, \ldots, 48\right\} = \{20, \ldots, 48\}$, then $k \nmid 5$ and $t \nmid 49$. By applying Corollary 3.3, we have the statement.

Now, let us compute $\mathrm{Sy}(\mathrm{PM}(10, 21))$. Since $B(10) = \{2, 5\}$ and $B(10, 21) = \{7\}$, from Proposition 3.2, we have that

$$\mathrm{Sy}(\mathrm{PM}(10, 21)) = \{\langle 10, 21, 32\rangle, \langle 10, 21, 95\rangle, \langle 10, 21, 49\rangle\}.$$

In the introduction, we noted that every irreducible PM-semigroup with embedding dimension equal to three is an M-semigroup. In the next result we prove this fact for symmetric PM-semigroups.

**Proposition 3.4.** (1) *If $k \in B(n_1)$, then*

$$\langle n_1, n_2, kn_2 - n_1\rangle = \left\{x \in \mathbb{N} \mid u\left(n_2 - \frac{n_1}{k}\right)x \mod n_1\left(n_2 - \frac{n_1}{k}\right) \le x\right\}.$$

(2) *If $t \in B(n_1, n_2)$, then*

$$\langle n_1, n_2, tn_1 - n_2\rangle = \left\{x \in \mathbb{N} \mid \frac{n_2}{t}(tu - v)x \mod \frac{n_2}{t}(tn_1 - n_2) \le x\right\}.$$

Proof. Let us observe that, if $d$ is a common divisor of $a$, $b$, $c$, then the inequalities $ax \mod b \le cx$ and $\frac{a}{d}x \mod \frac{b}{d} \le \frac{c}{d}x$ have the same set of solutions. From this fact and Proposition 2.10, we have the proof. □

In [8] it is shown that, if $S$ is a symmetric M-semigroup, then $\mathrm{e}(S) \le 3$. On the other hand, from Lemmas 2.8 and 2.9, $\langle n_1, n_2\rangle = \mathrm{S}\left(\left[\frac{n_1}{u}, \frac{n_2}{v}\right]\right) = \mathrm{S}\left(\left[\frac{n_1 n_2}{un_2}, \frac{n_1 n_2}{n_1 v}\right]\right) = \{x \in \mathbb{N} \mid un_2 x \mod n_1 n_2 \le x\}$. Therefore, $\langle n_1, n_2\rangle$ is an M-semigroup. Moreover, it is well known (see for instance [2]) that every numerical semigroup with embedding dimension equal to two is symmetric. From all this, we can enunciate the next result.

**Proposition 3.5.** *$S$ is a symmetric* M*-semigroup with* $\mathrm{m}(S) = n_1$ *and* $\mathrm{r}(S) = n_2$ *if and only if $S$ is in one of the following three cases,*

(1) $S = \langle n_1, n_2 \rangle$;

(2) $S = \langle n_1, n_2, kn_2 - n_1 \rangle$ *with* $k \in B(n_1)$;

(3) $S = \langle n_1, n_2, tn_1 - n_2 \rangle$ *with* $t \in B(n_1, n_2)$.

## 4. Pseudo-symmetric PM-semigroups

Our purpose is to study the pseudo-symmetric PM-semigroups with embedding dimension equal to three. For this aim, we begin with a series of lemmas.

Let $S$ be a numerical semigroup and let $\{m_1, m_2, m_3\}$ be its minimal system of generators. We define the numbers

$$c_i = \min \{x \in \mathbb{N} \setminus \{0\} \mid x m_i \in \langle \{m_1, m_2, m_3\} \setminus \{m_i\} \rangle\}, \ i \in \{1, 2, 3\}.$$

The next result is Theorem 10 in [10].

**Lemma 4.1.** *Let $S$ be a numerical semigroup with embedding dimension 3. Then $S$ is pseudo-symmetric if and only if, for some rearrangement of its generators $\{m_1, m_2, m_3\}$, we have that* $c_1 m_1 = (c_2 - 1)m_2 + m_3$, $c_2 m_2 = (c_3 - 1)m_3 + m_1$, *and* $c_3 m_3 = (c_1 - 1)m_1 + m_2$.

**Lemma 4.2.** *Let $S$ be a numerical semigroup with minimal system of generators $\{m_1, m_2, m_3\}$. If $k \in \mathbb{N}$ and $km_2 = m_1 + m_3$, then $k = c_2$.*

Proof. If $c_2 < k$, then we have that $c_2 m_2$ is either multiple of $m_1$ or multiple of $m_3$. Let us assume that $c_2 m_2 = \lambda m_1$, with $\lambda \in \mathbb{N}$. Then $m_1 + m_3 = km_2 = (k - c_2)m_2 + \lambda m_1$. Therefore, $m_3 = (k - c_2)m_2 + (\lambda - 1)m_1$. We conclude that $m_3 \in \langle m_1, m_2 \rangle$, in contradiction with the fact that $\{m_1, m_2, m_3\}$ is a minimal system of generators. □

The following result is Theorem 14 in [7].

**Lemma 4.3.** *Let $t$, $n$ be two positive integers such that $t < n$ and $t$ divides $n$. Then* $S = \left\langle \frac{n}{t}, t + 2, \frac{n+t+2}{2} \right\rangle$ *is a pseudo-symmetric modular numerical semigroup with Frobenius number $n - t - 2$ if and only if $\frac{n}{t}$ is odd and* $\gcd\left\{t + 2, \frac{n}{t}\right\} = 1$.

For our purpose, the next result is fundamental.

**Proposition 4.4.** *The following conditions are equivalent.*

(1) *$S$ is a pseudo-symmetric* PM*-semigroup with* $\mathrm{e}(S) = 3$.

(2) *There exists an arrangement* $m_1$, $m_2$, $m_3$ *of the minimal generators of* $S$ *such that* $m_1$ *is odd,* $\gcd\{m_1, m_2\} = 1$*, and* $m_3 = \frac{m_1+1}{2}(m_2 - 2) + 1$.

(3) $S$ *is a pseudo-symmetric* M*-semigroup with* $\mathrm{e}(S) = 3$.

Proof. (1) $\Rightarrow$ (2). By applying Lemmas 2.1 and 4.2, we deduce that there exists an arrangement $m_1$, $m_2$, $m_3$ of the minimal generators of $S$ such that $\gcd\{m_1, m_2\} = \gcd\{m_2, m_3\} = 1$ and $c_2 m_2 = m_1 + m_3$. Besides, as $S$ has embedding dimension 3, it is clear that $m_1$, $m_2$ and $m_3$ are greater than or equal to 3. As $c_2 m_2 = m_1 + m_3$, from Lemma 4.1 we deduce that either $c_1 = 2$ or $c_3 = 2$. Let us suppose, without loss of generality, that $c_3 = 2$. By using again Lemma 4.1, we have $c_2 m_2 = m_1 + m_3$ and $2m_3 = (c_1 - 1)m_1 + m_2$. Therefore, $2c_2 m_2 = 2m_1 + 2m_3 = 2m_1 + (c_1 - 1)m_1 + m_2$ and, in consequence, $(2c_2 - 1)m_2 = (c_1 + 1)m_1$. Since $\gcd\{m_1, m_2\} = 1$ and $1 \le c_1 \le m_2$, we have $c_1 + 1 = m_2$ and $m_1 = 2c_2 - 1$. Thus $m_1$ is odd. Moreover, $m_3 = c_2 m_2 - m_1 = c_2(c_1 + 1) - 2c_2 + 1 = c_2(c_1 - 1) + 1 = \frac{m_1+1}{2}(m_2 - 2) + 1$.

(2) $\Rightarrow$ (3). Let $t = m_2 - 2$ and $n = m_1(m_2 - 2)$. Then $\frac{n}{t} = m_1$ is odd and $\gcd\{t + 2, \frac{n}{t}\} = \gcd\{m_2, m_1\} = 1$. Thus, in view of Lemma 4.3, we have that $S = \langle m_1, m_2, m_3\rangle = \left\langle \frac{n}{t}, t + 2, \frac{n+t+2}{2} \right\rangle$ is a pseudo-symmetric modular numerical semigroup. To conclude the proof it suffices now to show that $S$ has embedding dimension 3. However, we can deduce it by the assert that all numerical semigroups with embedding dimension 2 are symmetric (see [4]). Therefore, these ones are not pseudo-symmetric.

(3) $\Rightarrow$ (1). It is obvious. $\square$

The next result gives us information about the arrangement of $m_1$, $m_2$, $m_3$ in the previous proposition.

**Lemma 4.5.** *Let* $m_1$, $m_2$, $m_3$ *be integers such that they are greater than or equal to 4 and* $m_3 = \frac{m_1+1}{2}(m_2 - 2) + 1$*. Then* $m_3 = \max\{m_1, m_2, m_3\}$.

Proof. It is easy to see that $m_3 = \frac{m_1+1}{2}(m_2-2)+1 \ge \frac{m_1+1}{2}2+1$. Therefore, $m_3 \ge m_1$. In the same way, $m_3 = \frac{m_1+1}{2}(m_2 - 2) + 1 \ge 2(m_2 - 2) + 1$, and $m_3 \ge m_2$. $\square$

From Theorem 7 in [9], we deduce the next result.

**Lemma 4.6.** $S$ *is a pseudo-symmetric numerical semigroup with* $\mathrm{m}(S) = \mathrm{e}(S) = 3$ *if and only if* $S$ *is minimally generated by* $\{3, x + 3, 2x + 3\}$*, where* $x$ *is a positive integer such that* $x \not\equiv 0 \mod 3$.

We denote $\mathrm{PSy}(\mathrm{PM}(n_1, n_2)) = \{S \in \mathrm{PM}(n_1, n_2) \mid S \text{ is pseudo-symmetric}\}$.

**Proposition 4.7.** *$S \in \mathrm{PSy}(\mathrm{PM}(n_1, n_2))$ if and only if ($n_1$ is odd and $S = \langle n_1, n_2, \frac{n_1+1}{2} n_2 - n_1 \rangle$) or ($n_2$ is odd, $n_1 \geq 4$, and $S = \langle n_1, n_2, \frac{n_2+1}{2} n_1 - n_2 \rangle$).*

Proof. If $k \in A(n_1)$, from Lemma 2.2, we know that $\langle n_1, n_2, kn_2 - n_1 \rangle$ is a PM-semigroup with $\mathrm{e}(S) = 3$ and $n_1 < n_2 < kn_2 - n_1$. If $n_1 = 3$, by applying Lema 4.6, we deduce that $\langle n_1, n_2, kn_2 - n_1 \rangle$ is pseudo-symmetric if and only if $n_2 - 3 = kn_2 - 3 - n_2$, and this equality is equivalent to $k = 2 = \frac{n_1+1}{2}$.

If $n_1 \geq 4$, from Proposition 4.4 and Lemma 4.5, we deduce that $\langle n_1, n_2, kn_2 - n_1 \rangle$ is pseudo-symmetric if and only if $n_1$ is odd and $kn_2 - n_1 = \frac{n_1+1}{2}(n_2 - 2) + 1$ or $n_2$ is odd and $kn_2 - n_1 = \frac{n_2+1}{2}(n_1 - 2) + 1$. The first case is possible if and only if $k = \frac{n_1+1}{2}$. The second case is not possible because, if $kn_2 - n_1 = \frac{n_2+1}{2}(n_1 - 2) + 1$, then $3n_1 = (2k - n_1 + 2)n_2$. Because $\gcd\{n_1, n_2\} = 1$, we have that 3 is a multiple of $n_2$, but $n_2 \geq 4$.

If $t \in A(n_1, n_2)$, from Lemma 2.3, we know that $\langle n_1, n_2, tn_1 - n_2 \rangle$ is a PM-semigroup with $\mathrm{e}(S) = 3$ and $n_1 < n_2 < tn_1 - n_2$. If $n_1 = 3$, again from Lemma 4.6, we deduce that $\langle n_1, n_2, tn_1 - n_2 \rangle$ is pseudo-symmetric if and only if $t = n_2 - 1$. Therefore, $\langle n_1, n_2, (n_2 - 1)n_1 - n_2 \rangle = \langle 3, n_2, 2n_2 - 3 \rangle = \langle n_1, n_2, \frac{n_1+1}{2} n_2 - n_1 \rangle$.

If $n_1 \geq 4$, again from Proposition 4.4 and Lemma 4.5, $\langle n_1, n_2, tn_1 - n_2 \rangle$ is pseudo-symmetric if and only if $n_1$ is odd and $tn_1 - n_2 = \frac{n_1+1}{2}(n_2 - 2) + 1$ or $n_2$ is odd and $tn_1 - n_2 = \frac{n_2+1}{2}(n_1 - 2) + 1$. Now, the second case is possible if and only if $t = \frac{n_2+1}{2}$. The first case is not possible because, if $tn_1 - n_2 = \frac{n_1+1}{2}(n_2 - 2) + 1$, then $3n_2 = (2t - n_2 + 2)n_1$. Because $\gcd\{n_1, n_2\} = 1$, now we have that 3 is a multiple of $n_1$, but $n_1 \geq 4$.

By applying Theorem 2.4, we finish the proof. □

As an immediate consequence of the previous proposition we have the next result.

**Corollary 4.8.** (1) *If $n_1 = 3$, then $\# \mathrm{PSy}(\mathrm{PM}(n_1, n_2)) = 1$.*

(2) *If $n_1 \geq 4$, then $\# \mathrm{PSy}(\mathrm{PM}(n_1, n_2)) = \#\{x \in \{n_1, n_2\} \mid x \text{ is odd}\}$.*

Because $\gcd\{n_1, n_2\} = 1$, then $\#\{x \in \{n_1, n_2\} \mid x \text{ is odd}\} \geq 1$. Therefore, $\mathrm{PSy}(\mathrm{PM}(n_1, n_2))$ is always nonempty.

*Example 4.1.* From Proposition 4.7, it is easy to check that

- $\mathrm{PSy}(\mathrm{PM}(5, 7)) = \{\langle 5, 7, 16 \rangle, \langle 5, 7, 13 \rangle\}$;
- $\mathrm{PSy}(\mathrm{PM}(5, 6)) = \{\langle 5, 6, 13 \rangle\}$;
- $\mathrm{PSy}(\mathrm{PM}(6, 7)) = \{\langle 6, 7, 17 \rangle\}$.

From Proposition 4.4, we know that every pseudo-symmetric PM-semigroup with embedding dimension equal to three is an M-semigroup. Therefore, for each numerical semigroup $S \in \mathrm{PSy}(\mathrm{PM}(n_1, n_2))$ there exist $a, b$ positive integers such that $S = \mathrm{S}(a, b, 1) = \{x \in \mathbb{N} \mid ax \mod b \leq x\}$. The next proposition give us a new proof of this fact. First we introduce a lemma which is Theorem 10 in [7].

**Lemma 4.9.** *Let $a$, $b$ be integers such that $2 \leq a < b$ and $\gcd\{a-1, b\} = 1$. Let $d = \gcd\{a, b\}$. Then $\mathrm{S}(a, b, 1)$ is a pseudo-symmetric numerical semigroup if and only if $0 < a(d+2) \mod b < d+2$. Moreover, if this is the case, then $\mathrm{S}(a, b, 1) = \langle \frac{b}{d}, d+2, \frac{b+d+2}{2} \rangle$.*

Now we are in conditions to show the announced result.

**Proposition 4.10.** (1) *If $n_1$ is odd, then*

$$\left\langle n_1, n_2, \frac{n_1+1}{2} n_2 - n_1 \right\rangle = \{x \in \mathbb{N} \mid u(n_2-2)x \mod n_1(n_2-2) \leq x\}.$$

(2) *If $n_1 \geq 4$ and $n_2$ is odd, then*

$$\left\langle n_1, n_2, \frac{n_2+1}{2} n_1 - n_2 \right\rangle = \{x \in \mathbb{N} \mid (n_2-v)(n_1-2)x \mod n_2(n_1-2) \leq x\}.$$

Proof. (1) Let us have $a = (n_2-2)u$, $b = (n_2-2)n_1$. Because $\gcd\{u, n_1\} = 1$, then $\gcd\{a, b\} = n_2 - 2$. Let us see that $\gcd\{a-1, b\} = 1$. Let us have $p = \gcd\{a-1, b\}$. Then $p|((n_2-2)u-1)$ and $p|((n_2-2)n_1)$. Therefore, $p|n_1$. Because $(n_2-2)u-1 = un_2-2u-1 = vn_1+1-2u-1 = vn_1-2u$, then $p|n_1$ and $p|(vn_1-2u)$. Therefore, $p|n_1$ and $p|(2u)$. Applying that $n_1$ is odd and $\gcd\{n_1, u\} = 1$, we have $p = 1$.

Now, because $(n_2-2)un_2 \mod ((n_2-2)n_1) = (n_2-2)(un_2 \mod n_1) = n_2-2$ and $0 < n_2 - 2 < n_2$, applying Lemma 4.9, we have that

$$\left\langle n_1, n_2, \frac{(n_2-2)n_1+n_2}{2} \right\rangle = \{x \in \mathbb{N} \mid u(n_2-2)x \mod n_1(n_2-2) \leq x\}.$$

In order to conclude, we must note that $\frac{(n_2-2)n_1+n_2}{2} = \frac{n_1+1}{2} n_2 - n_1$.

(2) If we consider $a = (n_1-2)(n_2-v)$ and $b = (n_1-2)n_2$, the proof of this case is analogous to the previous one. Therefore, we omit it. □

If $S$ is a pseudo-symmetric M-semigroup, in [7] it is proven that $\mathrm{e}(S) = 3$. Therefore, we can give the next result.

**Proposition 4.11.** *$S$ is a pseudo-symmetric* M*-semigroup with* $\mathrm{m}(S) = n_1$ *and* $\mathrm{r}(S) = n_2$ *if and only if one of following cases holds,*

(1) *$n_1$ is odd and $S = \left\langle n_1, n_2, \frac{n_1+1}{2}n_2 - n_1 \right\rangle$;*

(2) *$n_2$ is odd, $n_1 \geq 4$ and $S = \left\langle n_1, n_2, \frac{n_2+1}{2}n_1 - n_2 \right\rangle$.*

In the introduction we stated that the irreducible M-semigroups are just the irreducible PM-semigroups with embedding dimension less than or equal to three. As a consequence of Proposition 3.4, the commentary after it, and Proposition 4.4 (or Proposition 4.10), we show the above mentioned result.

**Proposition 4.12.** *Let $S$ be an irreducible numerical semigroup such that* $\mathrm{e}(S) \leq 3$*. Then $S$ is a* PM*-semigroup if and only if it is an* M*-semigroup.*

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