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Commutativity preservers via maximal centralizers

By GREGOR DOLINAR (Ljubljana) ALEXANDER GUTERMAN (Moscow)
BOJAN KUZMA (Koper) POLONA OBLAK (Ljubljana)

Abstract. Bijective maps on matrices over arbitrary fields with sufficiently many elements which preserve commutativity in both direction are classified.

1. Introduction and statement of the main result

The study of commutativity preserving maps on $M_n(\mathbb{F})$, the algebra of $n \times n$ matrices over a field $\mathbb{F}$ with additional assumption of linearity goes back to the work by Watkins [13]. He proved that for $n \geq 4$ every bijective linear commutativity preserving map Φ on $M_n(\mathbb{F})$, $\mathbb{F}$ algebraically closed field with characteristic zero, is of one of the following two standard forms: $\Phi(A) = cSAS^{-1} + f(A)I$ for all $A \in M_n(\mathbb{F})$ or $\Phi(A) = cSA^tS^{-1} + f(A)I$ for all $A \in M_n(\mathbb{F})$, where $0 \neq c \in \mathbb{F}$, $S \in M_n(\mathbb{F})$ is invertible, and f is a linear functional on $M_n(\mathbb{F})$. The research in this area was continued by many authors, see for example [3], [5], [9], [10], [11], [12] and references therein. In particular, Šemrl [12] removed the linearity assumption and characterized bijective maps on complex matrices which preserve commutativity in both directions, i.e., they preserve the set of commuting matrix pairs and the set of non-commuting matrix pairs. The main tool he used was the characterization of complex matrices with minimal and maximal centralizer. In the recent paper [4] we were able to extend Šemrl's result about minimal and maximal centralizers to matrix algebras over algebraically non-closed fields and in the present one we use our result to classify bijective maps, which preserve

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commutativity in both directions on matrices over algebraically non-closed fields thus extending the main result in [12].

Before stating our main theorem let us introduce some notations and let us recall some definitions. The *centralizer* of a matrix $A \in M_n(\mathbb{F})$ is the set $\mathcal{C}(A) = \{X \in M_n(\mathbb{F}) \mid AX = XA\}$ of all matrices that commute with A. The centralizer of a set $S \subseteq M_n(\mathbb{F})$ is $\mathcal{C}(S) = \{X \in M_n(\mathbb{F}) \mid AX = XA$ for every $A \in S\}$, the intersection of centralizers of all elements in S. Recall that the Centralizer Theorem (see, e.g., [8, p. 113, Corollary 1] and also [14, p. 106, Theorem 2]) states that for each matrix A we have $\mathcal{C}(\mathcal{C}(A)) = \mathbb{F}[A]$, where $\mathbb{F}[A]$ is the unital algebra over $\mathbb{F}$ generated by A.

Centralizer induces two natural relations on $M_n(\mathbb{F})$. The first one is a preorder defined by $A \preceq B$ if $\mathcal{C}(A) \subseteq \mathcal{C}(B)$. The second relation is the equivalence relation given by $A \sim B$ if $\mathcal{C}(A) = \mathcal{C}(B)$. We call such matrices $\mathcal{C}$-*equivalent*, and we say A, B are $\mathcal{C}$-*nonequivalent* (abbreviated $A \not\sim B$) if $\mathcal{C}(A) \neq \mathcal{C}(B)$. It is immediate that the preorder induces a partial order on a set of equivalence classes $M_n(\mathbb{F})/_\sim$.

A non-scalar matrix A is *minimal* if for every $X \in M_n(\mathbb{F})$ with $\mathcal{C}(A) \supseteq \mathcal{C}(X)$ it follows that $\mathcal{C}(A) = \mathcal{C}(X)$. A non-scalar matrix A is *maximal* if for every non-scalar $X \in M_n(\mathbb{F})$ with $\mathcal{C}(A) \subseteq \mathcal{C}(X)$ it follows that $\mathcal{C}(A) = \mathcal{C}(X)$. Minimal and maximal elements in the poset $M_n(\mathbb{F})/_\sim$ are the equivalence classes corresponding to the minimal and maximal matrices introduced above. Observe that a bijective map Φ preserves commutativity in both directions if and only if $\Phi(\mathcal{C}(A)) = \mathcal{C}(\Phi(A))$ holds for every A. Hence, such a map also preserves the partial order relation on equivalence classes.

By E_{ij} we denote the matrix with 1 on (i, j)-th position and 0 elsewhere, by 0_k and I_k we denote the zero $k \times k$ matrix and the identity $k \times k$ matrix, respectively. When clear from the context we omit the subscript. For a given scalar $\lambda \in \mathbb{F}$ define $J_n(\lambda) = \lambda I + \sum_{i=1}^{n-1} E_{i(i+1)}$ to be an elementary upper-triangular Jordan cell. We denote $J_n = J_n(0)$. A matrix $A \in M_n(\mathbb{F})$ is an *idempotent* if $A^2 = A$, it is a *nilpotent* if there exists an integer $k > 1$ such that $A^k = 0$. The smallest such k is its *nilpotency index*. The matrices with nilpotency index two are called *square-zero matrices*. For a monic polynomial $m(x) = m_0 + m_1 x + \cdots + m_{n-1}x^{n-1} + x^n \in \mathbb{F}[x]$ we let $C(m) = \sum_{i=1}^{n-1} E_{(i+1)i} - \sum_{i=1}^{n} m_{i-1}E_{in} \in M_n(\mathbb{F})$ be a companion matrix of m. A matrix is *non-derogatory* if its minimal polynomial equals its characteristic polynomial. Given a field homomorphism $\sigma : \mathbb{F} \to \mathbb{F}$ and a matrix $A \in M_n(\mathbb{F})$, we denote by A^σ a matrix obtained from A by applying σ to A entrywise. We denote by $\mathcal{D}_n(\mathbb{F}) \subset M_n(\mathbb{F})$ the subset of all diagonalizable matrices, and by $\mathcal{I}_n^1(\mathbb{F}) \subset M_n(\mathbb{F})$ the subset of all rank-one matrices.

The main result of the paper is as follows.

Theorem 1.1. *Let $n \geq 3$ be an integer and let $\mathbb{F}$ be a field with at least 2^{n-1} elements. If a bijective map $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserves commutativity in both directions, then there is a field homomorphism $\sigma : \mathbb{F} \to \mathbb{F}$ and an invertible matrix S such that one of the following holds.*

(i) $\Phi(A) = S\, p_A(A^\sigma) S^{-1}$ *for all* $A \in \mathcal{D}_n(\mathbb{F}) \cup \mathcal{I}_n^1(\mathbb{F})$.

(ii) $\Phi(A) = S\, p_A(A^\sigma)^{\mathrm{T}}\, S^{-1}$ *for all* $A \in \mathcal{D}_n(\mathbb{F}) \cup \mathcal{I}_n^1(\mathbb{F})$.

Here $p_A : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *is a matrix polynomial depending on* A.

Remark 1.2. Note that in the above theorem we cannot obtain a nice structure of Φ on the full matrix algebra, see for example [12, p. 22].

Remark 1.3. We do not know if the bound $|\mathbb{F}| \geq 2^{n-1}$ is optimal. It would be interesting to know if a similar result holds also for the fields with at least n elements. Note that n is the smallest possible cardinality of a field to ensure that there exists $n \times n$ non-derogatory diagonal matrix.

Remark 1.4. With the additional assumption of additivity more is known about commutativity preservers. See for example Banning and Mathieu [1] where bijections on semiprime rings which preserve commutativity are classified and also Brešar [3] for similar results on prime rings.

2. Proofs

The following Lemma will be used frequently. It is an easy consequence of the Centralizer Theorem.

Lemma 2.1. *Let $A, B \in M_n(\mathbb{F})$. Then $\mathcal{C}(A) \subseteq \mathcal{C}(B)$ if and only if $B \in \mathbb{F}[A]$. Consequently $\mathcal{C}(A) = \mathcal{C}(B)$ if and only if $\mathbb{F}[A] = \mathbb{F}[B]$.*

Let A and B be two matrices. We write $A \prec B$ if $\mathcal{C}(A) \subsetneq \mathcal{C}(B)$. A chain of *length* k from A to B is a set of $k+1$ matrices $A = X_0, X_1, \ldots, X_k = B$ such that $X_i \prec X_{i+1}$, $i = 0, \ldots, k-1$. We say that the *length* from A to B is k if there exists a chain of the length k from A to B but there is no chain of the length $k+1$ from A to B.

Lemma 2.2. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geq 2^{n-1}$. Let $A = C(m_1^{\alpha_1}) \oplus \cdots \oplus C(m_k^{\alpha_k}) \in M_n(\mathbb{F})$ be in its rational canonical form, where $k \geq 1$, $\alpha_1 \geq \cdots \geq \alpha_k \geq 1$ are integers and $m_1, \ldots, m_k$ are relatively prime irreducible polynomials.*

Assume that either $\alpha_1 \geq 4$ or $3 \geq \alpha_1 \geq \alpha_2 \geq 2$. Then there are at least 2^{n-1} pairwise $\mathcal{C}$-nonequivalent maximal matrices that commute with A.

Proof. First, suppose $\alpha_1 \geq 4$ and let us consider the polynomial $m = m_1 m_2^{\alpha_2} \dots m_k^{\alpha_k} \in \mathbb{F}[x]$. Matrix $\widehat{B} = m(A) = (m_1 m_2^{\alpha_2} \dots m_k^{\alpha_k})(A) \in \mathbb{F}[A]$ is nilpotent with index of nilpotency α_1. It easily follows that matrices $B_\gamma = \widehat{B}^{\alpha_1-2} + \gamma \widehat{B}^{\alpha_1-1} \in \mathbb{F}[A]$, $\gamma \in \mathbb{F}$, are non-scalar square-zero, hence maximal by [4, Theorem 3.2]. They all commute with A and are pairwise $\mathcal{C}$-nonequivalent. Since $|\mathbb{F}| \geq 2^{n-1}$, there are at least 2^{n-1} such matrices and the result holds.

Now, suppose $3 \geq \alpha_1 \geq \alpha_2 \geq 2$. For the sake of simplicity we denote $A_i = C(m_i^{\alpha_i})$. If $\alpha_1 = \alpha_2 = 3$, then for each $\gamma \in \mathbb{F}$ the matrices $L_\gamma = (m_1(A_1))^2 \oplus \gamma\, (m_2(A_2))^2 \oplus 0 \oplus \dots \oplus 0$ are non-scalar square-zero and $L_\gamma \in \mathbb{F}[A_1] \oplus \mathbb{F}[A_2] \oplus \dots \oplus \mathbb{F}[A_k] = \mathbb{F}[A]$, where the last equality follows by [2, Proposition 4.1]. Hence, the result holds. We treat the two remaining cases similarly by considering $M_\gamma = (m_1(A_1))^2 \oplus \gamma\, m_2(A_2) \oplus 0 \oplus \dots \oplus 0$ when $(\alpha_1, \alpha_2) = (3, 2)$, and $N_\gamma = m_1(A_1) \oplus \gamma\, m_2(A_2) \oplus 0 \oplus \dots \oplus 0$ when $(\alpha_1, \alpha_2) = (2, 2)$. □

Lemma 2.3. *Let $A = C(m_1^{\alpha_1}) \oplus \dots \oplus C(m_k^{\alpha_k}) \in M_n(\mathbb{F})$, $n \geq 3$, be in its rational canonical form, where $k \geq 1$, $\alpha_1 \geq \dots \geq \alpha_k \geq 1$ are integers and $m_1, \dots, m_k$ are relatively prime irreducible polynomials of the degrees $\mu_1, \dots, \mu_k$, respectively. Assume that $\alpha_1 \in \{2, 3\}$ and either $k = 1$ or $k \geq 2$, $\alpha_2 = \dots = \alpha_k = 1$. Then the number of pairwise $\mathcal{C}$-nonequivalent maximal matrices that commute with A differs from $2^{n-1} - 1$.*

Proof. For the sake of simplicity we write $A_i = C(m_i^{\alpha_i})$, $i = 1, \dots, k$.

1. Suppose first that $\alpha_1 = 3$ and either $k \geq 2$ with $\alpha_2 = \dots = \alpha_k = 1$ or $k = 1$. Then $A = A_1 \oplus D$ for $r \times r$ matrix $A_1 = C(m_1^{\alpha_1}) = C(m_1^3)$ and $(n-r) \times (n-r)$ matrix $D = C(m_2) \oplus \dots \oplus C(m_k)$, where D is omitted if $k = 1$.

Case 1. If $\mu_1 = \deg m_1 \geq 2$, then the matrix

$$B_\gamma = (\gamma I_r + A_1)(m_1(A_1))^2 \oplus 0_{n-r} \in \mathbb{F}[A_1] \oplus \mathbb{F}[D] = \mathbb{F}[A] \tag{1}$$

is square-zero, hence it is maximal for every $\gamma \in \mathbb{F}$. Let us show that B_{γ_1} and B_{γ_2} are $\mathcal{C}$-nonequivalent for $\gamma_1 \neq \gamma_2$. If $\mathcal{C}(B_{\gamma_1}) = \mathcal{C}(B_{\gamma_2})$ then $B_{\gamma_2} = p(B_{\gamma_1})$ for some polynomial $p \in \mathbb{F}[x]$. Since $B_{\gamma_1}^2 = 0$, we may assume $p(x) = \beta x + \delta$, $\beta, \delta \in \mathbb{F}$. Hence

$$\begin{aligned} 0 &= p(B_{\gamma_1}) - B_{\gamma_2} \\ &= \beta((\gamma_1 I_r + A_1)(m_1(A_1))^2 \oplus 0_{n-r}) + \delta I - (\gamma_2 I_r + A_1)(m_1(A_1))^2 \oplus 0_{n-r} \\ &= q(A_1) \oplus \delta I_{n-r}, \end{aligned}$$

where $q(x) = (\beta\gamma_1 - \gamma_2 + x(\beta - 1))(m_1(x))^2 + \delta$. Note that $\deg m_1 \geq 2$ by the condition of Case 1. Thus polynomial q is not divisible by polynomial m_1^3, a contradiction to $q(A_1) = 0_r$. Consequently, A commutes with at least $|\mathbb{F}|$ pairwise $\mathcal{C}$-nonequivalent maximal matrices B_γ, $\gamma \in \mathbb{F}$.

Case 2. If $\mu_1 = \deg m_1 = 1$, then without loss of generality we may assume $m_1(x) = x$. So $A_1 = J_3^{\mathrm{T}} \in M_3(\mathbb{F})$ is a lower triangular nilpotent Jordan cell. Since A is non-derogatory, $\mathcal{C}(A) = \mathbb{F}[A] = \mathbb{F}[J_3^{\mathrm{T}}] \oplus \mathbb{F}[D]$. Hence $\mathbb{F}[A]$ is not a field. Moreover, for any non-scalar matrix $X \in \mathbb{F}[A]$ there is $\lambda \in \mathbb{F}$ such that $X - \lambda I$ has a nilpotent right-upper 3×3 block. Hence $X - \lambda I$ is non-zero and non-invertible. Thus X cannot be similar to $C \oplus \cdots \oplus C$ for companion matrix $C = C(\tilde{m})$ of an irreducible polynomial $\tilde{m}$. Indeed, if the matrix is similar to $C \oplus \cdots \oplus C$, then the generated algebra is a field by the irreducibility of $\tilde{m}$, so it does not contain non-zero non-invertible matrices.

Hence by [4, Theorem 3.2], maximal matrices that commute with A are square-zero matrices and idempotents, up to $\mathcal{C}$-equivalence. Since the blocks $A_2, \ldots, A_k$ (if any) have irreducible minimal polynomials, $\mathbb{F}[A_i]$ is a field for $i = 2, \ldots, k$. Therefore, all non-scalar square-zero matrices in $\mathbb{F}(A)$ are $\mathcal{C}$-equivalent to $(J_3^{\mathrm{T}})^2 \oplus 0$ and each idempotent in $\mathbb{F}[A]$ is $\mathcal{C}$-equivalent to $0_3 \oplus P$ for some idempotent $P \in M_{n-3}(\mathbb{F})$. Then there are at most $2^{n-3} - 1$ pairwise $\mathcal{C}$-nonequivalent non-scalar idempotents in $\mathbb{F}[A]$ and thus 2^{n-3} $\mathcal{C}$-nonequivalent maximal matrices in $\mathbb{F}[A] = \mathcal{C}(A)$. Since $n \geq 3$ we have that $2^{n-3} < 2^{n-1} - 1$, and the result follows.

2. Suppose now that $\alpha_1 = 2$ and either $k \geq 2$ with $\alpha_2 = \ldots = \alpha_k = 1$ or $k = 1$. If $\mu_1 = \deg m_1 \geq 2$, then we define $B_\gamma = (\gamma I_r + A_1)m_1(A_1) \oplus 0_{n-r} \in \mathbb{F}[A_1] \oplus \mathbb{F}[D] = \mathbb{F}[A]$ as in (1) and proceed similarly as in the Case 1. If $\mu_1 = \deg m_1 = 1$, then without loss of generality $A_1 = J_2^{\mathrm{T}}$, and we proceed similarly as in the Case 2. $\square$

Lemma 2.4. *Let $A = C(m) \in M_n(\mathbb{F})$ be a companion matrix of an irreducible polynomial m. Then $\mathbb{F}[A]$ is a field and any chain of proper intermediate subfields between $\mathbb{F}$ and $\mathbb{F}[A]$ is finite with length $l \leq \log_2 n$.*

Proof. Since A is a companion matrix of an irreducible polynomial, $\mathbb{F}[A]$ is a field and the degree $[\mathbb{F}[A] : \mathbb{F}]$ of field extension $F[A] \supset \mathbb{F}$ equals n. Also, for every two subfields $L_k \supset L_i$ between $\mathbb{F}$ and $\mathbb{F}[A]$ we have $[L_k : L_i] \geq 2$. The result follows from $n = [\mathbb{F}[A] : \mathbb{F}] = [\mathbb{F}[A] : L_k] \cdot [L_k : L_i] \cdot [L_i : \mathbb{F}]$. $\square$

Lemma 2.5. *Let $A = C(m) \in M_n(\mathbb{F})$ be a companion matrix of an irreducible polynomial m. Then the length of each chain from A to any maximal matrix commuting with A is at most $\log_2 n - 1$.*

Proof. Since m is irreducible, $\mathcal{C}(A) = \mathbb{F}[A]$ is a field. Recall from Lemma 2.1 that for any $X_i, X_j \in M_n(\mathbb{F})$ it holds that $A \prec X_i \prec X_j$ if and only if $\mathbb{F}[A] \supsetneq \mathbb{F}[X_i] \supsetneq \mathbb{F}[X_j]$. Let $A = X_0 \prec X_1 \prec \cdots \prec X_k = M$ be a chain from A to a maximal matrix $M \in \mathcal{C}(A)$. This chain is in one-to-one correspondence with a chain of intermediate subfields $\mathbb{F}[A] = \mathbb{F}[X_0] \supsetneq \mathbb{F}[X_1] \supsetneq \cdots \supsetneq \mathbb{F}[X_k] = \mathbb{F}[M] \supsetneq \mathbb{F}$. It follows by Lemma 2.4 that each chain from A to a maximal matrix M has length at most $(\log_2 n) - 1$. $\square$

Lemma 2.6. *Let a non-diagonalizable $A = C(m_1) \oplus \cdots \oplus C(m_k) \in M_n(\mathbb{F})$ be in its rational canonical form, where $2 \leq k \leq n-1$ and $m_1, \ldots, m_k$ are relatively prime irreducible polynomials.*

(a) *Assume that all maximal matrices in $\mathcal{C}(A)$ are $\mathcal{C}$-equivalent to idempotents. Then the number of pairwise $\mathcal{C}$-nonequivalent maximal matrices in $\mathcal{C}(A)$ is at most $2^k - 2$.*

(b) *Assume some maximal matrix $M \in \mathcal{C}(A)$ is not $\mathcal{C}$-equivalent to idempotent. Then the length of any chain from A to M is at most $n - 3$.*

Proof. For the sake of simplicity we write $A_i = C(m_i)$, $i = 1, \ldots, k$.

To prove (a), observe that $\mathbb{F}[A_i]$ is a field for every i. So, each diagonal block of every idempotent in $\mathcal{C}(A) = \mathbb{F}[A_1] \oplus \cdots \oplus \mathbb{F}[A_k]$ has to be the zero matrix or the identity matrix. Hence, there are at most $2^k - 2$ non-scalar idempotents in $\mathcal{C}(A)$. But since A is non-diagonalizable we have $k \leq n-1$. Therefore the number of pairwise $\mathcal{C}$-nonequivalent maximal matrices that A commutes with is at most $2^k - 2 \leq 2^{n-1} - 2$.

To prove (b), note that $\mathcal{C}(A) = \mathbb{F}[A] = \mathbb{F}[A_1] \oplus \cdots \oplus \mathbb{F}[A_k]$ is a direct sum of fields, so there is no non-scalar nilpotent matrix commuting with A. By [4, Theorem 3.2], M is similar to $C \oplus \cdots \oplus C$ for some $C \in M_\ell(\mathbb{F})$ with properties as in Theorem [4, Theorem 3.2]. Note that $\ell \geq 2$, otherwise M is a scalar matrix. We claim that the length from A to M is at most $n - 3$.

Since $M \in \mathcal{C}(A) = \mathbb{F}[A_1] \oplus \cdots \oplus \mathbb{F}[A_k] \subseteq M_{d_1}(\mathbb{F}) \oplus \cdots \oplus M_{d_k}(\mathbb{F})$ we have $M = M_1 \oplus \cdots \oplus M_k$, where $M_j \in M_{d_j}(\mathbb{F})$. By [7, Theorem 11.20] the rational form of a matrix is unique up to permutation of blocks, so each M_j is similar to $\frac{d_j}{\ell}$ copies of C. It follows that each d_j is divisible by $\ell \geq 2$, so $d_j \geq 2$. Now, consider an arbitrary chain

$$A = X_0 \prec \cdots \prec X_i \prec X_{i+1} \prec \cdots \prec X_z = M. \tag{2}$$

Since $X_i \in \mathcal{C}(A) = \mathbb{F}[A_1] \oplus \cdots \oplus \mathbb{F}[A_k]$ it follows that $X_i = X_{i1} \oplus \cdots \oplus X_{ik}$ where $X_{ij} \in \mathbb{F}[A_j] \subseteq M_{d_j}(\mathbb{F})$. For each $j = 1, 2, \ldots, k$, $\mathbb{F}[A_j]$ is a field, so $\mathbb{F}[X_{ij}]$ is a subfield of $\mathbb{F}[A_j]$. As such, $\mathbb{F}[X_{ij}]$ contains neither non-scalar nilpotents nor non-scalar idempotents, so the rational form of X_{ij} consists of one or more identical cells that correspond to the same irreducible polynomial.

Each matrix X_i from the chain (2) induces a partition $\mathcal{P}_i$ of the set $\{1, \ldots, k\}$, where two indices j_1 and j_2 are in the same cell of $\mathcal{P}_i$ if the blocks X_{ij_1} and X_{ij_2} have the same minimal polynomial. For example, $\mathcal{P}_0 = \{\{1\}, \ldots, \{k\}\}$ and $\mathcal{P}_z = \{\{1, \ldots, k\}\}$. Recall that the partition $\{S_j\}_{1\le j\le s}$ of the set $S = \{1, 2, \ldots, k\}$ is the collection of non-empty disjoint subsets $S_1, \ldots, S_s \subseteq S$, called cells, such that $\cup_{j=1}^{s} S_j = S$. Moreover, a partition $\{S_j\}_{1\le j\le s}$ is coarser than a partition $\{\tilde{S}_j\}_{1\le j\le \tilde{s}}$ if every cell $\tilde{S}_j$ is a subset of some cell S_ℓ, and it is strictly coarser if it is coarser but not equal to $\{\tilde{S}_j\}_{1\le j\le \tilde{s}}$.

By Lemma 2.1, $X_{i+1} \in \mathbb{F}[X_i]$ for all $i = 1, 2, \ldots, z-1$. This implies that $X_{(i+1)j} \in \mathbb{F}[X_{ij}]$ for each j, and since the rational form of each block X_{ij} consists of direct sum of identical cells, $\mathcal{P}_{i+1}$ is coarser than $\mathcal{P}_i$.

Now, $X_i \prec X_{i+1}$ implies that either there exists j, such that $\mathbb{F}[X_{(i+1)j}] \subsetneq \mathbb{F}[X_{ij}]$, or else $\mathbb{F}[X_{(i+1)j}] = \mathbb{F}[X_{ij}]$ for each j and the induced partition $\mathcal{P}_{i+1}$ is strictly coarser than $\mathcal{P}_i$. To see this, assume erroneously that $\mathbb{F}[X_{ij}] = \mathbb{F}[X_{(i+1)j}]$ for all $j = 1, \ldots, k$ and that the induced partitions of X_i and X_{i+1} are equal. Write them as $\mathcal{P}_i = \{S_1, \ldots, S_{s_i}\} = \mathcal{P}_{i+1}$ and devise another block-decomposition of X_i

$$X_i = \hat{X}_{i1} \oplus \cdots \oplus \hat{X}_{is_i}, \quad \hat{X}_{it} = \bigoplus_{j\in S_t} X_{ij} \in M_{n_t}(\mathbb{F}),$$

where $n_t = \sum_{j\in S_t} d_j$, and likewise for

$$X_{i+1} = \hat{X}_{(i+1)1} \oplus \cdots \oplus \hat{X}_{(i+1)s_i}, \quad \hat{X}_{(i+1)t} = \bigoplus_{j\in S_t} X_{(i+1)j} \in M_{n_t}(\mathbb{F}).$$

Clearly, $X_{i+1} \in \mathbb{F}[X_i]$ so to gain a contradiction we only have to show that $X_i \in \mathbb{F}[X_{i+1}]$, which will imply that $\mathcal{C}(X_{i+1}) = \mathcal{C}(X_i)$. Since $X_{ij} \in \mathbb{F}[A_j]$ and $\mathbb{F}[A_j]$ is a field, i.e., has no non-scalar idempotents nor nilpotents, the rational form of X_{ij} consists of identical cells which are all companion matrices of the same irreducible polynomial. The definition of induced partition further implies that the rational form of each $\hat{X}_{it}$ consists of identical cells, so that $\hat{X}_{it} = \bigoplus_{r=1}^{n_t/\ell_t} C_{it}$, where $C_{it} = C(m_{it}) \in M_{\ell_t}(\mathbb{F})$ is a companion matrix corresponding to the same irreducible polynomial m_{it}. Applying a suitable similarity, we can assume that $\hat{X}_{it} = \bigoplus_{r=1}^{n_t/\ell_t} C_{it}$ actually equals its rational form. Then $X_{i+1} \in \mathbb{F}[X_i]$ implies $X_{i+1} = f(X_i) = f(\hat{X}_{i1}) \oplus \cdots \oplus f(\hat{X}_{is_i})$, so

$$\hat{X}_{(i+1)t} = f(\hat{X}_{it}) = \bigoplus_{r=1}^{n_t/\ell_t} f(C_{it}), \quad t = 1, \ldots, s_i. \tag{3}$$

Fix now $t \in \{1, \dots, s_i\}$ and $j \in S_t$. Since $\mathbb{F}[X_{ij}] = \mathbb{F}[X_{(i+1)j}]$ there is a polynomial g_t such that $X_{ij} = g_t(X_{(i+1)j})$. Note that $X_{(i+1)j} = f(X_{ij}) = \bigoplus_{r=1}^{d_j/\ell_t} f(C_{it})$, wherefrom $g_t(f(C_{it})) = C_{it}$. Because of (3), all block constituents $\hat{X}_{(i+1)t}$ are the same, so $g_t(\hat{X}_{(i+1)t}) = \hat{X}_{it}$.

Given distinct t, t', the blocks $\hat{X}_{(i+1)t}$ and $\hat{X}_{(i+1)t'}$ have relatively prime minimal polynomials, since otherwise the partition $\mathcal{P}_{i+1}$ would be strictly coarser that $\mathcal{P}_i$. This implies then $\mathbb{F}[X_{i+1}] = \mathbb{F}[\hat{X}_{(i+1)1}] \oplus \cdots \oplus \mathbb{F}[\hat{X}_{(i+1)s_i}]$, and we can find a polynomial g with $g(\hat{X}_{(i+1)t}) = g_t(\hat{X}_{(i+1)t}) = \hat{X}_{it}$ for each $t = 1, \dots, s_i$. Thus, $X_i = g(X_{i+1}) \in \mathbb{F}[X_{i+1}]$, a contradiction.

Let us find the upper bound for the length of chain (2). By Lemma 2.5 there exist at most $\log_2 d_j - 1$ nested proper intermediate subfields $\mathbb{F} \subsetneq \mathbb{F}[M_j] = \mathbb{F}[X_{zj}] \subsetneq \cdots \subsetneq \mathbb{F}[X_{(i+1)j}] \subsetneq \mathbb{F}[X_{ij}] \subsetneq \cdots \subsetneq \mathbb{F}[A_j]$. Also, we obtain the final induced partition $\mathcal{P}_z = \{\{1, \dots, k\}\}$ from the initial one $\mathcal{P}_0 = \{\{1\}, \dots, \{k\}\}$ in at most $k-1$ steps, where at each step the partition is strictly coarser than the previous one. All together, the upper bound for the length of chain (2) is

$$\begin{aligned}\big((\log_2 d_1 - 1) + \cdots + (\log_2 d_k - 1)\big) + (k-1) &= \log_2 d_1 + \cdots + \log_2 d_k - 1 \\ &\le (d_1 - 1) + \cdots + (d_k - 1) - 1 \le n - 2. \quad (4)\end{aligned}$$

If $k \ge 2$, then the last inequality in (4) is strict. If $k = 1$, then $3 \le n = d_1$ implies the first inequality in (4) is strict. Whatever the case, the length is at most $n-3$. □

Remark 2.7. Suppose $\mathbb{F}$ has characteristic 0. Then, the splitting field of an irreducible polynomial m is Galois over $\mathbb{F}$ [6, Corrolary 1, p. 91 and Theorem 3.4, p. 92]. Let $A = C(m)$. Then, with the help of the Fundamental Theorem of Galois theory [6, Theorem 2.10] it is easy to see the number of pairwise $\mathcal{C}$-nonequivalent maximal matrices in $\mathcal{C}(A)$ coincides with the number of maximal subgroups of Galois group of field extension $(\mathbb{F}[A] : \mathbb{F})$.

Lemma 2.8. *Let $n \ge 3$. Assume $\mathbb{F}$ is a field with $|\mathbb{F}| \ge 2^{n-1}$. Let $A \in M_n(\mathbb{F})$ be non-derogatory. Then the following statements are equivalent.*

(i) *A is diagonalizable over $\mathbb{F}$.*

(ii) *Up to $\mathcal{C}$-equivalence there exists exactly $2^{n-1} - 1$ maximal matrices that commute with A and the length from A to each of them is $n-2$.*

Proof. Let us prove that (i) implies (ii). We may assume without loss of generality that A is diagonal. Since A is non-derogatory it has n different eigenvalues. If a maximal matrix $B \in M_n(\mathbb{F})$ commutes with A, then B is also diagonal and by Theorem [4, Theorem 3.2] it is $\mathcal{C}$-equivalent to an idempotent. Since non-scalar diagonal idempotents are exactly diagonal maximal matrices and diagonal idempotents have only 0 and 1 entries on the main diagonal, B must be equal to one of the $2^n - 2$ diagonal non-scalar idempotents. Note that for an arbitrary idempotent P the only idempotent $\mathcal{C}$-equivalent to it is $I - P$. Therefore there exists $2^{n-1} - 1$ pairs of $\mathcal{C}$-nonequivalent diagonal idempotents and hence exactly $2^{n-1}-1$ pairwise $\mathcal{C}$-nonequivalent maximal matrices that commute with A.

Let B be a maximal diagonal matrix which commutes with A. Let us determine the length of a chain from A to B. If

$$A = X_0 \prec \cdots \prec X_k = B$$

is an arbitrary chain from A to B, then each X_i commutes with A, so it is diagonal. Moreover, $X_i \prec X_{i+1}$, i.e., $\mathcal{C}(X_i) \subsetneq \mathcal{C}(X_{i+1})$ implies $X_{i+1} \in \mathbb{F}[X_i]$, see Lemma 2.1. Thus X_{i+1} cannot have more distinct eigenvalues than X_i. Also, if X_i and X_{i+1} would have the same number of distinct eigenvalues, then the repeated eigenvalues of X_i and X_{i+1} would be at the same positions implying $\mathcal{C}(X_{i+1}) = \mathcal{C}(X_i)$, a contradiction. Therefore in each chain from A to B the number of distinct eigenvalues is strictly decreasing. Since A has n distinct eigenvalues and B has 2 different eigenvalues, every chain has length at most $n-2$. Clearly, the chain with length $n-2$ does exist.

We will prove that (ii) implies (i) by assuming that non-derogatory matrix A is not diagonalizable and showing that (ii) is not true in that case. We can assume that $A = C(m_1^{\alpha_1}) \oplus \cdots \oplus C(m_k^{\alpha_k})$ is in its rational form, where $k \geq 1$, $\alpha_1 \geq \cdots \geq \alpha_k \geq 1$ are integers and $m_1, \ldots, m_k$ are relatively prime irreducible polynomials.

If $\alpha_1 \geq 4$ or $3 \geq \alpha_1 \geq \alpha_2 \geq 2$, then by Lemma 2.2 there exist at least 2^{n-1} pairwise $\mathcal{C}$-nonequivalent maximal matrices that commute with A, so (ii) does not hold. Thus it remains to consider the cases $\alpha_1 \leq 3$ and $\alpha_i = 1$ for $i \geq 2$. If $\alpha_1 \in \{2, 3\}$, then (ii) does not hold by Lemma 2.3. The case $k = 1$ and $\alpha_1 = 1$ contradicts (ii) by Lemma 2.5 since $\log_2 n - 1 < n - 2$ for $n \geq 3$. The case $k \geq 2$ and $\alpha_1 = 1$ contradicts (ii) by Lemma 2.6. □

Proof of main Theorem. We will prove the theorem in several steps.

1. We start by showing that Φ maps diagonal matrices bijectively onto diagonal ones. Let $D \in M_n(\mathbb{F})$ be a non-derogatory diagonal matrix. By [4, Proposition 2.3] non-derogatory matrices are exactly minimal ones, and since Φ preserves the partial order relation on equivalence classes we obtain that $\Phi(D)$ is also non-derogatory. It follows by Lemma 2.8 that $\Phi(D)$ is diagonalizable. By applying a suitable similarity transformation, we may assume that $\Phi(D)$ is already diagonal.

Since D and $\Phi(D)$ are diagonal non-derogatory matrices, each of them has distinct diagonal entries and therefore $\mathcal{C}(D)$ and $\mathcal{C}(\Phi(D))$ consist of diagonal matrices only. Hence Φ maps diagonal matrices bijectively onto diagonal ones.

2. We next show that Φ has the form (i) or (ii) on the scalar multiples of idempotents of rank one. Repeating the arguments on pages 73 and 74 in [5] we obtain that a diagonal idempotent of rank-one is mapped by Φ onto a diagonal idempotent of rank-one up to $\mathcal{C}$-equivalence. (The main idea is to compute the cardinality of the set

$$\mathcal{S}_P = \{\mathcal{C}(A) \mid \mathcal{C}(A) \subseteq \mathcal{C}(P), \quad A \text{ diagonal}\}$$

for various maximal diagonal matrices P. It is proved in [5, p. 73–74] that $|\mathcal{S}_P|$ is maximal if and only if P is $\mathcal{C}$-equivalent to a rank-one idempotent.) Using the standard arguments we obtain that idempotents of rank-one are mapped to idempotents of rank-one up to $\mathcal{C}$-equivalence.

Note that each $\mathcal{C}$-equivalence class contains at most one rank-one idempotent. Indeed by Lemma 2.1 two matrices A and B are $\mathcal{C}$-equivalent if and only if $\mathbb{F}[A] = \mathbb{F}[B]$ and clearly a subalgebra of $M_n(\mathbb{F})$, $n \geq 3$, generated by an idempotent of rank one contains only one idempotent of rank one. So Φ induces a bijective map on the set of idempotents of rank-one which preserves commutativity in both directions. Since two idempotents of rank-one commute if and only if they are either orthogonal or equal, the induced bijection preserves orthogonality among rank-one idempotents. By [12, Theorem 2.3] there exists an invertible T and a field isomorphism $\sigma : \mathbb{F} \to \mathbb{F}$ such that $\Phi(P) \sim TP^\sigma T^{-1}$ for every rank-one idempotent P or $\Phi(P) \sim T(P^\sigma)^{\mathrm{T}} T^{-1}$ for every rank-one idempotent P. Moreover, since Φ preserves $\mathcal{C}$-equivalence among matrices it also holds that $\Phi(\alpha P) \sim \alpha TP^\sigma T^{-1}$ for every rank-one idempotent P and every $\alpha \in \mathbb{F} \setminus \{0\}$ or $\Phi(\alpha P) \sim \alpha T(P^\sigma)^{\mathrm{T}} T^{-1}$ for every rank-one idempotent P and every $\alpha \in \mathbb{F}\setminus\{0\}$.

By appropriately modifying Φ we may assume that $\Phi(\alpha P) \sim \alpha P$ for every scalar multiple of rank-one idempotent P. By Lemma 2.1, $\Phi(\alpha P) = \gamma_\alpha \alpha P + \delta_\alpha I$, so we may further assume that Φ fixes every scalar multiple of rank-one idempotent P. The new map may not be bijective any longer, but it still preservers commutativity in both directions.

3. Action of Φ on the square-zero rank-one matrices. Let N be a square-zero rank-one matrix. Since it is similar to E_{12} we may assume without loss of generality that $N = E_{12}$. It is easy to see that the linear span of rank-one idempotents in $\mathcal{C}(E_{12})$ contains all matrices with zero entries in the upper left 2×2 block, and in the second row and the first column. Indeed, $E_{ji} = (E_{ji} +$

$E_{ii}) - E_{ii}$, $E_{ik} = (E_{ik} + E_{ii}) - E_{ii}$, where $i = 3, \ldots, n$, $j \neq 2$, $k \neq 1$, and $E_{ji} + E_{ii}, E_{ik} + E_{ii}, E_{ii} \in \mathcal{C}(E_{12})$. Note that Φ fixes rank-one idempotents. Hence $\Phi(E_{12})$ commutes with every matrix having zeros in the second row and in the first column. This is possible only if $\Phi(E_{12}) = \alpha E_{12} + \beta I$ for some α, β. Since Φ maps maximal matrices to maximal matrices, α must be non-zero, and appropriately modifying Φ again, we can assume $\Phi(E_{12}) = E_{12}$. So, Φ fixes each sqare-zero rank-one matrix.

4. Action of Φ on the diagonalizable matrices. Let $A = \sum_{i=1}^{n} \lambda_i E_{ii}$. Then A commutes with E_{ij} if and only if $\lambda_i = \lambda_j$. Since each E_{ij} is fixed, we see that $\Phi(A) \sim A$ for every diagonal matrix A, and by the standard arguments also for every diagonalizable matrix. By Lemma 2.1, $\Phi(A) \sim A$ if and only if $\mathbb{F}[\Phi(A)] = \mathbb{F}[A]$, and in particular, $\Phi(A)$ is a polynomial in A for every diagonalizable matrix A. $\square$

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GREGOR DOLINAR
FACULTY OF ELECTRICAL ENGINEERING
UNIVERSITY OF LJUBLJANA
TRŽAŠKA CESTA 25
SI-1000 LJUBLJANA
SLOVENIA

E-mail: gregor.dolinar@fe.uni-lj.si

BOJAN KUZMA
UNIVERSITY OF PRIMORSKA
GLAGOLJAŠKA 8
SI-6000 KOPER
SLOVENIA
AND
IMFM
JADRANSKA 19
SI-1000 LJUBLJANA
SLOVENIA

E-mail: bojan.kuzma@famnit.upr.si

ALEXANDER GUTERMAN
FACULTY OF ALGEBRA
DEPARTMENT OF MATHEMATICS
AND MECHANICS
MOSCOW STATE UNIVERSITY
GSP-1, 119991 MOSCOW
RUSSIA

E-mail: guterman@list.ru

POLONA OBLAK
FACULTY OF COMPUTER
AND INFORMATION SCIENCE
UNIVERSITY OF LJUBLJANA
TRŽAŠKA CESTA 25
SI-1000 LJUBLJANA
SLOVENIA

E-mail: polona.oblak@fri.uni-lj.si

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