Publ. Math. Debrecen
84/3-4 (2014), 451–461
DOI: 10.5486/PMD.2014.580

Strong version of the Stečkin–Lenski approximation theorem

By LÁSZLÓ LEINDLER (Szeged)

Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthdays

Abstract. Very recently W. Lenski [5] verified the pointwise analog of Stečkin's theorem on approximation of Fourier series by de la Vallée Poussin means. In [3] we extended the same Stečkin's theorem to strong approximation. In this paper we show that the pointwise expansion of our sharper result also holds.

1. Introduction

The fundamental Fejér and Lebesgue theorems have been extended countless different ways by various authors. E.g. results pertaining to the approximation rate are known as Bernstein theorems, regarding the strong summability we can mention the result due to Hardy and Littlewood [2], and nearly fifty years later G. Alexits [1] raised some analogous problems in connection with the strong approximation. Nikolskiĭ [6] and Stečkin [7] using the de la Vallée Poussin means, proved again theorems regarding ordinary approximation. They gave estimates for the order of approximation with the aid of the best approximation $E_n(f)$ given by trigonometric polynomials. In [7] Stečkin proved the most general result including or improving the previous ones. His result states:

If $f(x)$ is a continuous function, then

$$\|f(x) - \sigma_{n,m}(x)\| \leqq K \sum_{\nu=0}^{n} \frac{E_{n-m+\nu}(f)}{m+\nu+1} \quad (0 \leqq m \leqq n,\ n = 0, 1, \ldots),$$

Mathematics Subject Classification: 41A25, 41A45.
Key words and phrases: pointwise approximation, strong approximation.

where

$$\sigma_{n,m}(x) := \frac{1}{m+1}\sum_{\nu=n-m}^{n} s_\nu(x),$$

and $\|\cdot\|$ *denotes the maximum norm.*

In [3] we sharpened this result using strong means, that is, it is proved that

$$\Big\|\frac{1}{m+1}\sum_{\nu=n-m}^{n}|s_\nu(x)-f(x)|\Big\| \leqq K\sum_{\nu=0}^{n}\frac{E_{n-m+\nu}(f)}{m+\nu+1}$$

also holds.

Very recently Lenski [5] proved the pointwise analog of the Stečkin theorem.

The aim of the present paper is to prove the pointwise version of our theorem cited above.

2. Notions and notations

To establish Lenski's theorem and our one, we remember some notions and notations.

Let L^p $(1 \leqq p < \infty)$ [respect C] be the class of 2π-periodic real functions integrable with p-th power [continuous] over $Q = [-\pi, \pi]$, and let $X^p = L^p$ if $1 \leqq p < \infty$ and $X^p = C$ for $p = \infty$.

Let

$$\|f\| = \|f\|_{X^p} := \begin{cases} \left\{\int_Q |f(x)|^p dx\right\}^{1/p}, & \text{if } 1 \leqq p < \infty,\\ \sup\limits_{x\in Q}|f(x)|, & \text{if } p = \infty,\end{cases}$$

and

$$\|f\|_{x,\delta} = \|f\|_{X^p,x,\delta} := \begin{cases} \sup\limits_{0<h\leqq\delta}\left\{\frac{1}{2h}\int_{x-h}^{x+h}|f(t)|^p dt\right\}^{1/p}, & 0 \leqq p < \infty,\\ \sup\limits_{0<h\leqq\delta}\left\{\sup\limits_{0<|t|\leqq h}|f(x+t)|\right\}, & p = \infty.\end{cases}$$

If $f(x) \in X^p$ then let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \tag{2.1}$$

be its Fourier series. Denote by $s_n = s_n(x) = s_n(f,x)$ the n-th partial sum of

(2.1), and the de la Vallée Poussin sums by

$$\sigma_{n,m} := \sigma_{n,m}(x) := \frac{1}{m+1}\sum_{\nu=n-m}^{n} s_\nu(x) \quad (0 \leqq m \leqq n,\ n = 0, 1, 2, \ldots).$$

The strong de la Vallée Poussin means of differences are defined as follows:

$$V_{n,m} := V_{n,m}(x) := \frac{1}{m+1}\sum_{\nu=n-m}^{n} |s_\nu(x) - f(x)|.$$

It is clear that

$$V_{n,m}(x) \geqq |\sigma_{n,m}(x) - f(x)|.$$

Let $T := T(x)$ denote trigonometric polynomial of the degree at most n $(T \in H_n)$.

The pointwise best approximation of f are defined as follows:

$$E_n(x,\delta) := E_n(f,x,\delta) := E_n(f,x,\delta)_{X^p}$$

$$:= \begin{cases} \inf\limits_{T\in H_n}\left\{\sup\limits_{0<h\leqq\delta}\left(\dfrac{1}{2h}\displaystyle\int_{-h}^{h}|f(x+t)-T(x+t)|^p dt\right)^{1/p}\right\}, & 1 \leq p < \infty, \\ \inf\limits_{T\in H_n}\left\{\sup\limits_{0<h\leqq\delta}|f(x+h)-T(x+h)|\right\}, & p = \infty; \end{cases}$$

and their arithmetic means will be denoted by

$$F_{n,m} := F_{n,m}(x) := F_{n,m}(f,x)_{X^p} := \frac{1}{m+1}\sum_{k=0}^{m} E_n\left(x, \frac{\pi}{k+1}\right).$$

We note plus that

$$E_n(f,x,0) := |f(x) - T_n(x)|,$$

where $T_n(x)$ is the trigonometric polynomial of the degree at most n of the best approximation of f.

As usual, $\mathbb{N}_0$ denotes the set of nonnegative integers, and

$$D_n(t) := \frac{1}{2} + \sum_{\nu=1}^{n}\cos\nu t$$

the Dirichlet kernel of order n.

We shall also use the following notation: $L \ll R$ if there exists a positive constant K such that $L \ll KR$, but not necessarily the same K at each occurrence.

3. Theorems

First we recall Lenski's main theorem.

Theorem L. *If $f \in X^p$, then for any positive integers $m \leqq n$ and all real x*

$$|\sigma_{n,m}(x) - f(x)| \ll \sum_{\nu=0}^{n} \frac{F_{n-m+\nu,m}(x) + F_{n-m+\nu,\nu}(x)}{m+\nu+1} + E_{2n-m}(f,x,0) \qquad (3.1)$$

holds.

Analyzing the proof of Theorem 1 verified in a preceding paper of Lenski [4], it is easy to see that, implicitly, he proved the following inequality, too.

Corollary L. *If $f \in X^p$, then for any positive integers $m \leqq n$ and all real x*

$$V_{n,m}(x) \ll F_{n-m,m}(x)\left(1 + \ln\frac{n+1}{m+1}\right) + E_{n-m}(f,x,0).$$

In our proof we shall use this corollary.

Roughly speaking, our result is the statement, that the left hand side of (3.1) can be replaced by $V_{n,m}(x)$, as well.

More precisely, our result reads as follows:

Theorem. *If $f \in X^p$, then for any positive integers $m \leqq n$ and all real x*

$$V_{n,m}(x) \ll \sum_{\nu=0}^{n} \frac{F_{n-m+\nu,m}(x) + F_{n-m+\nu,\nu}(x)}{m+\nu+1} + E_{2n-m}(f,x,0)$$

holds true.

4. Lemmas

To prove our theorem we shall use two lemmas of [5] in unchanged form, and three of then, we modify to strong means.

Lemma 1. *If T_n is the trigonometric polynomial of the degree at most n of the best approximation of $f \in X^p$ with respect to the norm $\|\cdot\|_{X^p}$, then it is also the trigonometric polynomial of the degree at most n of the best approximation of $f \in X^p$ with respect to the norm $\|\cdot\|_{X^p,x,\delta}$ for any $\delta \in [0,\pi]$.*

Lemma 2. *If $n \in \mathbb{N}_0$ and $\delta > 0$ then $E_n(f,x,\delta)_{X^p}$ is nonincreasing function of n and nondecreasing function of δ. These imply that for $m,n \in \mathbb{N}$ the function $F_{n,m}(f,x)_{X^p}$ is nonincreasing function of n and m simultaneously.*

Lemma 3. *Let $m, n, q \in \mathbb{N}_0$ such that $m \leqq n$ and $q \geqq m+1$. If $f \in X^p$ then*

$$|V_{n+q,m}(x) - V_{n,m}(x)| \ll F_{n-m,m}(x) \sum_{\nu=0}^{q-1} \frac{1}{m+\nu+1}.$$

Proof. An easy consideration shows that with $V_{n,m} := V_{n,m}(x)$

$$(m+1)|V_{n+q,m} - V_{n,m}| = \left| \sum_{k=n+q-m}^{n+q} |s_k - f| - \sum_{k=n-m}^{n} |s_k - f| \right|$$
$$\leqq \sum_{k=n-m}^{n} \Big| |s_{k+q} - f| - |s_k - f| \Big| \leqq \sum_{k=n-m}^{n} |s_{k+q} - s_k|.$$

Let $T_\nu := T_\nu(f; x)$ denote the polynomial of best trigonometric approximation of order ν for f. Since the kernels $D_{k+q}(x) - D_k(x)$, for $k \geqq n-m$, are orthogonal to the trigonometric polynomial T_{n-m}, thus we have that

$$\sum_{k=n-m}^{n} |s_{k+q}(x) - s_k(x)|$$
$$= \sum_{k=n-m}^{n} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+t) - T_{n,m}(x+t)](D_{k+q}(t) - D_k(t))dt \right| =: \sum.$$

Lenski [5] in his Lemma 7 proved that

$$\sum \ll F_{n-m,m}(x) \sum_{\nu=0}^{q-1} \frac{1}{m+\nu+1},$$

herewith our Lemma 3 is also verified. □

Before formulating our fourth lemma we define a new difference. Let $m, n \in \mathbb{N}_0$ and $m \leqq n$. Denote

$$\tau_{n,m} := \tau_{n,m}(x) := \tau_{n,m}(f, x) := (m+1)(V_{n+m+1,m} - V_{n,m}).$$

Lemma 4. *Let $m, n, \mu \in \mathbb{N}_0$ such that $2\mu \leqq m \leqq n$. If $f \in X^p$ then*

$$|\tau_{n,m}(x) - \tau_{n-\mu,m-\mu}(x)| \ll \mu F_{n-\mu+1,\mu-1}(x) \ln \frac{m}{\mu}.$$

Proof. An easy consideration gives the following equality:

$$\tau_{n,m} - \tau_{n-\mu,m-\mu} = \left(\sum_{k=n+m-2\mu+2}^{n+m+1} -2 \sum_{k=n-\mu+1}^{n} \right) |s_k - f|.$$

Next we use Lemma 3 to estimate the following differences:

$$\left| \left(\sum_{k=n+m-2\mu+2}^{n+m-\mu+1} - \sum_{k=n-\mu+1}^{n} \right) |s_k - f| \right| = \mu |V_{n+m-\mu+1,\mu-1} - V_{n,\mu-1}| \\ \ll \mu F_{n-\mu+1,\mu-1}(x) \ln \frac{m}{\mu}$$

and

$$\left| \left(\sum_{k=n+m-\mu+2}^{n+m+1} - \sum_{k=n-\mu+1}^{n} \right) |s_k - f| \right| = \mu |V_{n+m+1,\mu-1} - V_{n,\mu-1}| \\ \ll \mu F_{n-\mu+1,\mu-1}(x) \ln \frac{m}{\mu},$$

herewith Lemma 4 is proved. □

Lemma 5. *Let $m, n \in \mathbb{N}_0$ and $m \leqq n$. If $f \in X^p$ then*

$$|\tau_{n,m}(x)| \ll \sum_{k=n-m}^{n} F_{k,k-n+m}(x). \tag{4.1}$$

Proof. We may assume that $m \geqq 2$, otherwise (4.1) is trivial. Next we present the sequence $\{m_s\}$ constructed by Stečkin [7]. Let

$$m_0 = m, \ m_s := m_{s-1} - [m_{s-1}/2] \qquad (s = 1, 2, \ldots),$$

where $[y]$ denotes the integral part of y. Clearly there exists an index $t \geqq 1$ such that

$$m = m_0 > m_1 > \ldots > m_t = 1.$$

By the definition of m_s we have

$$m_s \geqq m_{s-1}/2,$$

$$m_{s-1} - m_s = [m_{s-1}/2] \geqq m_{s-1}/3 \quad (s = 1, 2, \ldots, t), \tag{4.2}$$

thus

$$m_{t-1} = 2 \quad \text{and} \quad m_{t-1} - m_t = 1, \tag{4.3}$$

furthermore

$$m_{s-1} - m_s \leqq m_s \leqq 3(m_s - m_{s+1}) \quad (s = 1, \dots, t-1). \tag{4.4}$$

Using the equality

$$\tau_{n,m}(x) = \sum_{s=1}^{t} \{\tau_{n-m+m_{s-1},m_{s-1}} - \tau_{n-m+m_s,m_s}\} + \tau_{n-m+m_t,m_t},$$

and that $m_t = 1$ we get

$$|\tau_{n,m}(x)| \leqq \sum_{s=1}^{t} |\tau_{n-m+m_{s-1},m_{s-1}} - \tau_{n-m+m_s,m_s}| + |\tau_{n-m+1,1}|. \tag{4.5}$$

To estimate the terms in the sum $\sum_{s=1}^{t}$ in (4.5) we use Lemma 4 with $\mu = m_{s-1} - m_s$ and $m = m_{s-1}$, whence

$$\begin{aligned} &|\tau_{n-m+m_{s-1},m_{s-1}} - \tau_{n-m+m_s,m_s}| \\ &\qquad \ll (m_{s-1} - m_s) F_{n-m+m_s+1,m_{s-1}-m_s} \ln \frac{m_{s-1}}{m_{s-1} - m_s}, \quad (s \leqq t-1) \end{aligned} \tag{4.6}$$

follows, furthermore by Lemma 5,

$$|\tau_{n-m+i,i}| \ll F_{n-m,i}, \quad i = 1, 2, \tag{4.7}$$

holds.

In virtue of (4.2)–(4.7) we obtain that

$$|\tau_{n,m}(x)| \ll \left\{ \sum_{s=1}^{t-1} (m_s - m_{s+1}) F_{n-m+m_s+1,m_s} + F_{n-m+2,m-2} + F_{n-m,1} \right\},$$

whence, due to the monotonicity of $F_{n,m}$,

$$\begin{aligned} |\tau_{n,m}(x)| &\ll \left\{ \sum_{s=1}^{t-1} \sum_{\nu=m_{s+1}+1}^{m_s} F_{n-m+\nu+1,\nu} + F_{n-m+2,m-2} + F_{n-m,1} \right\} \\ &\ll \sum_{k=n-m}^{n} F_{k,k-n+m}, \end{aligned}$$

which proves Lemma 5. $\square$

5. Proof of Theorem

The proof is a unified version used in the following three papers [7], [3] and [5]. Let $n > 0$ and $0 \leqq m \leqq n$ be fixed, and following Stečkin's idea, construct an increasing sequence $\{n_s\}$ $(s = 0, 1, \ldots, t)$ in the following way. Set $n_0 = n$. If $n_0, \ldots, n_s$ are already defined and $n_s < 2n$, we define n_{s+1} as follows in (5.2): Let $\nu_s = \nu_s(x)$ denote the smallest natural number such that

$$F_{n-m+\nu_s,\nu}(x) \leqq \frac{1}{2} F_{n_s-m,\nu}(x) \quad (\nu = 0, 1, \ldots, n). \tag{5.1}$$

According to the magnitude of ν_s let

$$n_{s+1} := \begin{cases} n_s + m + 1, & \text{if } \ \nu_s \leqq m, \\ n_s + \nu_s, & \text{if } \ m + 1 \leqq \nu_s < 2n + m - n_s, \\ 2n + m, & \text{if } \ \nu_s \geqq 2n + m - n_s. \end{cases} \tag{5.2}$$

In $n_{s+1} < 2n$, the we continue the procedure, and if $n_{s+1} \geqq 2n$, the we stop, and define $t := s + 1$.

This sequence $\{n_s\}$ has the following properties:

$$t \geqq 1, \ n = n_0 < n_1 < \ldots < n_t, \quad 2n \leqq n_t \leqq 2n + m,$$

and

$$n_{s+1} - n_s \geqq m + 1, \quad (s = 0, 1, \ldots, t - 1). \tag{5.3}$$

By (5.1) and (5.3) we also have the relations

$$F_{n_{s+1}-m,\nu}(x) \leqq \frac{1}{2} F_{n_s-m,\nu}(x), \quad \text{if} \quad s \leqq t - 2; \tag{5.4}$$

furthermore if $n_{s+1} - n_s > m + 1$ and $s \leqq t - 1$, then

$$\frac{1}{2} F_{n_s-m,\nu}(x) \leqq F_{n_{s+1}-m-1,\nu}(x). \tag{5.5}$$

Namely, if $n_t = n_{s+1} = 2n + m$, then $\nu_s \geqq 2n + m - n_s$, or equivalently, $\nu_s - m + \nu_s - 1 \geqq 2n - 1 = n_{s+1} - m - 1$, thus, by (5.1), $F_{n_{s+1}-m-1}(x) \geqq F_{n_s-m+(\nu_s-1),\nu} > \frac{1}{2} F_{n_s-m,\nu}$ clearly holds.

Now, by means of this sequence $\{n_s\}$ we can write

$$V_{n,m} = \sum_{s=0}^{t-1} \{V_{n_s,m} - V_{n_{s+1},m}\} + V_{n_t,m}. \tag{5.6}$$

First we estimate the extra term. Using Corollary L and that $2n \leqq n_t \leqq 2n+m$ we get

$$\begin{aligned}|V_{n_t,m}(x)| &\ll F_{n_t-m,m}(x)\left(1+\ln\frac{n_t+1}{m+1}\right)+E_{n_t-m}(x,0)\\ &\ll F_{2n-m,n}(x)\ln\frac{n+m}{m+1}+E_{2n-m}(x,0)\\ &\ll \sum_{\nu=0}^{n}\frac{F_{n-m+\nu,m}(x)}{m+\nu+1}+E_{2n-m}(x,0). \end{aligned} \tag{5.7}$$

Newt we estimate the terms of the sum. If $n_{s+1}-n_s=m+1$, then

$$V_{n_{s+1},m}(x)-V_{n_s,m}(x)=\frac{1}{m+1}\tau_{n_s,m}(x).$$

Applying Lemma 5 we obtain that

$$\begin{aligned}|V_{n_{s+1},m}(x)-V_{n_s,m}(x)| &\ll \frac{1}{m+1}\sum_{k=n_s-m}^{n_s}F_{k,k-n_s+m}(x)\\ &\ll \sum_{\nu=0}^{n_{s+1}-n_s-1}\frac{F_{n_s-m+\nu,\nu}(x)}{m+\nu+1}. \end{aligned} \tag{5.8}$$

If $n_{s+1}-n_s>m+1$, then we use Lemma 3 and the inequality (5.5), and arrive at the inequalities:

$$\begin{aligned}|V_{n_{s+1},m}(x)-V_{n_s,m}(x)| &\ll F_{n_s-m,m}(x)\sum_{\nu=0}^{n_{s+1}-n_s-1}\frac{1}{m+\nu+1}\\ \ll F_{n_{s+1}-m-1,m}(x)\sum_{\nu=0}^{n_{s+1}-n_s-1}\frac{1}{m+\nu+1} &\ll \sum_{\nu=0}^{n_{s+1}-n_s-1}\frac{F_{n_s-m+\nu,m}(x)}{m+\nu+1}. \end{aligned} \tag{5.9}$$

The inequalities (5.8) and (5.9) give

$$|V_{n_s,m}(x)-V_{n_{s+1},m}(x)| \ll \sum_{\nu=0}^{n_{s+1}-n_s-1}\frac{F_{n_s-m+\nu,m}(x)+F_{n_s-m+\nu,\nu}(x)}{m+\nu+1} \tag{5.10}$$

for any $s=0,1,\dots,t-1$.

Applying (5.10) we get

$$\sum\nolimits_1^* := \sum_{s=0}^{t-1} |V_{n_s,m}(x) - V_{n_{s+1},m}(x)|$$
$$\ll \sum_{s=0}^{t-1} \sum_{\nu=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+\nu,m}(x) + F_{n_s-m+\nu,\nu}(x)}{m+\nu+1} =: \sum\nolimits_2^* . \tag{5.11}$$

Since $n_{s+1} - n_s \leqq 2n + m - n - 1 = n + m - 1$ for all $s \leqq t-1$, and if we change the order of summation in $\sum_2^*$, then we get the inequality

$$\sum\nolimits_2^* \leqq \sum_{\nu=0}^{n+m-1} \frac{1}{m+\nu+1} \sum_{s:n_{s+1}-n_s>\nu} (F_{n_s-m+\nu,m}(x) + F_{n_s-m+\nu,\nu}(x)). \tag{5.12}$$

The non-void inner sums of (5.12) can be estimated as follows: Let p denote the smallest index s having the property $n_{s+1} - n_s > \nu$. Then

$$\begin{aligned}\sum_{s:n_{s+1}-n_s>\nu} &(F_{n_s-m+\nu,m}(x) + F_{n_s-m+\nu,\nu}(x)) = F_{n_p-m+\nu,m}(x) + F_{n_p-m+\nu,\nu}(x)\\ &+ \sum_{s\geqq p+1:n_{s+1}-n+s>\nu} [F_{n_s-m+\nu,m}(x) + F_{n_s-m+\nu,\nu}(x)]\\ &\leqq F_{n_p-m+\nu,m}(x) + F_{n_p-m+\nu,\nu}(x)\\ &+ \sum_{s:s\geqq p+1} [F_{n_s-m+\nu,m}(x) + F_{n_s-m+\nu,\nu}(x)] =: \sum\nolimits_3^* .\end{aligned} \tag{5.13}$$

Now using the inequalities (5.4) and that $F_{n,m}(x)$ are nonincreasing, we get

$$\sum\nolimits_3^* \leqq 3[F_{n_p-m+\nu,m}(x) + F_{n_p-m+\nu,\nu}(x)]. \tag{5.14}$$

Finally, collecting our partial results, (5.6), (5.7) and (5.10)–(5.14), we arrive at (5.1), and hereby our theorem is proved.

References

[1] G. Alexits, Sur les bornes de la théorie de l'approximation des fonctions continues par polynomes, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), 329–340.

[2] G. H. Hardy and J. E. Littlewood, Sur la série de Fourier d'une fonction à carré sommable, *Comptes Rendus, Paris* **156** (1913), 1307–1309.

[3] L. Leindler, Sharpening of Stečkin's theorem to strong approximation, *Anal. Math.* **16** (1990), 27–38.

[4] W. Lenski, Pointwise best approximation by de la Vallée Poussin means, *East Journal on Approximation* **14** (2008), 131–136.

[5] W. Lenski, Pointwise analog of the Stečkin approximation theorem, *J. of Mathematics* **2013** (2013), Art. ID426347, 8 pades, `http://dx.dvi.org/10.1155/2013/426347`.

[6] S.M. Nikol'skiĭ, On some approximation methods by trigonometric sums, *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **4** (1940), 509–520 (in *Russian*).

[7] S. B. Stečkin, On the approximation of periodic functions by de la Vallée Poussin sums, *Anal. Math.* **4** (1978), 61–74.

LÁSZLÓ LEINDLER
BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED
HUNGARY

E-mail: leindler@math.u-szeged.hu

(Received March 12, 2013)