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Characterizations of peripherally multiplicative mappings between real function algebras

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Abstract. Let X be a compact Hausdorff space; let $\tau : X \to X$ be a topological involution; and let $\mathcal{A} \subset C(X,\tau)$ be a real function algebra. Given $f \in \mathcal{A}$, the peripheral spectrum of f is the set $\sigma_\pi(f)$ of spectral values of f of maximum modulus. We demonstrate that if $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ are surjective mappings between real function algebras $\mathcal{A} \subset C(X,\tau)$ and $\mathcal{B} \subset C(Y,\varphi)$ that satisfy $\sigma_\pi(T_1(f)T_2(g)) = \sigma_\pi(S_1(f)S_2(g))$ for all $f, g \in \mathcal{A}$, then there exists a homeomorphism $\psi : \mathrm{Ch}(\mathcal{B}) \to \mathrm{Ch}(\mathcal{A})$ between the Choquet boundaries such that $(\psi \circ \varphi)(y) = (\tau \circ \psi)(y)$ for all $y \in \mathrm{Ch}(\mathcal{B})$, and there exist functions $\kappa_1, \kappa_2 \in \mathcal{B}$, with $\kappa_1^{-1} = \kappa_2$, such that $T_j(f)(y) = \kappa_j(y) S_j(f)(\psi(y))$ for all $f \in \mathcal{A}$, all $y \in \mathrm{Ch}(\mathcal{B})$, and $j = 1, 2$. As a corollary, it is shown that if either $\mathrm{Ch}(\mathcal{A})$ or $\mathrm{Ch}(\mathcal{B})$ is a minimal boundary (with respect to inclusion) for its corresponding algebra, then the same result holds for surjective mappings $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ that satisfy $\sigma_\pi(T_1(f)T_2(g)) \cap \sigma_\pi(S_1(f)S_2(g)) \neq \varnothing$ for all $f, g \in \mathcal{A}$.

1. Introduction and background

Given a compact Hausdorff space X, a topological involution is a continuous mapping $\tau : X \to X$ such that $\tau(\tau(x)) = x$ for all $x \in X$. Define $C(X,\tau) = \{f \in C(X) : f \circ \tau = \overline{f}\}$, then a real function algebra is a uniformly closed, real subalgebra $\mathcal{A} \subset C(X,\tau)$ that separates points and contains the real constant functions. Kulkarni and Limaye introduced real function algebras and gave a thorough account of the theory in [6]. Although they are analogous to uniform algebras, real function algebras are strictly real Banach algebras.

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There has been a recent surge of work done on analyzing mappings between Banach algebras of continuous functions that preserve certain spectral properties, and these problems are known as *spectral preserver problems*. MOLNÁR began interest in such problems in [13] by characterizing surjective maps $T : C(X) \to C(X)$, where X is a first-countable, compact Hausdorff space, that satisfy $\sigma(T(f)T(g)) = \sigma(fg)$ for all $f, g \in C(X)$. A wide range of spectral preserver problems have now been studied in a variety of settings (see [3] for a recent survey), but such problems have yet to be investigated for real function algebras. This setting offers new challenges, as the algebraic structure has been restricted to real scalars (affecting the spectral structure), and the topological involution τ adds a new layer of structure to be analyzed. In this work, we address these issues and answer a particular spectral preserver problem in real function algebras.

Given a real function algebra $\mathcal{A}$ and an $f \in \mathcal{A}$, the spectrum of f is the non-empty, compact set $\sigma(f) = \{a + ib \in \mathbb{C} : (f - a)^2 + b^2 \notin \mathcal{A}^{-1}\}$ (cf. [6]), where $\mathcal{A}^{-1}$ is the collection of multiplicatively invertible elements of $\mathcal{A}$. The set of spectral values of f of maximum modulus is known as the *peripheral spectrum* of f and it is denoted by

$$\sigma_\pi(f) = \Big\{\lambda \in \sigma(f) : |\lambda| = \max_{z \in \sigma(f)} |z|\Big\}.$$

A foursome of surjective mappings $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ between real function algebras that satisfy $\sigma_\pi(T_1(f)T_2(g)) = \sigma_\pi(S_1(f)S_2(g))$ for all $f, g \in \mathcal{A}$ are called *jointly peripherally multiplicative mappings*. We characterize such mappings, and prove that T_1 and T_2 are essentially weighted composition operators. In particular, the pre-composition mapping is between the Choquet boundaries, which is the set $\mathrm{Ch}(\mathcal{A})$ of points x such that $\operatorname{Re} e_x$, where e_x is the point-evaluation at x, is an extreme point of the state space of $\mathcal{A}$.

Main Theorem. *Let X and Y be compact Hausdorff spaces; let $\tau : X \to X$ and $\varphi : Y \to Y$ be topological involutions; and let $\mathcal{A} \subset C(X, \tau)$ and $\mathcal{B} \subset C(Y, \varphi)$ be real function algebras. If $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ are surjective mappings that satisfy*

$$\sigma_\pi(T_1(f)T_2(g)) = \sigma_\pi(S_1(f)S_2(g))$$

for all $f, g \in \mathcal{A}$, then there exists a homeomorphism $\psi : \mathrm{Ch}(\mathcal{B}) \to \mathrm{Ch}(\mathcal{A})$ between the Choquet boundaries such that $(\psi \circ \varphi)(y) = (\tau \circ \psi)(y)$ for all $y \in \mathrm{Ch}(\mathcal{B})$, and there exist functions $\kappa_1, \kappa_2 \in \mathcal{B}$ that satisfy $\kappa_1^{-1} = \kappa_2$ and

$$T_j(f)(y) = \kappa_j(y) S_j(f)(\psi(y))$$

for all $f \in \mathcal{A}$, all $y \in \mathrm{Ch}(\mathcal{B})$, and $j = 1, 2$.

Note that when S_1 and S_2 are identity mappings and $T_1(1) = T_2(1) = 1$, then $T_1 = T_2$ and T_1 is an isometric algebra isomorphism. This mirrors the previous results on peripherally multiplicative mappings (e.g. [2], [11]), with the addition of demonstrating that the pre-composition mapping ψ is a topological conjugacy between φ and τ. Furthermore, studying multiple mappings that jointly satisfy spectral conditions has recently received attention [2], [9], [14], and doing so offers the benefit of answering a wide range of possible questions at once.

Four surjective mappings T_1, $T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ between real function algebras $\mathcal{A} \subset C(X,\tau)$ and $\mathcal{B} \subset C(Y,\varphi)$ that satisfy $\sigma_\pi(T_1(f)T_2(g)) \cap \sigma_\pi(S_1(f)S_2(g)) \neq \varnothing$ for all $f, g \in \mathcal{A}$ are known as *jointly weakly peripherally multiplicative*. In the setting of uniform algebras, it is still an open question as to whether or not such mappings are weighted composition operators. One approach has been to impose additional topological conditions on the underlying domain of the functions, and this is done to guarantee that the Choquet boundary has further structure (see [9], [14]). Following in this vein, we demonstrate that the conclusion of Main Theorem is true for jointly weakly peripherally multiplicative mappings, provided that either $\mathrm{Ch}(\mathcal{A})$ or $\mathrm{Ch}(\mathcal{B})$ is a minimal boundary for its respective algebra. This is to say that every function attains its maximum modulus on the Choquet boundary, and no proper subset has this property.

Corollary 5.1. *Let X and Y be compact Hausdorff spaces; let $\tau : X \to X$ and $\varphi : Y \to Y$ be topological involutions; and let $\mathcal{A} \subset C(X,\tau)$ and $\mathcal{B} \subset C(Y,\varphi)$ be real function algebras. If either $\mathrm{Ch}(\mathcal{A})$ is a minimal boundary for $\mathcal{A}$ or $\mathrm{Ch}(\mathcal{B})$ is a minimal boundary for $\mathcal{B}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ are surjective mappings that satisfy*

$$\sigma_\pi(T_1(f)T_2(g)) \cap \sigma_\pi(S_1(f)S_2(g)) \neq \varnothing$$

for all $f, g \in \mathcal{A}$, then there exists a homeomorphism $\psi : \mathrm{Ch}(\mathcal{B}) \to \mathrm{Ch}(\mathcal{A})$ such that $(\psi \circ \varphi)(y) = (\tau \circ \psi)(y)$ for all $y \in \mathrm{Ch}(\mathcal{B})$ and there exist functions $\kappa_1, \kappa_2 \in \mathcal{B}$ that satisfy $\kappa_1^{-1} = \kappa_2$ and

$$T_j(f)(y) = \kappa_j(y) S_j(f)(\psi(y))$$

for all $f \in \mathcal{A}$, all $y \in \mathrm{Ch}(\mathcal{B})$, and $j = 1, 2$.

We begin in Section 2 with the basic material on real function algebras that will be needed throughout. This includes a real function algebra version of a result known as *Bishop's lemma*, and its relevant applications. Results that are needed to characterize (weakly) peripherally multiplicative maps are demonstrated in Section 3, and the proof of the Main Theorem is given in Section 4. The Main Theorem is then used in Section 5 to prove Corollary 5.1.

2. Preliminary and prior results

Throughout this section we assume that X is a compact Hausdorff space, τ is a topological involution on X, and $\mathcal{A} \subset C(X,\tau)$ is a real function algebra. The complexification of $\mathcal{A}$ is the uniform algebra $\mathcal{A}_{\mathbb{C}} = \{f + ig : f,\ g \in \mathcal{A}\}$ [6, Theorem 1.3.20]. Given an $f \in \mathcal{A}$, then $\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda \notin \mathcal{A}_{\mathbb{C}}^{-1}\}$ (cf. [6, Remark 1.1.11]). The peripheral range of $f \in \mathcal{A}_{\mathbb{C}}$ is the set $\mathrm{Ran}_\pi(f) = \{f(x) : x \in X, |f(x)| = \|f\|\}$, where $\|\cdot\|$ denotes the uniform norm. For uniform algebras, it is known that the peripheral spectrum and peripheral range coincide (see [11, Lemma 1]), thus it follows that

$$\sigma_\pi(f) = \mathrm{Ran}_\pi(f)$$

for all $f \in \mathcal{A}$. Note that $\sigma_\pi(f)$ is closed under complex conjugation for any $f \in \mathcal{A}$.

Given an $f \in \mathcal{A}_{\mathbb{C}}$, the maximizing set of f is the non-empty, compact set $M(f) = \{x \in X : |f(x)| = \|f\|\}$. If $f \in \mathcal{A}$, then the set $M(f)$ is τ-invariant, which is to say that $\tau[M(f)] = M(f)$. A function $h \in \mathcal{A}_{\mathbb{C}}$ is a peaking function if and only if $\mathrm{Ran}_\pi(h) = \{1\}$. The collection of all peaking functions of $\mathcal{A}$ and of $\mathcal{A}_{\mathbb{C}}$ are denoted by $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A}_{\mathbb{C}})$, respectively. Clearly, $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}(\mathcal{A}_{\mathbb{C}})$ and given a $k \in \mathcal{P}(\mathcal{A}_{\mathbb{C}})$, there exists an $h \in \mathcal{P}(\mathcal{A})$ such that $M(h) = M(k) \cup \tau[M(k)]$ [6, Theorem 2.2.11]. Moreover, given an $f \in \mathcal{A} \setminus \{0\}$ and an $x_0 \in M(f)$, then

$$k = \frac{f}{2[f(x_0)]} + \frac{f^2}{2[f(x_0)]^2}$$

belongs to $\mathcal{P}(\mathcal{A}_{\mathbb{C}})$ and satisfies $M(k) = \{x \in X : f(x) = f(x_0)\}$. Consequently, there exists an $h \in \mathcal{P}(\mathcal{A})$ such that $M(h) = M(k) \cup \tau[M(k)] = f^{-1}[\{f(x_0)\}] \cup f^{-1}[\{f(\tau(x_0))\}]$.

A subset $F \subset X$ is a p-set for $\mathcal{A}$ if $F = \bigcap_{h \in \mathcal{S}} M(h)$ for some family $\mathcal{S} \subset \mathcal{P}(\mathcal{A})$. Given an $F \subset X$, let $\mathcal{P}_F(\mathcal{A}) = \{h \in \mathcal{P}(\mathcal{A}) : F \subset M(h)\}$. A non-empty subset $F \subset X$ is a p-set for $\mathcal{A}$ if and only if given an open neighborhood U of F, then there exists an $h \in \mathcal{P}_F(\mathcal{A})$ such that $M(h) \subset U$ [6, Theorem 2.2.3]. These definitions and results also apply to $\mathcal{A}_{\mathbb{C}}$, and we shall use $\mathcal{P}_x(\mathcal{A})$ instead of $\mathcal{P}_{\{x\}}(\mathcal{A})$.

A τ-invariant subset $B \subset X$ is a boundary for $\mathcal{A}$ if $M(f) \cap B \neq \varnothing$ for all $f \in \mathcal{A}$, and the Choquet boundary is a boundary for $\mathcal{A}$ [6, Theorem 4.2.5]. Additionally, the Choquet boundary coincides with collection of $x \in X$ such that $\{x, \tau(x)\}$ is a p-set, i.e. $\mathrm{Ch}(\mathcal{A}) = \{x \in X : \{x, \tau(x)\}$ is a p-set for $\mathcal{A}\} = \{x \in X : \{x\}$ is a p-set for $\mathcal{A}_{\mathbb{C}}\}$ [6, Theorems 4.2.4 and 4.3.7].

An *(i)-peaking function* is a function $h \in \mathcal{A}$ such that $\mathrm{Ran}_\pi(h) = \{i, -i\}$. We denote the collection of all (i)-peaking functions by $i\mathcal{P}(\mathcal{A})$. In particular, given a subset $F \subset X$ define $i\mathcal{P}_F(\mathcal{A}) = \{h \in i\mathcal{P}(\mathcal{A}) : h\big|_F \equiv i\}$. These types of functions were studied extensively in [12], where it was demonstrated that $i\mathcal{P}_x(\mathcal{A}) \neq \varnothing$ for any $x \in \mathrm{Ch}(\mathcal{A})$ with $x \neq \tau(x)$.

Given an $x \in X$, define $x_\tau = \{x, \tau(x)\}$. In particular, let $\mathrm{Ch}_\tau(\mathcal{A}) = \{x_\tau : x \in \mathrm{Ch}(\mathcal{A})\}$. For $F \subset X$, define $\mathcal{M}_F(\mathcal{A}) = \big\{f \in \mathcal{A} : \|f\| = 1,\ |f|\big|_F \equiv 1\big\}$, and note that $i\mathcal{P}_F(\mathcal{A}), \mathcal{P}_F(\mathcal{A}) \subset \mathcal{M}_F(\mathcal{A})$. These sets are useful as they can identify elements of $\mathrm{Ch}_\tau(\mathcal{A})$.

Lemma 2.1. *Let $\mathcal{A} \subset C(X,\tau)$ be a real function algebra; let $x_\tau \in \mathrm{Ch}_\tau(\mathcal{A})$; and let $y \in X$. Then $x_\tau = y_\tau$ if and only if $\mathcal{M}_{x_\tau}(\mathcal{A}) \subset \mathcal{M}_{y_\tau}(\mathcal{A})$.*

The proof of this lemma follows exactly as the proof of [9, Lemma 2].

2.1. Bishop's lemma for real function algebras and its applications. A classic result from uniform algebra theory is a result known as Bishop's lemma (see [1, Theorem 2.4.1]), and the conclusion of this lemma also holds for real function algebras.

Lemma 2.2 (Bishop's lemma for real function algebras)**.** *Let $\mathcal{A} \subset C(X,\tau)$ be a real function algebra; let $F \subset X$ be a p-set for $\mathcal{A}$; and let $f \in \mathcal{A}$ be such that $f \not\equiv 0$ on F. Then there exists an $h \in \mathcal{P}_F(\mathcal{A})$ such that $M(fh) \cap F \neq \varnothing$.*

Proof. As $f \not\equiv 0$ on F, there exists a $k \in \mathcal{P}_F(A_{\mathbb{C}})$ such that $M(fk) \cap F \neq \varnothing$ [8, Lemma 3]. Set $\widehat{k} = \overline{k \circ \tau}$, then it is straightforward to demonstrate that $\widehat{k} \in \mathcal{P}_F(\mathcal{A}_{\mathbb{C}})$ and $h = k \cdot \widehat{k} \in \mathcal{P}_F(\mathcal{A})$. Moreover,

$$\|fh\| \leq \|fk\| = |f(x)k(x)| = |f(x)k(x)\widehat{k}(x)| = |f(x)h(x)| \leq \|fh\|$$

for any $x \in M(fk) \cap F$. Therefore, $M(fh) \cap F \neq \varnothing$. □

Since x_τ is a p-set for $\mathcal{A}$ for any $x \in \mathrm{Ch}(\mathcal{A})$, the following improvement can be made to Bishop's lemma:

Lemma 2.3. *Let $\mathcal{A} \subset C(X,\tau)$ be a real function algebra; let $x \in \mathrm{Ch}(\mathcal{A})$; and let $f \in \mathcal{A}$. If $f(x) \neq 0$, then there exists an $h \in \mathcal{P}_x(\mathcal{A})$ such that $\sigma_\pi(fh) = \{f(x), f(\tau(x))\}$. If $f(x) = 0$, then given an $\varepsilon > 0$, there exists an $h \in \mathcal{P}_x(\mathcal{A})$ such that $\|fh\| < \varepsilon$.*

Proof. Suppose that $f(x) \neq 0$, then Lemma 2.2 implies that there exists an $h_1 \in \mathcal{P}_x(\mathcal{A})$ such that $x \in M(fh_1)$. Thus there exists an $h_2 \in \mathcal{P}_x(\mathcal{A})$ such that $M(h_2) = (fh_1)^{-1}[\{f(x)\}] \cup (fh_1)^{-1}[\{f(\tau(x))\}]$. Setting $h = h_1 \cdot h_2 \in \mathcal{P}_x(\mathcal{A})$, it then follows that $\sigma_\pi(fh) = \mathrm{Ran}_\pi(fh) = \{f(x), f(\tau(x))\}$.

Now, suppose that $f(x)=0$ and let $\varepsilon>0$. Since $\{x\}$ is a p-set for $\mathcal{A}_{\mathbb{C}}$, there exists a $k\in\mathcal{P}_x(\mathcal{A}_{\mathbb{C}})$ such that $\|fk\|<\varepsilon$ [2, Lemma 2.1]. Set $\widehat{k}=\overline{k\circ\tau}\in\mathcal{P}_x(\mathcal{A}_{\mathbb{C}})$, and let $h=k\cdot\widehat{k}\in\mathcal{P}_x(\mathcal{A})$. Therefore, $\|fh\|=\|fk\widehat{k}\|\leq\|fk\|<\varepsilon$. □

Using Lemma 2.3, the following result can be obtained:

Lemma 2.4. *Let $\mathcal{A}\subset C(X,\tau)$ be a real function algebra and let $f,g\in\mathcal{A}$. If $\|fh\|\leq\|gh\|$ for all $h\in\mathcal{P}(\mathcal{A})$, then $|f(x)|\leq|g(x)|$ for all $x\in\mathrm{Ch}(\mathcal{A})$.*

The proof of this lemma follows exactly as that of [11, Lemma 2]. Given an $h\in\mathcal{P}_x(\mathcal{A})$ and a $k\in i\mathcal{P}_x(\mathcal{A})$, it is clear that $hk\in i\mathcal{P}_x(\mathcal{A})$, and this yields an (i)-peaking version of Bishop's lemma:

Corollary 2.1. *Let $\mathcal{A}\subset C(X,\tau)$ be a real function algebra; let $x\in\mathrm{Ch}(\mathcal{A})$ be such that $\tau(x)\neq x$; and let $f\in\mathcal{A}$. If $f(x)\neq 0$, then there exists an $h\in i\mathcal{P}_x(\mathcal{A})$ such that $\sigma_\pi(fh)=\{if(x),-if(\tau(x))\}$.*

It is worth noting that if $\tau(x)=x$, then $i\mathcal{P}_x(\mathcal{A})=\varnothing$. Another useful application of Bishop's lemma is the following result, which we will use repeatedly.

Lemma 2.5. *Let $\mathcal{A}\subset C(X,\tau)$ be a real function algebra; let $x\in\mathrm{Ch}(\mathcal{A})$; and let $f\in\mathcal{A}$. If $M(f)\cap\mathrm{Ch}(\mathcal{A})=x_\tau$, then $M(f)=x_\tau$.*

Proof. Suppose that $M(f)\cap\mathrm{Ch}(\mathcal{A})=x_\tau=\{x,\tau(x)\}$. We will demonstrate that if $y\in M(f)$ and $y\notin\{x,\tau(x)\}$, then there exists a $g\in\mathcal{A}$ such that $g(y)=1$ and $0=g(x)=\|g\|$, which is clearly a contradiction.

Indeed, suppose that $y\in M(f)$ and $y\notin x_\tau$, then there exists an $s\in\mathcal{A}$ such that $s(y)=1$ and $s(x)=0$ [6, Lemma 1.3.9]. Additionally, we can find a peaking function q such that $M(q)=f^{-1}[\{f(y)\}]\cup f^{-1}[\{f(\tau(y)\}]$. As s is non-zero on $M(q)$, Lemma 2.2 implies that there exists an $h\in\mathcal{P}_{M(q)}(\mathcal{A})$ such that $M(sh)\cap M(q)\neq\varnothing$. Set $g=shq$, and note that $g(y)=1$ and $g(x)=0$. As $M(g)\subset M(f)$, it must be that $M(g)\cap\mathrm{Ch}(\mathcal{A})=x_\tau$, hence $0=g(x)=\|g\|$. □

Using this lemma, we now given a criterion for $\mathrm{Ch}(\mathcal{A})$ to be a minimal boundary, which is to say that no proper subset of $\mathrm{Ch}(\mathcal{A})$ is again a boundary for $\mathcal{A}$.

Lemma 2.6 (Real function algebra version of Proposition 7.1.1 [10])**.** *Let $\mathcal{A}\subset C(X,\tau)$ be a real function algebra. Then $\mathrm{Ch}(\mathcal{A})$ is a minimal boundary for $\mathcal{A}$ if and only if for each $x\in\mathrm{Ch}(\mathcal{A})$, there exists an $f\in\mathcal{A}$ such that $M(f)=x_\tau$.*

Proof. Suppose that $\mathrm{Ch}(\mathcal{A})$ is a minimal boundary for $\mathcal{A}$. Given an $x\in\mathrm{Ch}(\mathcal{A})$, then $\mathrm{Ch}(\mathcal{A})\setminus\{x,\tau(x)\}$ is not a boundary for $\mathcal{A}$. Therefore there exists an $f\in\mathcal{A}$ such that $M(f)\cap\mathrm{Ch}(\mathcal{A})=x_\tau$, and Lemma 2.5 implies that $M(f)=x_\tau$.

For the reverse direction, suppose that for each $x \in \mathrm{Ch}(\mathcal{A})$ there exists an $f \in \mathcal{A}$ such that $M(f) = x_\tau$. This implies that given any boundary B for $\mathcal{A}$, then $\{x, \tau(x)\} \subset B$ for any $x \in \mathrm{Ch}(\mathcal{A})$. Therefore $\mathrm{Ch}(\mathcal{A}) \subset B$, and it follows that $\mathrm{Ch}(\mathcal{A})$ is a minimal boundary for $\mathcal{A}$. □

In general, $\mathrm{Ch}(\mathcal{A})$ need not be a minimal boundary. If $M(h) = x_\tau$ for $x \in \mathrm{Ch}(\mathcal{A})$ and $h \in \mathcal{A}$, then there exists a $k \in \mathcal{A}_{\mathbb{C}}$ such that $M(k) = \{x\}$ (cf. [6, Theorem 2.2.11]). In other words, x is a peak point for the uniform algebra $\mathcal{A}_{\mathbb{C}}$, and it is well-known that such points need not exist.

When $\mathrm{Ch}(\mathcal{A})$ is a minimal boundary for $\mathcal{A}$, then we have the following improvement of Lemma 2.3 and Corollary 2.1:

Corollary 2.2. *Let $\mathcal{A} \subset C(X, \tau)$ be a real function algebra such that* $\mathrm{Ch}(\mathcal{A})$ *is a minimal boundary for $\mathcal{A}$; let $x \in \mathrm{Ch}(\mathcal{A})$; and let $f \in \mathcal{A}$ be such that $f(x) \neq 0$. Then there exists an $h \in \mathcal{P}_x(\mathcal{A})$ such that $M(fh) = M(h) = x_\tau$. Moreover, if $x \neq \tau(x)$, then there exists a $k \in i\mathcal{P}_x(\mathcal{A})$ such that $M(fk) = M(k) = x_\tau$.*

3. Jointly norm multiplicative maps

In the study of (weakly) peripherally multiplicative mappings, the first task is to investigate mappings that multiplicatively preserve the uniform norm (e.g. [8, Section 3]). Four mappings $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ between real function algebras $\mathcal{A} \subset C(X, \tau)$ and $\mathcal{B} \subset C(Y, \varphi)$ that satisfy $\|T_1(f)T_2(g)\| = \|S_1(f)S_2(g)\|$ for all $f, g \in \mathcal{A}$ are known as *jointly norm multiplicative mappings*, and the following proposition will be proven in this section:

Proposition 3.1. *Let $\mathcal{A} \subset C(X, \tau)$ and $\mathcal{B} \subset C(Y, \varphi)$ be real function algebras and let $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ be surjective mappings such that*

$$\|T_1(f)T_2(g)\| = \|S_1(f)S_2(g)\| \tag{1}$$

for all $f, g \in \mathcal{A}$. Then there exists a bijective mapping $\Psi : \mathrm{Ch}_\tau(\mathcal{A}) \to \mathrm{Ch}_\varphi(\mathcal{B})$. Moreover, given an $x \in \mathrm{Ch}(\mathcal{A})$, then $|S_1(f)(x)S_2(g)(x)| = |T_1(f)(y)T_2(g)(y)|$ for all $f, g \in \mathcal{A}$ and all $y \in \Psi(x_\tau)$.

For the remainder of this section, let $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ be surjective mappings between real function algebras $\mathcal{A} \subset C(X, \tau)$ and $\mathcal{B} \subset C(Y, \tau)$ that satisfy (1). The proof of Proposition 3.1 will follow via a sequence of lemmas.

Lemma 3.1. *Let $f, g \in \mathcal{A}$ and $j \in \{1, 2\}$. Then $|S_j(f)(x)| \leq |S_j(g)(x)|$ for all $x \in \mathrm{Ch}(\mathcal{A})$ if and only if $|T_j(f)(y)| \leq |T_j(g)(y)|$ for all $y \in \mathrm{Ch}(\mathcal{B})$.*

This lemma follows from Lemma 2.4, and the proof is exactly as that of [9, Lemma 7]. Given an $x \in \mathrm{Ch}(\mathcal{A})$, set $A_j(x) = S_j^{-1}[\mathcal{M}_{x_\tau}(\mathcal{A})]$, where $j = 1, 2$. Given $h \in A_1(x)$ and $k \in A_2(x)$, then $S_1(h)S_2(k) \in \mathcal{M}_{x_\tau}(\mathcal{A})$. Combining this fact with (1) implies that $\|T_1(h)T_2(k)\| = \|S_1(h)S_2(k)\| = 1$. Define

$$A_{x_\tau} = \bigcap_{h \in A_1(x),\ k \in A_2(x)} M(T_1(h)T_2(k)). \tag{2}$$

Lemma 3.2. *Let $x \in \mathrm{Ch}(\mathcal{A})$. Then the set A_{x_τ} defined by (2) is non-empty.*

Proof. Let $h_1, \ldots, h_n \in A_1(x)$ and $k_1, \ldots, k_n \in A_2(x)$, then $S_1(h_1) \cdot \ldots \cdot S_1(h_n)$ and $S_2(k_1) \cdot \ldots \cdot S_2(k_n)$ both belong to $\mathcal{M}_{x_\tau}(\mathcal{A})$. By the surjectivity of S_1 and S_2, there exist $h \in A_1(x)$ and $k \in A_2(x)$ such that $S_1(h) = S_1(h_1) \cdot \ldots \cdot S_1(h_n)$ and $S_2(k) = S_2(k_1) \cdot \ldots \cdot S_2(k_n)$. Then

$$|S_1(h)(\zeta)| \leq |S_1(h_j)(\zeta)| \quad \text{and} \quad |S_2(k)(\zeta)| \leq |S_2(k_j)(\zeta)|$$

for all $\zeta \in \mathrm{Ch}(\mathcal{A})$ and each $1 \leq j \leq n$, and Lemma 3.1 implies that

$$|T_1(h)(\eta)| \leq |T_1(h_j)(\eta)| \quad \text{and} \quad |T_2(k)(\eta)| \leq |T_2(k_j)(\eta)|$$

for all $\eta \in \mathrm{Ch}(\mathcal{B})$ and each $1 \leq j \leq n$. As $1 = \|S_1(h)S_2(k)\| = \|T_1(h)T_2(k)\|$, there exists a $y \in \mathrm{Ch}(\mathcal{B})$ such that $|T_1(h)(y)T_2(k)(y)| = 1$. We then have

$$\begin{aligned} 1 &= |T_1(h)(y)T_2(k)(y)| \leq |T_1(h_j)(y)T_2(k_j)(y)| \\ &\leq \|T_1(h_j)T_2(k_j)\| = \|S_1(h_j)S_2(k_j)\| = 1 \end{aligned}$$

for each $1 \leq j \leq n$. This yields that $y \in \bigcap_{j=1}^n M(T_1(h_j)T_2(k_j))$. Therefore, by the finite intersection property, A_{x_τ} is non-empty. $\square$

Since A_{x_τ} is a non-empty intersection of maximizing sets, it meets the Choquet boundary (cf. [7]).

Lemma 3.3. *Let $x \in \mathrm{Ch}(\mathcal{A})$; let $y \in A_{x_\tau} \cap \mathrm{Ch}(\mathcal{B})$; and let $f, g \in \mathcal{A}$. Then $T_1(f)T_2(g) \in \mathcal{M}_{y_\varphi}(\mathcal{B})$ if and only if $S_1(f)S_2(g) \in \mathcal{M}_{x_\tau}(\mathcal{A})$.*

Proof. Suppose that $T_1(f)T_2(g) \in \mathcal{M}_{y_\varphi}(\mathcal{B})$. Since $1 = \|T_1(f)T_2(g)\| = \|S_1(f)S_2(g)\|$, it is only to show that $|S_1(f)(x)S_2(g)(x)| = 1$. If $S_1(f)(x)S_2(g)(x) = 0$, then we can assume without loss of generality that $S_1(f)(x) = 0$ and Lemma 2.3 implies that there exists an $h \in \mathcal{P}_x(\mathcal{A})$ such that $\|S_1(f)h\| < 1/\|S_2(g)\|$. Let $p_1, p_2 \in \mathcal{A}$ be such that $S_1(p_1) = S_2(p_2) = h$, then $p_1 \in A_1(x)$ and $p_2 \in A_2(x)$, hence $y \in M(T_1(p_1)T_2(p_2))$. This yields that $T_1(p_1)T_2(p_2) \in \mathcal{M}_{y_\varphi}(\mathcal{B})$, hence

$$1 = \|T_1(f)T_2(g)T_1(p_1)T_2(p_2)\| \leq \|T_1(f)T_2(p_2)\| \cdot \|T_1(p_1)T_2(g)\|$$

$$= \|S_1(f)h\| \cdot \|hS_2(g)\| < \frac{1}{\|S_2(g)\|} \cdot \|S_2(g)\| = 1,$$

which is a contradiction. Thus $S_1(f)(x) \neq 0$ and $S_2(g)(x) \neq 0$, so Lemma 2.3 implies that there exist $h_1, h_2 \in \mathcal{P}_x(\mathcal{A})$ such that $\|S_1(f)h_1\| = |S_1(f)(x)|$ and $\|S_2(g)h_2\| = |S_2(g)(x)|$. Let $k_1, k_2 \in \mathcal{A}$ be such that $S_2(k_1) = h_1$ and $S_1(k_2) = h_2$, then $T_1(k_2)T_2(k_1) \in \mathcal{M}_{y_\varphi}(\mathcal{B})$ and

$$\begin{aligned} |S_1(f)(x)S_2(g)(x)| &= \|S_1(f)h_1\| \cdot \|S_2(g)h_2\| = \|T_1(f)T_2(k_1)\| \cdot \|T_1(k_2)T_2(g)\| \\ &\geq \|T_1(f)T_2(g)T_1(k_2)T_2(k_1)\| = 1 = \|S_1(f)S_2(g)\|. \end{aligned}$$

Therefore $S_1(f)S_2(g) \in \mathcal{M}_{x_\tau}(\mathcal{A})$, and the converse is proved in a similar fashion. $\square$

Using this lemma, it is now shown that $A_{x_\tau} \cap \mathrm{Ch}(\mathcal{B})$ is at most a doubleton.

Lemma 3.4. *Let $x \in \mathrm{Ch}(\mathcal{A})$, and let $y \in A_{x_\tau} \cap \mathrm{Ch}(\mathcal{B})$. Then $A_{x_\tau} \cap \mathrm{Ch}(\mathcal{B}) = y_\varphi = \{y, \varphi(y)\}$.*

Proof. Since $y \in M(T_1(h)T_2(k))$ for any $h \in A_1(x)$ and $k \in A_2(x)$, it follows that $\varphi(y) \in M(T_1(h)T_2(k))$. This implies that $y_\varphi \subset A_{x_\tau} \cap \mathrm{Ch}(\mathcal{B})$. Let $z \in A_{x_\tau} \cap \mathrm{Ch}(\mathcal{B})$, and suppose that $z_\varphi \cap y_\varphi = \varnothing$. Then there exists an open set U such that $y_\varphi \subset U$ and $z_\varphi \subset X \setminus U$. As $y_\varphi = \{y, \varphi(y)\}$ is a p-set, there exists a $k \in \mathcal{P}_y(\mathcal{B}) \subset \mathcal{M}_{y_\varphi}(\mathcal{B})$ such that $M(k) \subset U$. If $h_1, h_2 \in \mathcal{A}$ are such that $T_1(h_1) = T_2(h_2) = k$, then $T_1(h_1)T_2(h_2) = k^2 \in \mathcal{M}_{y_\varphi}(\mathcal{B})$. Thus Lemma 3.3 implies that $S_1(h_1)S_2(h_2) \in \mathcal{M}_{x_\tau}(\mathcal{A})$. As $z \in A_{x_\tau} \cap \mathrm{Ch}(\mathcal{B})$, applying Lemma 3.3 again yields that $k^2 = T_1(h_1)T_2(h_2) \in \mathcal{M}_{z_\varphi}(\mathcal{B})$. Hence $|k(z)| = 1$, which contradicts that $M(k) \subset U$. Therefore, $z_\varphi \cap y_\varphi \neq \varnothing$. Consequently, $y_\varphi = z_\varphi$. $\square$

In light of Lemma 3.4, we define the mapping $\Psi : \mathrm{Ch}_\tau(\mathcal{A}) \to \mathrm{Ch}_\varphi(\mathcal{B})$, where $\mathrm{Ch}_\tau(\mathcal{A}) = \{x_\tau : x \in \mathrm{Ch}(\mathcal{A})\}$ and $\mathrm{Ch}_\varphi(\mathcal{B}) = \{y_\varphi : y \in \mathrm{Ch}(\mathcal{B})\}$, by

$$\Psi(x_\tau) = A_{x_\tau} \cap \mathrm{Ch}(\mathcal{B}) = y_\varphi = \{y, \varphi(y)\}. \tag{3}$$

Lemma 3.5. *The mapping defined by* (3) *is a bijection.*

Proof. Let $x_\tau, z_\tau \in \mathrm{Ch}_\tau(\mathcal{A})$, and suppose that $\Psi(x_\tau) = \Psi(z_\tau)$. Given $h \in \mathcal{M}_{x_\tau}(\mathcal{A})$, let $k_1, k_2 \in \mathcal{A}$ be such that $S_1(k_1) = S_2(k_2) = h$, then $S_1(k_1)S_2(k_2) = h^2 \in \mathcal{M}_{x_\tau}(\mathcal{A})$. Lemma 3.3 implies that $T_1(k_1)T_2(k_2) \in \mathcal{M}_{\Psi(x_\tau)}(\mathcal{B}) = \mathcal{M}_{\Psi(z_\tau)}(\mathcal{B})$. Applying Lemma 3.3 again gives that $h^2 = S_1(k_1)S_2(k_2) \in \mathcal{M}_{z_\tau}(\mathcal{A})$, thus $h \in \mathcal{M}_{z_\tau}(\mathcal{A})$. It follows that $\mathcal{M}_{x_\tau}(\mathcal{A}) \subset \mathcal{M}_{z_\tau}(\mathcal{A})$, and $x_\tau = z_\tau$ by Lemma 2.1. Therefore Ψ is injective.

Now, let $y \in \mathrm{Ch}(\mathcal{B})$. Define $B_1(y) = T_1^{-1}[\mathcal{M}_{y_\varphi}(\mathcal{B})]$, $B_2(y) = T_2^{-1}[\mathcal{M}_{y_\varphi}(\mathcal{B})]$, and

$$B_{y_\varphi} = \bigcap_{h \in B_1(y),\quad k \in B_2(y)} M(S_1(h)S_2(k)).$$

This set is non-empty, and the proof is analogous to the proof of Lemma 3.2. As B_{y_φ} is a non-empty intersection of maximizing sets, it meets the Choquet boundary $\mathrm{Ch}(\mathcal{A})$, thus there exists an $x \in B_{y_\varphi} \cap \mathrm{Ch}(\mathcal{A})$. Given $k \in \mathcal{M}_{y_\varphi}(\mathcal{B})$, let $h_1 \in B_1(y)$ and $h_2 \in B_2(y)$ be such that $T_1(h_1) = T_2(h_2) = k$. It follows that $S_1(h_1)S_2(h_2) \in \mathcal{M}_{x_\tau}(\mathcal{A})$, thus Lemma 3.3 implies that $k^2 = T_1(h_1)T_2(h_2) \in \mathcal{M}_{\Psi(x_\tau)}(\mathcal{B})$, hence $k \in \mathcal{M}_{\Psi(x_\tau)}(\mathcal{B})$. Therefore $\mathcal{M}_{y_\varphi}(\mathcal{B}) \subset \mathcal{M}_{\Psi(x_\tau)}(\mathcal{B})$, and Lemma 2.1 yields that $\Psi(x_\tau) = y_\varphi$. Consequently, Ψ is surjective. $\square$

We now complete the proof of Proposition 3.1 with the following lemma:

Lemma 3.6. *Let $x \in \mathrm{Ch}(\mathcal{A})$ and let $y \in \Psi(x_\tau)$. Then $|S_1(f)(x)S_2(g)(x)| = |T_1(f)(y)T_2(g)(y)|$ for all $f, g \in \mathcal{A}$.*

Proof. Let $f, g \in \mathcal{A}$. If any of $S_1(f), S_2(g), T_1(f)$, or $T_2(g)$ is identically 0, then the result follows from (1). So we assume that $S_1(f), S_2(g), T_1(f), T_2(g) \neq 0$. Now, suppose that $S_1(f)(x)S_2(g)(x) = 0$, then, without loss of generality, we can assume that $S_1(f)(x) = 0$. Given an $\varepsilon > 0$, Lemma 2.3 implies that there exists an $h \in \mathcal{P}_x(\mathcal{A}) \subset \mathcal{M}_{x_\tau}(\mathcal{A})$ such that $\|S_1(f)h\| < \varepsilon/\|S_2(g)\|$. Given $p_1, p_2 \in \mathcal{A}$ such that $S_1(p_1) = S_2(p_2) = h$, then $S_1(p_1)S_2(p_2) \in \mathcal{M}_{x_\tau}(\mathcal{A})$. Lemma 3.3 yields that $T_1(p_1)T_2(p_2) \in \mathcal{M}_{y_\varphi}(\mathcal{B})$, thus

$$\begin{aligned}
|T_1(f)(y)T_2(g)(y)| &= |T_1(f)(y)T_2(p_2)(y)T_1(p_1)(y)T_2(g)(y)| \\
&\leq \|T_1(f)T_2(p_2)T_1(p_1)T_2(g)\| \leq \|T_1(f)T_2(p_2)\| \cdot \|T_1(p_1)T_2(g)\| \\
&= \|S_1(f)S_2(p_2)\| \cdot \|S_1(p_1)S_2(g)\| = \|S_1(f)h\| \cdot \|S_2(g)h\| \\
&< \frac{\varepsilon}{\|S_2(g)\|} \cdot \|S_2(g)\| = \varepsilon.
\end{aligned}$$

As ε was chosen arbitrarily, $T_1(f)(y)T_2(g)(y) = 0$. A similar argument implies that if $T_1(f)(y)T_2(g)(y) = 0$, then $S_1(f)(x)S_2(g)(x) = 0$. In either situation, $|S_1(f)(x)S_2(g)(x)| = |T_1(f)(y)T_2(g)(y)|$.

Suppose that $S_1(f)(x)S_2(g)(x) \neq 0$. Then $S_1(f)(x) \neq 0$, $S_2(g)(x) \neq 0$, and Lemma 2.2 implies that there exist functions $h_1, h_2 \in \mathcal{P}_x(\mathcal{A}) \subset \mathcal{M}_{x_\tau}(\mathcal{A})$ such that $\|S_1(f)h_1\| = |S_1(f)(x)|$ and $\|S_2(g)h_2\| = |S_2(g)(x)|$. Let $k_1, k_2 \in \mathcal{A}$ be such that $S_1(k_1) = h_2$ and $S_2(k_2) = h_1$, then $S_1(k_1)S_2(k_2) \in \mathcal{M}_{x_\tau}(\mathcal{A})$ and Lemma 3.3 gives that $T_1(k_1)T_2(k_2) \in \mathcal{M}_{y_\varphi}(\mathcal{B})$. Thus

$$|T_1(f)(y)T_2(g)(y)| = |T_1(f)(y)T_2(k_2)(y)T_1(k_1)(y)T_2(g)(y)|$$

$$\begin{aligned}&\le \|T_1(f)T_2(k_2)T_1(k_1)T_2(g)\| \le \|T_1(f)T_2(k_2)\| \cdot \|T_1(k_1)T_2(g)\|\\&= \|S_1(f)S_2(k_2)\| \cdot \|S_1(k_1)S_2(g)\| = \|S_1(f)h_1\| \cdot \|S_2(g)h_2\|\\&= |S_1(f)(x)S_2(g)(x)|.\end{aligned}$$

An analogous argument gives the reverse inequality. □

We conclude this section with the following corollary to this lemma.

Corollary 3.1. *Let $f, g \in \mathcal{A}$; let $x \in \mathrm{Ch}(\mathcal{A})$; and let $y \in \Psi(x_\tau)$. Then $x \in M(S_1(f)S_2(g))$ if and only if $y \in M(T_1(f)T_2(g))$.*

4. Jointly peripherally multiplicative maps

Suppose that $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ are surjective mappings between real function algebras $\mathcal{A} \subset C(X, \tau)$ and $\mathcal{B} \subset C(Y, \varphi)$ that satisfy

$$\sigma_\pi(T_1(f)T_2(g)) = \sigma_\pi(S_1(f)S_2(g)) \tag{4}$$

for all $f, g \in \mathcal{A}$. Any such foursome satisfies (1), thus Proposition 3.1 yields that there exists a bijective mapping $\Psi : \mathrm{Ch}_\tau(\mathcal{A}) \to \mathrm{Ch}_\varphi(\mathcal{B})$, and given an $x \in \mathrm{Ch}(\mathcal{A})$, then $|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|$ for all $f, g \in \mathcal{A}$, and all $y \in \Psi(x_\tau)$. The proof of the Main Theorem will follow from a sequence of lemmas.

Lemma 4.1. *Let $h, k \in \mathcal{A}$ be such that $S_1(h) = S_2(k) = 1$. Then $T_1(h)T_2(k) = 1$.*

Proof. By (4), $\sigma_\pi(T_1(h)T_2(k)) = \sigma_\pi(S_1(h)S_2(k)) = \{1\}$. Given $y \in \mathrm{Ch}(\mathcal{B})$, there exists an $x \in \mathrm{Ch}(\mathcal{A})$ such that $y_\varphi = \Psi(x_\tau)$. As $x \in M(S_1(h)S_2(k))$, Corollary 3.1 implies that $y \in M(T_1(h)T_2(k))$, thus $T_1(h)(y)T_2(k)(y) = 1$. Therefore, $T_1(h)T_2(k) \equiv 1$ on $\mathrm{Ch}(\mathcal{B})$, hence $T_1(h)T_2(k) = 1$. □

Given any pair $h, k \in \mathcal{A}$ such that $S_1(h) = S_1(k) = 1$ and any $f \in \mathcal{A}$ such that $S_2(f) = 1$, Lemma 4.1 implies that $T_1(h)T_2(f) = 1 = T_1(k)T_2(f)$. This yields that $T_1(h) = T_1(k)$. A similar argument implies that $T_2(h) = T_2(k)$ for any $h, k \in \mathcal{A}$ with $S_2(h) = S_2(k) = 1$. Define the function $\kappa_j \in \mathcal{B}$ to be

$$\kappa_j = T_j(h), \tag{5}$$

where $h \in \mathcal{A}$ is such that $S_j(h) = 1$, and $j = 1, 2$. This definition is independent of the choice of h, and $\kappa_1 \cdot \kappa_2 = 1$.

Define $\widetilde{T}_1, \widetilde{T}_2 : \mathcal{A} \to \mathcal{B}$ by

$$\widetilde{T}_1(f) = T_1(f)\kappa_2 \quad \text{and} \quad \widetilde{T}_2(f) = \kappa_1 T_2(f). \tag{6}$$

It is straightforward to verify that $\widetilde{T}_1$ and $\widetilde{T}_2$ are surjective mappings, and, as $\widetilde{T}_1 \cdot \widetilde{T}_2 = T_1 \cdot T_2$, the mappings $\widetilde{T}_1, \widetilde{T}_2, S_1$, and S_2 satisfy (4). Moreover, $\sigma_\pi(\widetilde{T}_j(f)) = \sigma_\pi(S_j(f))$ for all $f \in \mathcal{A}$ and $j = 1, 2$, and this implies that $S_j(f) \in \mathcal{P}(\mathcal{A})$ if and only if $\widetilde{T}_j(f) \in \mathcal{P}(\mathcal{B})$, where $j = 1, 2$. The same statement holds for the (i)-peaking functions. Given an $x \in \mathrm{Ch}(\mathcal{A})$ and $y \in \Psi(x_\tau)$, then $|\widetilde{T}_j(f)(y)| = |S_j(f)(x)|$ holds for all $f \in \mathcal{A}$ and $j = 1, 2$ by Lemma 3.6. This yields $S_j(h) \in \mathcal{P}_x(\mathcal{A})$ if and only if $\widetilde{T}_j(h) \in \mathcal{P}_y(\mathcal{B})$, where $j = 1, 2$.

Lemma 4.2. *Let $x \in \mathrm{Ch}(\mathcal{A})$ and let $y \in \Psi(x_\tau)$. Then $\tau(x) = x$ if and only if $\varphi(y) = y$.*

Proof. Suppose that $\tau(x) = x$, and let $g \in \mathcal{B}$. If $g(y) = 0$, then $g(y) = g(\varphi(y))$. If $g(y) \neq 0$, then Lemma 2.3 implies that there exists a $k \in \mathcal{P}_y(\mathcal{B})$ such that $\sigma_\pi(gk) = \{g(y), g(\varphi(y))\}$. Let $f, h \in \mathcal{A}$ be such that $\widetilde{T}_1(f) = g$ and $\widetilde{T}_2(h) = k$, then (4) implies that $\sigma_\pi(S_1(f)S_2(h)) = \{g(y), g(\varphi(y))\}$. As $y \in M(\widetilde{T}_1(f)\widetilde{T}_2(h)) = M(T_1(f)T_2(h))$, Corollary 3.1 yields that $x \in M(S_1(f)S_2(h))$, hence $S_1(f)(x)S_2(h)(x)$ belongs to $\sigma_\pi(S_1(f)S_2(h)) = \{g(y), g(\varphi(y))\}$. Since $\tau(x) = x$, we have that $S_1(f)(x)S_2(h)(x)$ is a real number, so either $g(y)$ or $g(\varphi(y))$ is real. In either case, $g(y) = g(\varphi(y))$. Therefore, as g was chosen arbitrarily and as $\mathcal{B}$ separates points, $y = \varphi(y)$.

A similar argument demonstrates the converse. □

Given $x \in \mathrm{Ch}(\mathcal{A})$ with $\tau(x) \neq x$, Lemma 4.2 implies that $\varphi(y) \neq y$, where $y \in \Psi(x_\tau)$. Consequently, both $i\mathcal{P}_x(\mathcal{A})$ and $i\mathcal{P}_y(\mathcal{B})$ are non-empty. Following an argument similar to [4, Proposition 2.8], we now demonstrate that $\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y) = -1$ for any choice of $h \in S_1^{-1}[i\mathcal{P}_x(\mathcal{A})]$ and $k \in S_2^{-1}[i\mathcal{P}_x(\mathcal{A})]$, where $y \in \Psi(x_\tau)$. This type of result is paramount in the study of spectral preservers, especially for non-unital algebras (see [5]).

Lemma 4.3. *Let $x \in \mathrm{Ch}(\mathcal{A})$ be such that $x \neq \tau(x)$ and let $y \in \Psi(x_\tau)$. Then $\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y) = -1$ for all $h \in S_1^{-1}[i\mathcal{P}_x(\mathcal{A})]$ and $k \in S_2^{-1}[i\mathcal{P}_x(\mathcal{A})]$.*

Proof. $h \in S_1^{-1}[i\mathcal{P}_x(\mathcal{A})]$ and $k \in S_2^{-1}[i\mathcal{P}_x(\mathcal{A})]$. Since $\widetilde{T}_1(h), \widetilde{T}_2(k) \in i\mathcal{P}(\mathcal{B})$ and $|\widetilde{T}_1(h)(y)| = |S_1(h)(x)| = 1 = |S_2(k)(x)| = |\widetilde{T}_2(k)(y)|$, it follows that $\widetilde{T}_1(h)(y), \widetilde{T}_2(k)(y) = \pm i$. Thus $\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y) = \pm 1$, hence $\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y) = \widetilde{T}_1(h)(\varphi(y))\widetilde{T}_2(k)(\varphi(y))$. Lemma 2.3 implies that there exists a $g \in \mathcal{P}_y(\mathcal{B})$ such that $\sigma_\pi(\widetilde{T}_1(h)\widetilde{T}_2(k)g) = \{\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y)\}$. Moreover, $\sigma_\pi(\widetilde{T}_1(h)\widetilde{T}_2(k)g^2) =$

$\{\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y)\}$. Let $f_1, f_2 \in \mathcal{A}$ be such that $\widetilde{T}_1(f_1) = \widetilde{T}_1(h)g$ and $\widetilde{T}_2(f_2) = \widetilde{T}_2(k)g$, then (4) yields that

$$\sigma_\pi(S_1(f_1)S_2(f_2)) = \sigma_\pi(S_1(f_1)S_2(k)) = \sigma_\pi(S_1(h)S_2(f_2)) = \{\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y)\}.$$

As y belongs to $M(\widetilde{T}_1(f_1)\widetilde{T}_2(k))$ and $M(\widetilde{T}_1(h)\widetilde{T}_2(f_2))$, Corollary 3.1 implies that x belongs to $M(S_1(f_1)S_2(k))$ and $M(S_1(h)S_2(f_2))$. This yields that $iS_1(f_1)(x) \in \sigma_\pi(S_1(f_1)S_2(k))$ and $iS_2(f_2)(x) \in \sigma_\pi(S_1(h)S_2(f_2))$. Hence $\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y) = iS_1(f_1)(x) = iS_2(f_2)(x)$, thus $1 = [\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y)]^2 = -S_1(f_1)(x)S_2(f_2)(x)$. Since $y \in M(\widetilde{T}_1(f_1)\widetilde{T}_2(f_2))$, it must be that $x \in M(S_1(f_1)S_2(f_2))$, thus $-1 \in \sigma_\pi(S_1(f_1)S_2(f_2))$. Consequently, $\widetilde{T}_1(h)(y)\widetilde{T}_2(k)(y) = -1$. $\square$

Given $x \in \mathrm{Ch}(\mathcal{A})$, with $x \neq \tau(x)$, $h, k \in S_1^{-1}[i\mathcal{P}_x(\mathcal{A})]$, and $f \in S_2^{-1}[i\mathcal{P}_x(\mathcal{A})]$, Lemma 4.3 implies that $\widetilde{T}_1(h)(y)\widetilde{T}_2(f)(y) = -1 = \widetilde{T}_1(k)(y)\widetilde{T}_2(f)(y)$ for any $y \in \Psi(x_\tau)$, hence $\widetilde{T}_1(h)(y) = \widetilde{T}_1(k)(y)$. As $\widetilde{T}_1(h), \widetilde{T}_1(k) \in i\mathcal{P}(\mathcal{B})$, it follows that there exists a unique point $y' \in \Psi(x_\tau)$ such that $\widetilde{T}_1(h)(y') = i$ for all $h \in S_1^{-1}[i\mathcal{P}_x(\mathcal{A})]$. A similar argument implies that there exists a unique $y'' \in \Psi(x_\tau)$ such that $\widetilde{T}_2(h)(y'') = i$ for all $h \in S_2^{-1}[i\mathcal{P}_x(\mathcal{A})]$, and Lemma 4.3 implies that $y' = y''$.

Define the mapping $\theta : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$ as follows:

$$\begin{aligned} &\{\theta(x)\} = \Psi(x_\tau) && \text{if } x = \tau(x), \\ &\theta(x) \in \Psi(x_\tau) \text{ such that } \widetilde{T}_j(h)(\theta(x)) = i && \\ &\qquad\qquad \text{for all } h \in S_j^{-1}[i\mathcal{P}_x(\mathcal{A})] \ (j = 1, 2) && \text{if } x \neq \tau(x). \end{aligned} \tag{7}$$

The mapping θ is well defined by Lemmas 4.2 and 4.3. Moreover, θ is a surjective mapping, since Ψ is surjective.

Lemma 4.4. *Let $x \in \mathrm{Ch}(\mathcal{A})$; let $f \in \mathcal{A}$; and let $j \in \{1, 2\}$. Then $\widetilde{T}_j(f)(\theta(x)) = S_j(f)(x)$.*

Proof. Fix $n \in \{1, 2\}$ such that $n \neq j$. Suppose that $S_j(f)(x) = 0$, then $|\widetilde{T}_j(f)(\theta(x))| = |S_j(f)(x)| = 0$, hence $\widetilde{T}_j(f)(\theta(x)) = 0 = S_j(f)(x)$. If $S_j(f)(x) \neq 0$, then Lemma 2.3 implies that there exists an $h \in \mathcal{P}_x(\mathcal{A})$ such that $\sigma_\pi(S_j(f)h) = \{S_j(f)(x), S_j(f)(\tau(x))\}$. Let $g_1 \in \mathcal{A}$ be such that $S_n(g_1) = h$, then $S_n(g_1) \in \mathcal{P}_x(\mathcal{A})$, which implies that $\widetilde{T}_n(g_1) \in \mathcal{P}_{\theta(x)}(\mathcal{B})$. Since $x \in M(S_j(f)h)$, Corollary 3.1 implies that $\theta(x) \in M(\widetilde{T}_j(f)\widetilde{T}_n(g_1))$, thus

$$\widetilde{T}_j(f)(\theta(x)) \in \sigma_\pi(\widetilde{T}_j(f)\widetilde{T}_n(g_1)) = \sigma_\pi(S_j(f)S_n(g_1)) = \{S_j(f)(x), S_j(f)(\tau(x))\}.$$

So either $\widetilde{T}_j(f)(\theta(x)) = S_j(f)(x)$ or $\widetilde{T}_j(f)(\theta(x)) = S_j(f)(\tau(x))$.

If $\tau(x) = x$ or $S_j(f)(x) = S_j(f)(\tau(x))$, then $\widetilde{T}_j(f)(\theta(x)) = S_j(f)(x)$. Thus, we suppose that $\tau(x) \neq x$ and $S_j(f)(x) \neq S_j(f)(\tau(x))$. Corollary 2.1 implies that there exists a $k \in i\mathcal{P}_x(\mathcal{A})$ such that $\sigma_\pi(S_j(f)k) = \{iS_j(f)(x), -iS_j(f)(\tau(x))\}$. Let $g_2 \in \mathcal{A}$ be such that $S_n(g_2) = k$. As $S_n(g_2) \in i\mathcal{P}_x(\mathcal{A})$, the definition of θ implies that $\widetilde{T}_n(g_2) \in i\mathcal{P}_{\theta(x)}(\mathcal{B})$. Moreover, since $x \in M(S_j(f)k)$, it must be that $\theta(x) \in M(\widetilde{T}_j(f)\widetilde{T}_n(g_2))$, hence

$$\begin{aligned} i\widetilde{T}_j(f)(\theta(x)) \in \sigma_\pi(\widetilde{T}_j(f)\widetilde{T}_n(g_2)) &= \sigma_\pi(S_j(f)S_n(g_2)) \\ &= \{iS_j(f)(x), -iS_j(f)(\tau(x))\}. \end{aligned}$$

Now, if $\widetilde{T}_j(f)(\theta(x)) = S_j(f)(\tau(x))$, then $i\widetilde{T}_j(f)(\theta(x)) = iS_j(f)(\tau(x))$, which implies that $iS_j(f)(\tau(x)) \in \{iS_j(f)(x), -iS_j(f)(\tau(x))\}$. This is a contradiction, so it must be that $\widetilde{T}_j(f)(\theta(x)) = S_j(f)(x)$. $\square$

We now have the tools necessary to prove the Main Theorem.

Main Theorem. *Let X and Y be compact Hausdorff spaces; let $\tau : X \to X$ and $\varphi : Y \to Y$ be topological involutions; and let $\mathcal{A} \subset C(X, \tau)$ and $\mathcal{B} \subset C(Y, \varphi)$ be real function algebras. If $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ are surjective mappings that satisfy*

$$\sigma_\pi(T_1(f)T_2(g)) = \sigma_\pi(S_1(f)S_2(g))$$

for all $f, g \in \mathcal{A}$, then there exists a homeomorphism $\psi : \mathrm{Ch}(\mathcal{B}) \to \mathrm{Ch}(\mathcal{A})$ such that $(\psi \circ \varphi)(y) = (\tau \circ \psi)(y)$ for all $y \in \mathrm{Ch}(\mathcal{B})$, and there exists functions $\kappa_1, \kappa_2 \in \mathcal{B}$ that satisfy $\kappa_1^{-1} = \kappa_2$ and

$$T_j(f)(y) = \kappa_j(y)S_j(f)(\psi(y))$$

for all $f \in \mathcal{A}$, all $y \in \mathrm{Ch}(\mathcal{B})$, and $j = 1, 2$.

Proof. Suppose that $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ are surjective mappings that satisfy (4). Let $\kappa_1, \kappa_2 \in \mathcal{B}$ be the functions defined by (5), and let $\theta : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$ be the mapping defined by (7). Then Lemma 4.4 implies that $T_j(f)(\theta(x)) = \kappa_j(\theta(x))S_j(f)(x)$ for all $x \in \mathrm{Ch}(\mathcal{A})$, for all $f \in \mathcal{A}$, and for $j \in \{1, 2\}$. We will now demonstrate that θ is a bijection, and that its formal inverse is the homeomorphism we seek.

Indeed, let $x \in \mathrm{Ch}(\mathcal{A})$ and let $g \in \mathcal{B}$. If $f \in \mathcal{A}$ is such that $\widetilde{T}_1(f) = g$, then

$$\begin{aligned} g(\varphi(\theta(x))) = \widetilde{T}_1(f)(\varphi(\theta(x))) &= \overline{\widetilde{T}_1(f)(\theta(x))} = \overline{S_1(f)(x)} \\ &= S_1(f)(\tau(x)) = \widetilde{T}_1(f)(\theta(\tau(x))) = g(\theta(\tau(x))). \end{aligned}$$

Since g was arbitrary and $\mathcal{B}$ separates points, it follows that $\varphi(\theta(x)) = \theta(\tau(x))$. Now, suppose that $\theta(x) = \theta(x')$ for $x, x' \in \mathrm{Ch}(\mathcal{A})$, and let $f \in \mathcal{A}$. If $g \in \mathcal{A}$ is such that $f = S_1(g)$, then

$$f(x) = S_1(g)(x) = \widetilde{T}_1(g)(\theta(x)) = \widetilde{T}_1(g)(\theta(x')) = S_1(g)(x') = f(x').$$

As $\mathcal{A}$ separates points, it follows that $x = x'$. This implies that θ is injective, hence θ is a bijection.

Let $\psi = \theta^{-1}$, then $\psi : \mathrm{Ch}(\mathcal{B}) \to \mathrm{Ch}(\mathcal{A})$ satisfies $(\psi \circ \varphi)(y) = (\tau \circ \psi)(y)$ for all $y \in \mathrm{Ch}(\mathcal{B})$ and $T_j(f)(y) = \kappa_j(y)S_j(f)(\psi(y))$ for all $f \in \mathcal{A}$, all $y \in \mathrm{Ch}(\mathcal{B})$, and $j = 1, 2$. It only to show that ψ and θ are continuous. Let $x \in \mathrm{Ch}(\mathcal{A})$ and let $\{x_\lambda\}_{\lambda \in \Lambda} \subset \mathrm{Ch}(\mathcal{A})$ be a net such that $x_\lambda \to x$. If $g \in \mathcal{B}$ and $f \in \mathcal{A}$ are such that $T_1(f) = g$, then

$$g(\theta(x_\lambda)) = T_1(f)(\theta(x_\lambda)) = S_1(f)(x_\lambda) \to S_1(f)(x) = T_1(f)(\theta(x)) = g(\theta(x))$$

As $\mathcal{B}$ separates points, the topology on Y coincides with the weak topology generated by $\mathcal{B}$ (see [10, Lemma 1, p. 3]). Since $g(\theta(x_\lambda)) \to g(\theta(x))$ for all $g \in \mathcal{B}$, it follows that $\theta(x_\lambda) \to \theta(x)$. Therefore, θ is continuous, and an analogous argument yields that ψ is continuous. □

5. Jointly weakly peripherally multiplicative maps

Throughout this section, $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ are surjective mappings between real function algebras $\mathcal{A} \subset C(X, \tau)$ and $\mathcal{B} \subset C(Y, \varphi)$ that satisfy

$$\sigma_\pi(T_1(f)T_2(g)) \cap \sigma_\pi(S_1(f)S_2(g)) \neq \varnothing \tag{8}$$

for all $f, g \in \mathcal{A}$. Any such four mappings satisfy (1), thus Proposition 3.1 yields that there exists a bijective mapping $\Psi : \mathrm{Ch}_\tau(\mathcal{A}) \to \mathrm{Ch}_\varphi(\mathcal{B})$, and given an $x \in \mathrm{Ch}(\mathcal{A})$, then $|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|$ for all $f, g \in \mathcal{A}$, and all $y \in \Psi(x_\tau)$.

Lemma 5.1. *Let $f, g \in \mathcal{A}$ and let $x \in \mathrm{Ch}(\mathcal{A})$. Then $M(S_1(f)S_2(g)) = x_\tau = \{x, \tau(x)\}$ if and only if $M(T_1(f)T_2(g)) = \Psi(x_\tau)$, in which case $\sigma_\pi(S_1(f)S_2(g)) = \sigma_\pi(T_1(f)T_2(g))$.*

Proof. Since $M(S_1(f)S_2(g)) = x_\tau$ if and only if $M(T_1(f)T_2(g)) \cap \mathrm{Ch}(\mathcal{B}) = \Psi(x_\tau)$, Lemma 2.5 implies that $M(S_1(f)S_2(g)) = x_\tau$ if and only $M(T_1(f)T_2(g)) = \Psi(x_\tau)$. In this case, $\sigma_\pi(S_1(f)S_2(g)) = \{\lambda, \overline{\lambda}\}$, where $\lambda = S_1(f)(x)S_2(g)(x)$. Since

$M(T_1(f)T_2(g)) = \Psi(x_\tau)$, the set $\sigma_\pi(T_1(f)T_2(g))$ is at most a doubleton. By (8), either $\lambda \in \sigma_\pi(T_1(f)T_2(g))$ or $\overline{\lambda} \in \sigma_\pi(T_1(f)T_2(g))$. As $\sigma_\pi(T_1(f)T_2(g))$ is closed under conjugation, it follows that $\sigma_\pi(T_1(f)T_2(g)) = \{\lambda, \overline{\lambda}\} = \sigma_\pi(S_1(f)S_2(g))$. $\square$

An immediate consequence of this lemma and Lemma 2.6 is that $\mathrm{Ch}(\mathcal{A})$ is a minimal boundary for $\mathcal{A}$ if and only if $\mathrm{Ch}(\mathcal{B})$ is a minimal boundary for $\mathcal{B}$. Thus for the remainder of this section, we will assume that either $\mathrm{Ch}(\mathcal{A})$ or $\mathrm{Ch}(\mathcal{B})$ is a minimal boundary for its respective algebra.

Given $f, g \in \mathcal{A}$ and $x \in \mathrm{Ch}(\mathcal{A})$, Lemma 5.1 implies that that $S_1(f)S_2(g) \in \mathcal{P}(\mathcal{A})$ with $M(S_1(f)S_2(g)) = x_\tau = \{x, \tau(x)\}$ if and only if $T_1(f)T_2(g) \in \mathcal{P}(\mathcal{B})$ with $M(T_1(f)T_2(g)) = \Psi(x_\tau)$. The same result holds if $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$ are replaced with $i\mathcal{P}(\mathcal{A})$ and $i\mathcal{P}(\mathcal{B})$. Additionally, if $T_1(f), T_2(g) \in i\mathcal{P}_y(\mathcal{B})$, where $y \in \Psi(x_\tau)$, are such that $M(T_1(f)T_2(g)) = \Psi(x_\tau)$, then $\sigma_\pi(S_1(f)S_2(g)) = \sigma_\pi(T_1(f)T_2(g)) = \{-1\}$ and $M(S_1(f)S_2(g)) = x_\tau$. An equivalent result holds for $S_1(f), S_2(g) \in i\mathcal{P}_x(\mathcal{A})$. Moreover, we have the following lemma:

Lemma 5.2. *Let $h_1, h_2, k_1, k_2 \in \mathcal{A}$; let $y \in \mathrm{Ch}(\mathcal{B})$ be such that $\varphi(y) \neq y$; and let $x \in \mathrm{Ch}(\mathcal{A})$ be such that $\Psi(x_\tau) = y_\varphi$. If $T_1(h_1), T_2(h_2) \in \mathcal{P}_y(\mathcal{B})$ and $T_1(k_1), T_2(k_2) \in i\mathcal{P}_y(\mathcal{B})$ with $M(T_1(h_1)) = M(T_2(h_2)) = M(T_1(k_1)) = M(T_2(k_2)) = y_\varphi$, then there exists an $x_0 \in x_\tau$ such that $S_1(k_1)S_2(h_2)$ and $S_1(h_1)S_2(k_2)$ belong to $i\mathcal{P}_{x_0}(\mathcal{A})$.*

Proof. Suppose that $T_1(h_1), T_2(h_2) \in \mathcal{P}_y(\mathcal{B})$ and $T_1(k_1), T_2(k_2) \in i\mathcal{P}_y(\mathcal{B})$ and that $M(T_1(h_1)) = M(T_2(h_2)) = M(T_1(k_1)) = M(T_2(k_2)) = y_\varphi$. Since both $T_1(h_1)T_2(k_2)$ and $T_1(k_1)T_2(h_2)$ belong to $i\mathcal{P}_y(\mathcal{B})$ with $M(T_1(h_1)T_2(k_2)) = M(T_1(k_1)T_2(h_2)) = y_\varphi$, it follows that $S_1(k_1)S_2(h_2)$ and $S_1(h_1)S_2(k_2)$ belong to $i\mathcal{P}(\mathcal{A})$. Moreover, $S_1(h_1)S_2(h_2) \in \mathcal{P}_x(\mathcal{A})$ and $\sigma_\pi(S_1(k_1)S_2(k_2)) = \{-1\}$, thus

$$\begin{aligned}[S_1(k_1)(x)S_2(h_2)(x)] \cdot [S_1(h_1)(x)S_2(k_2)(x)] \\ = [S_1(h_1)(x)S_2(h_2)(x)] \cdot [S_1(k_1)(x)S_2(k_2)(x)] = (1) \cdot (-1) = -1.\end{aligned}$$

It follows that $S_1(k_1)(x)S_2(h_2)(x) = S_1(h_1)(x)S_2(k_2)(x) = \pm i$. Therefore there exists an $x_0 \in x_\tau$ such that $S_1(k_1)S_2(h_2), S_1(h_1)S_2(k_2) \in i\mathcal{P}_{x_0}(\mathcal{A})$. $\square$

We note that the analogous lemma, where $x \in \mathrm{Ch}(\mathcal{A})$ is such that $\tau(x) \neq x$ and the functions h_1, h_2, k_1, and k_2 satisfy $S_1(h_1), S_2(h_2) \in \mathcal{P}_x(\mathcal{A})$ and $S_1(k_1)$, $S_2(k_2) \in i\mathcal{P}_x(\mathcal{B})$ with $M(S_1(h_1)) = M(S_2(h_2)) = M(S_1(k_1)) = M(S_2(k_2)) = x_\tau$, is also true and proven similarly.

Using these facts, we now prove the main result of this section: any jointly weakly peripherally multiplicative foursome of mappings is jointly peripherally multiplicative, assuming either $\mathrm{Ch}(\mathcal{A})$ or $\mathrm{Ch}(\mathcal{B})$ is minimal.

Lemma 5.3. *The mappings S_1, S_2, T_1, and T_2 satisfy* (4).

Proof. Let $f, g \in \mathcal{A}$ and let $\lambda \in \sigma_\pi(T_1(f)T_2(g))$. If $\lambda = 0$, then $T_1(f)T_2(g) = 0$, and it follows that $\sigma_\pi(S_1(f)S_2(g)) = \{0\} = \sigma_\pi(T_1(f)T_2(g))$.

Suppose that $\lambda \neq 0$, then there exists a $y_0 \in M(T_1(f)T_2(g))$ such that $\lambda = T_1(f)(y_0)T_2(g)(y_0)$. In fact, we may choose $y_0 \in \mathrm{Ch}(\mathcal{B})$, since there exists a peaking function $q \in \mathcal{P}(\mathcal{A})$ such that $M(q) = (T_1(f)T_2(g))^{-1}[\{\lambda\}] \cup (T_1(f)T_2(g))^{-1}[\{\overline{\lambda}\}]$ and $M(q) \cap \mathrm{Ch}(\mathcal{B}) \neq \varnothing$.

Now, let $x \in \mathrm{Ch}(\mathcal{A})$ be such that $\Psi(x_\tau) = \{y_0, \varphi(y_0)\}$, then it follows that $x \in M(S_1(f)S_2(g))$, hence $S_1(f)(x)S_2(g)(x) \in \sigma_\pi(S_1(f)S_2(g))$. We claim that $S_1(f)(x)S_2(g)(x)$ is equal to either λ or $\overline{\lambda}$.

Set $\alpha = T_1(f)(y_0) \neq 0$ and $\beta = T_2(g)(y_0) \neq 0$. Then Corollary 2.2 implies that there exist $h_1, h_2 \in \mathcal{A}$ such that $T_1(h_1), T_2(h_2) \in \mathcal{P}_{y_0}(\mathcal{B})$ and

$$M(T_1(h_1)) = M(T_2(h_2)) = M(T_1(f)T_2(h_2)) = M(T_2(g)T_1(h_1)) = \{y_0, \varphi(y_0)\}.$$

As $T_1(h_1)T_2(h_2) \in \mathcal{P}_{y_0}(\mathcal{B})$ with $M(T_1(h_1)T_2(h_2)) = \{y_0, \varphi(y_0)\}$, we have that $S_1(h_1)S_2(h_2) \in \mathcal{P}_x(\mathcal{A})$ with $M(S_1(h_1)S_2(h_2)) = x_\tau$. Additionally, Lemma 5.1 implies that

$$\begin{aligned} &M(S_1(f)S_2(h_2)) = M(S_1(h_1)S_2(g)) = x_\tau, \\ &\sigma_\pi(S_1(f)S_2(h_2)) = \sigma_\pi(T_1(f)T_2(h_2)) = \{\alpha, \overline{\alpha}\}, \quad \text{and} \\ &\sigma_\pi(S_1(h_1)S_2(g)) = \sigma_\pi(T_1(h_1)T_2(g)) = \{\beta, \overline{\beta}\}. \end{aligned}$$

This yields that $S_1(f)(x)S_2(h_2)(x)$ is equal to either α or $\overline{\alpha}$ and $S_1(h_1)(x)S_2(g)(x)$ is equal to either β or $\overline{\beta}$. As,

$$S_1(f)(x)S_2(g)(x) = [S_1(f)(x)S_2(h_2)(x)] \cdot [S_1(h_1)(x)S_2(g)(x)],$$

it follows that $S_1(f)(x)S_2(g)(x)$ is equal to $\alpha\beta$, $\overline{\alpha}\beta$, $\alpha\overline{\beta}$, or $\overline{\alpha\beta}$.

If $y_0 = \varphi(y_0)$, then $\alpha = \overline{\alpha}$ and $\beta = \overline{\beta}$, which yields that $S_1(f)(x)S_2(g)(x)$ is equal to either $\lambda = \alpha\beta$ or $\overline{\lambda} = \overline{\alpha\beta}$. Now, suppose that $y_0 \neq \varphi(y_0)$. Corollary 2.2 implies that there exist $k_1, k_2 \in \mathcal{A}$ such that $T_1(k_1), T_2(k_2) \in i\mathcal{P}_{y_0}(\mathcal{B})$ and

$$M(T_1(k_1)) = M(T_2(k_2)) = M(T_1(f)T_2(k_2)) = M(T_2(g)T_1(k_1)) = \{y_0, \varphi(y_0)\}.$$

Moreover, Lemma 5.1 yields that

$$\begin{aligned} &M(S_1(f)S_2(k_2)) = M(S_1(k_1)S_2(g)) = x_\tau, \\ &\sigma_\pi(S_1(f)S_2(k_2)) = \sigma_\pi(T_1(f)T_2(k_2)) = \{i\alpha, -i\overline{\alpha}\}, \quad \text{and} \\ &\sigma_\pi(S_1(k_1)S_2(g)) = \sigma_\pi(T_1(k_1)T_2(g)) = \{i\beta, -i\overline{\beta}\}. \end{aligned}$$

By Lemma 5.2, there exists an $x_0 \in x_\tau$ such that $S_1(h_1)S_2(k_2)$ and $S_1(k_1)S_2(h_2)$ belong to $i\mathcal{P}_{x_0}(\mathcal{A})$. Note that $S_1(f)(x_0)S_2(h_2)(x_0) = \alpha$ or $S_1(f)(x_0)S_2(h_2)(x_0) = \overline{\alpha}$. In the latter case,

$$iS_1(f)(x_0) = S_1(f)(x_0)S_1(k_1)(x_0)S_2(h_2)(x_0) = \overline{\alpha}S_1(k_1)(x_0).$$

As $\sigma_\pi(T_1(k_1)T_2(k_2)) = \{-1\}$, it follows that $S_1(k_1)(x_0)S_2(k_2)(x_0) = -1$, hence $-\overline{\alpha} = iS_1(f)(x_0)S_2(k_2)(x_0)$. This implies that

$$i\overline{\alpha} = S_1(f)(x_0)S_2(k_2)(x_0) \in \sigma_\pi(S_1(k_1)S_2(g)) = \{i\alpha, -i\overline{\alpha}\},$$

which yields that $\alpha = \overline{\alpha}$. Consequently, it must be that $S_1(f)(x_0)S_2(h_2)(x_0) = \alpha$, and a similar argument implies that $S_1(h_1)(x_0)S_2(g)(x_0) = \beta$. If $x = x_0$, then $S_1(f)(x)S_2(g)(x) = \lambda$. Likewise, if $\tau(x) = x_0$, then $S_1(f)(x)S_2(g)(x) = \overline{\lambda}$.

As $S_1(f)(x)S_2(g)(x) \in \sigma_\pi(S_1(f)S_2(g))$, it must be that $\lambda \in \sigma_\pi(S_1(f)S_2(g))$. Therefore, $\sigma_\pi(T_1(f)T_2(g)) \subset \sigma_\pi(S_1(f)S_2(g))$, and the reverse inclusion is proven similarly. □

Under the assumption that at least one of the Choquet boundaries is a minimal boundary for its respective algebra, any foursome of mappings that satisfy (8) automatically satisfy (4). Therefore, we can apply the Main Theorem and arrive at the following corollary:

Corollary 5.1. *Let X and Y be compact Hausdorff spaces; let $\tau : X \to X$ and $\varphi : Y \to Y$ be topological involutions; and let $\mathcal{A} \subset C(X, \tau)$ and $\mathcal{B} \subset C(Y, \varphi)$ be real function algebras. If either* $\mathrm{Ch}(\mathcal{A})$ *is a minimal boundary for $\mathcal{A}$ or* $\mathrm{Ch}(\mathcal{B})$ *is a minimal boundary for $\mathcal{B}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ are surjective mappings that satisfy*

$$\sigma_\pi(T_1(f)T_2(g)) \cap \sigma_\pi(S_1(f)S_2(g)) \neq \varnothing$$

for all $f, g \in \mathcal{A}$, then there exists a homeomorphism $\psi : \mathrm{Ch}(\mathcal{B}) \to \mathrm{Ch}(\mathcal{A})$ such that $(\psi \circ \varphi)(y) = (\tau \circ \psi)(y)$ for all $y \in \mathrm{Ch}(\mathcal{B})$, and there exist functions $\kappa_1, \kappa_2 \in \mathcal{B}$ that satisfy $\kappa_1^{-1} = \kappa_2$ and

$$T_j(f)(y) = \kappa_j(y)S_j(f)(\psi(y))$$

for all $f \in \mathcal{A}$, all $y \in \mathrm{Ch}(\mathcal{B})$, and $j = 1, 2$.

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