Publ. Math. Debrecen
84/3-4 (2014), 351–356
DOI: 10.5486/PMD.2014.5610

On the m-ary expansion of a multiple of $(m^k - 1)/(m - 1)$

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Abstract. We prove an extremal result on the m-ary expansion of $n(m^k - 1)/(m - 1)$.

1. Introduction

Recently, Z.-W. Sun [3] proved that for any n, $k \geq 1$,

$$\frac{1}{(2^k - 2)n + 1}\binom{(2^k - 1)n}{n}\binom{2(2^k - 1)n}{(2^k - 1)n}$$

is divisible by

$$2^{k-1}\binom{2n}{n}.$$

One key of Sun's proof is the following curious lemma:

For positive integers n and k, the number of 1's in the binary expansion of $n(2^k - 1)$ is at least k.

In fact, he got a stronger result [3, Lemma 3.2]:

Theorem 1. *For a prime p and positive integers n and k, the sum of all digits in the expansion of $n(p^k - 1)$ in base p is at least $k(p - 1)$.*

Motivated by Theorem 1, Sun made the following conjecture.

Conjecture 1. *Suppose that n, m, k are positive integers and $m \geq 2$.*

Mathematics Subject Classification: Primary: 11A63; Secondary: 11A07.
Key words and phrases: m-ary expansion; sum of digits.
The author is supported by National Natural Science Foundation of China (Grant No. 11271185).

(i) *There are at least k non-zero digits in the m-ary expansion of* $n(m^k-1)/(m-1)$.

(ii) *The sum of all digits in the m-ary expansion of $n(m^k-1)$ is at least $k(m-1)$.*

In fact, Conjecture 1 is not new. A solution of Part (i) of Conjecture 1 for the base $m = 10$ has been given in [1]. Furthermore, the second part of Conjecture 1 for $m = 10$ is also proved in [2]. Of course, those two proofs can be extended to any base $m \geq 2$ without any difficulty.

However, in this short note, we shall prove the following stronger result.

Theorem 2. *Suppose that $m \geq 2$, $k \geq 1$ and $a_1, a_2, \dots, a_k \in \mathbb{N} = \{0, 1, 2, \dots\}$ are not all zero.*

(i) *If $a_k m^{k-1} + a_{k-1} m^{k-2} + \cdots + a_2 m + a_1$ is a multiple of $(m^k-1)/(m-1)$, then*

$$\sum_{j=1}^{k} \left\lceil \frac{a_j}{m} \right\rceil \geq k,$$

where $\lceil x \rceil = \min\{z \in \mathbb{Z} : z \geq x\}$.

(ii) *Suppose that*

$$a_k m^{k-1} + a_{k-1} m^{k-2} + \cdots + a_2 m + a_1 \equiv 0 \pmod{m^k - 1}.$$

Then

$$\sum_{j=1}^{k} \left\lfloor \frac{a_j}{m-1} \right\rfloor \geq k,$$

where $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$.

Let us give an explanation why Theorem 2 implies Conjecture 1. Write

$$n(m^k-1)/(m-1) = b_h m^{h-1} + b_{h-1} m^{h-2} + \cdots + b_2 m + b_1,$$

where $0 \leq b_i < m$. Let

$$a_j = \sum_{\substack{1 \leq i \leq h \\ i \equiv j \pmod{k}}} b_i.$$

Since all b_i are less than m, we have

$$\left\lceil \frac{a_j}{m-1} \right\rceil \leq |\{1 \leq i \leq h : i \equiv j \pmod{k},\ b_i > 0\}|.$$

On the other hand,

$$\sum_{i=1}^{h} b_i m^{i-1} \equiv \sum_{j=1}^{k} m^{j-1} \sum_{\substack{1\le i\le h\\ i\equiv j \pmod{k}}} b_i = \sum_{j=1}^{k} a_j m^{j-1} \equiv 0 \pmod{(m^k-1)/(m-1)}.$$

It follows from Part (i) of Theorem 2 that

$$|\{1\le i\le h : b_i>0\}| \ge \sum_{j=1}^{k}\left\lceil \frac{a_j}{m-1}\right\rceil \ge \sum_{j=1}^{k}\left\lceil \frac{a_j}{m}\right\rceil \ge k.$$

Similarly, if $n(m^k-1) = b_h m^{h-1} + \cdots + b_1$ with $0\le b_i<m$ and let

$$a_j = \sum_{\substack{1\le i\le h\\ i\equiv j \pmod{k}}} b_i,$$

by Part (ii) of Theorem 2, we have

$$\sum_{i=1}^{h} b_i = \sum_{j=1}^{k} a_j \ge (m-1)\sum_{j=1}^{k}\left\lfloor \frac{a_j}{m-1}\right\rfloor \ge (m-1)k.$$

We shall prove Theorem 2 in the next section.

2. Proof of Theorem 2

Proposition 1. *Suppose that $m\ge 2$, $k\ge 1$ and $a_1,a_2,\dots,a_k\in\mathbb{N}$ are not all zero. Let q be a divisor of m^k-1. Suppose that $\tau(x_1,\dots,x_k)$ is a nonnegative integer-valued symmetric function satisfying that*

$$\tau(x_1,x_2,\dots,x_{k-1},md+b) \ge \tau(x_1+d,x_2,\dots,x_{k-1},b)$$

for every $0\le b<m$ and $d\ge 1$. If

$$a_k m^{k-1} + a_{k-1}m^{k-2} + \cdots + a_2 m + a_1 \equiv 0 \pmod{q},$$

then

$$\tau(a_1,a_2,\dots,a_k) \ge \min_{1\le t\le (m^k-1)/q}\{\tau^*(tq)\},$$

where

$$\tau^*(h) = \tau(c_1,c_2,\dots,c_k)$$

provided $0\le h<m^k$ has the m-ary expansion $h = c_k m^{k-1} + c_{k-1}m^{k-2} + \cdots + c_1$.

Proof. Let

$$S=\left\{(a_1,\dots,a_k):a_1,\dots,a_k\in\mathbb{N}\text{ are not all zero, }\sum_{j=1}^{k}a_jm^{j-1}\equiv 0\pmod q\right\}.$$

For $\mathbf{x}=(a_1,\dots,a_k)$, define

$$\sigma(\mathbf{x})=a_1+a_2+\cdots+a_k.$$

Let

$$t_0=\min_{(a_1,\dots,a_k)\in S}\tau(a_1,\dots,a_k)$$

and

$$T=\{(a_1,\dots,a_k)\in S:\tau(a_1,\dots,a_k)=t_0\}.$$

Clearly T is non-empty. Choose an $\mathbf{x}^\circ=(a_1^\circ,\dots,a_k^\circ)\in T$ such that

$$\sigma(\mathbf{x}^\circ)=\min_{(a_1,\dots,a_k)\in T}\sigma(a_1,\dots,a_k).$$

Since

$$m\cdot\sum_{j=1}^{k}a_jm^{j-1}=\sum_{j=1}^{k}a_jm^{j}\equiv a_k+\sum_{j=1}^{k-1}a_jm^{j}\pmod q,$$

$(a_1,a_2,\dots,a_k)\in S$ implies $(a_k,a_1,a_2,\dots,a_{k-1})\in S$. Furthermore, note that both τ and σ are symmetric. So by the definition of T and the choice of $\mathbf{x}^\circ$, without loss of generality, we may assume that $a_k^\circ\ge\max\{a_1^\circ,a_2^\circ,\dots,a_{k-1}^\circ\}$. We shall prove that $a_k^\circ<m$. Assume on the contrary that $a_k^\circ\ge m$. Write $a_k^\circ=md+b$ with $0\le b<m$ and $d\ge 1$. Then

$$\sum_{j=1}^{k}a_j^\circ m^{j-1}=(md+b)m^{k-1}+\sum_{j=1}^{k-1}a_j^\circ m^{j-1}\equiv bm^{k-1}+(d+a_1^\circ)+\sum_{j=2}^{k-1}a_j^\circ m^{j-1}\pmod q.$$

Hence $\mathbf{x}^\triangle:=(a_1^\circ+d,a_2^\circ,\dots,a_{k-1}^\circ,b)\in S$. Note that now

$$\tau(a_1^\circ,\dots,a_k^\circ)=\tau(a_1^\circ,a_2^\circ,\dots,a_{k-1}^\circ,md+b)\ge\tau(a_1^\circ+d,a_2^\circ,\dots,a_{k-1}^\circ,b).$$

It follows that $\mathbf{x}^\triangle$ also lies in T. But clearly

$$\sigma(\mathbf{x}^\circ)-\sigma(\mathbf{x}^\triangle)=a_1^\circ+a_k^\circ-(a_1^\circ+d+b)=(m-1)d\ge 1,$$

i.e., $\sigma(\mathbf{x}^\triangle)<\sigma(\mathbf{x}^\circ)$. This evidently leads to a contradiction with our choice of $\mathbf{x}^\circ$.

So we must have $a_k^\circ<m$, i.e.,

$$\max\{a_1^\circ,a_2^\circ,\dots,a_k^\circ\}\le m-1.$$

Thus $a_k^\circ m^{k-1}+\cdots+a_1^\circ=t_0q$ for some $1\le t_0\le(m^k-1)/q$. And

$$\tau(a_1^\circ,a_2^\circ,\dots,a_k^\circ)=\tau^*(t_0q)\ge\min_{1\le t\le(m^k-1)/q}\{\tau^*(tq)\}.\qquad\square$$

Now let

$$\tau_1(x_1,\dots,x_k)=\sum_{i=1}^{k}\left\lceil\frac{x_j}{m}\right\rceil$$

and

$$\tau_2(x_1,\dots,x_k)=\sum_{i=1}^{k}\left\lfloor\frac{x_j}{m-1}\right\rfloor.$$

We shall verify τ_1 and τ_2 satisfy the requirements of Proposition 1. Evidently τ_1 and τ_2 are symmetric. Note that

$$\left\lceil\frac{md+b}{m}\right\rceil=d+\left\lceil\frac{b}{m}\right\rceil,$$

and

$$\left\lceil\frac{x_1+d}{m}\right\rceil\le\left\lceil\frac{x_1}{m}\right\rceil+\left\lceil\frac{d}{m}\right\rceil\le\left\lceil\frac{x_1}{m}\right\rceil+d.$$

Clearly it follows that

$$\tau_1(x_1,x_2,\dots,x_{k-1},md+b)\ge\tau_1(x_1+d,x_2,\dots,x_{k-1},b).$$

Moreover, if $m>2$, then

$$\left\lfloor\frac{md+b}{m-1}\right\rfloor\ge d+\left\lfloor\frac{b}{m-1}\right\rfloor,$$

and

$$\left\lfloor\frac{x_1+d}{m-1}\right\rfloor\le\left\lfloor\frac{x_1}{m-1}\right\rfloor+\left\lfloor\frac{d}{m-1}\right\rfloor+1\le\left\lfloor\frac{x_1}{m-1}\right\rfloor+d.$$

That is,

$$\tau_2(x_1,x_2,\dots,x_{k-1},md+b)\ge\tau_2(x_1+d,x_2,\dots,x_{k-1},b).$$

Of course, apparently the above inequality also holds when $m=2$. Finally, note that for each $1\le t\le m-1$,

$$\tau_1^*\left(t\cdot\frac{m^k-1}{m-1}\right)=\tau_1^*\left(\sum_{j=1}^{k}t\cdot m^{k-j}\right)=\sum_{j=1}^{k}\left\lceil\frac{t}{m}\right\rceil=k,$$

and

$$\tau_2^*(m^k-1)=\tau_2^*\left(\sum_{j=1}^{k}(m-1)\cdot m^{k-j}\right)=\sum_{j=1}^{k}\left\lfloor\frac{m-1}{m-1}\right\rfloor=k.$$

Thus applying Proposition 1, we immediately get Theorem 2.

ACKNOWLEDGMENT. I am grateful to two anonymous referees for their very useful comments on this paper. In particular, one of the referees informed me of the references [1] and [2]. I also thank Professor ZHI-WEI SUN for his helpful discussions.

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(Received July 26, 2012; revised June 22, 2013)