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Conformal vector fields on a locally projectively flat Randers manifold

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Abstract. We study and characterize conformal vector fields on a Randers manifold of projectively isotropic flag curvature. In particular, we prove that any conformal vector field on a non-Riemannian locally projectively flat Randers manifold of dimension $n \geq 3$ must be homothetic and completely determine conformal vector fields on a locally projectively flat Randers manifold.

1. Introduction

It is well known that there are a vast amount of literatures on the conformal (resp. homothetic or killing) vector fields on a Riemannian manifold M. They are closely related with the conformal (resp. homothetic or isometric) transformation group on M, which plays a very important role in geometry and physics.

Conformal (resp. homothetic or killing) vector fields on a Riemannian manifold have been successfully applied to the study of non-Riemannian Finsler manifolds. For example, a Randers metric of weakly isotropic (resp. constant) flag curvature can be generated by a Riemannian metric h of isotropic (resp. constant) sectional curvature K and its conformal (resp. homothetic) vector fields.

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This leads to the classification of Randers metrics of weakly isotropic (resp. constant) flag curvature ([BRS], [SY]). The key idea is to express a Randers metric $F = \alpha + \beta$ in terms of a Riemannian metric $h = \sqrt{h_{ij}(x)y^i y^j}$ and a vector field $W = W^i \frac{\partial}{\partial x^i}$ by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \quad W_0 := W_i y^i, \tag{1.1}$$

where $W_i := h_{ij} W^j$ and $\lambda := 1 - \|W_x\|_h^2$. (h, W) is called *the navigation data* of F. However, this idea can not be used to classify Randers metric of scalar flag curvature. In fact, it is an open problem to classify Randers metric of scalar flag curvature up to now.

In [Sh], the first author in the present paper discussed the navigation problem in a more general setting. It is shown that the shortest time paths on a Finsler manifold (M, F) under the influence of a force field V with $F(x, -V_x) < 1$ are just the geodesics of the new Finsler metric $\widetilde{F} = \widetilde{F}(x, y)$ defined by the following equation:

$$F\left(x, \frac{y}{\widetilde{F}(x,y)} - V_x\right) = 1, \quad y \in T_x M. \tag{1.2}$$

Note that if $F = h$ is a Riemannian metric h and $V = W$ is a vector field W on M, then the new metric $\tilde{F}$ defined by (1.2) is a Randers metric expressed in (1.1). If F is a Randers metric expressed by (h, W) in (1.1) and V is a vector field on M, then the new metric $\tilde{F}$ defined by (1.2) is still a Randers metric expressed by (1.1) with W replaced by $W + V$ (cf. [ShX1]). X. Mo and L. Huang have established the relation between the flag curvature of a given Finsler metric F and that of the new Finsler metric $\tilde{F}$ generated by (F, V) if V is a conformal (resp. homothetic) vector field on a Finsler manifold (M, F) (cf. [MH1], [MH2]). To have a better understanding on Finsler metrics with some flag curvature properties, it is very important to study the conformal (resp. homothetic or killing) vector fields on a Finsler manifold.

Recently, some progress has been made in the study of the conformal (resp. homothetic or killing) vector fields on a Finsler manifold (cf. [JB], [Kang] etc.). In particular, we have completely determined all conformal vector fields on a Randers manifold of weakly isotropic flag curvature and constructed a new class of Randers metrics of scalar flag curvature (cf. [ShX1], [ShX2]). For a Randers metric $F = \alpha + \beta$, if there is a closed 1-form η such that $\bar{F} := \alpha + \bar{\beta}(\bar{\beta} := \beta - \eta)$ is of weakly isotropic flag curvature, then $F = \alpha + \beta$ is projectively equivalent to $\bar{F} = \alpha + \bar{\beta}$ and hence it is of scalar flag curvature. Randers metrics $F = \alpha + \beta$ with such property are said to be *of projectively isotropic flag curvature*. Obviously, every locally projectively flat Randers metric or every Randers metric of weakly isotropic flag curvature is of projectively isotropic flag curvature and consequently of scalar flag curvature.

In this paper, we shall study and characterize conformal vector fields on a Randers manifold of projectively isotropic flag curvature. In particular, we completely determine all conformal fields on a locally projectively flat Randers manifold.

Theorem 1.1. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M $(n \geq 3)$ such that $\bar{F} := \alpha + \bar{\beta}$ is a Randers metric of weakly isotropic flag curvature and $\eta := \beta - \bar{\beta}$ is closed. Let $(\bar{h}, \bar{W})$ be the navigation data of $\bar{F}$ and V be a conformal vector field on (M, F) with conformal factor $c(x)$. Assume $\eta = \eta_i y^i \neq 0$ and V satisfies*

$$V^j \eta_{i;j} + \eta^j V_{j;i} = 2c\eta_i, \tag{1.3}$$

where ";" is the covariant derivative with respect to the Levi–Civita connection of $\bar{h}$. Then V must be homothetic with respect to F.

Under the restriction (1.3), V is a conformal vector field on (M, F) with the conformal factor $c(x)$ if and only if V is a conformal vector field on $(M, \bar{F})$ with the same conformal factor $c(x)$, which is regarded as the geometric meaning of the equation (1.3) (see Lemma 3.1). Theorem 1.1 shows that V must be homothetic with respect to F and $\bar{F}$ respectively in this case. Thus, one can determine the homothetic vector fields on (M, F) from those on $(M, \bar{F})$. Furthermore, in Theorem 1.1, since $\bar{F}$ is of weakly isotropic flag curvature and $\dim M \geq 3$, at any point, there is a local coordinate system $(U, (x^i))$, in which $\bar{h}$ is of constant sectional curvature μ and $\bar{W}$ is a conformal vector field with conformal factor $\sigma(x)$ with respect to $\bar{h}$ according to [SY]. Explicitly, $\bar{h}$ and $\bar{W}$ are respectively expressed by:

$$\bar{h} = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1 + \mu|x|^2}, \quad y \in T_x R^n \tag{1.4}$$

$$\begin{aligned} \bar{W} = -2\left\{\left(\delta\sqrt{1 + \mu|x|^2} + \langle v, x\rangle\right)x - \frac{|x|^2 v}{1 + \sqrt{1 + \mu|x|^2}}\right\} \\ + xQ + d + \mu\langle d, x\rangle x, \end{aligned} \tag{1.5}$$

where δ is a constant, Q is a skew symmetric matrix independent of $x \in R^n$ and $v, d \in R^n$ are constant vectors (cf. [SX]). In this case,

$$\sigma = \frac{\delta + \langle v, x\rangle}{\sqrt{1 + \mu|x|^2}}. \tag{1.6}$$

Thus we have

Theorem 1.2. *Let F and $\bar{F}$ be as in Theorem 1.1. Assume $(\bar{h}, \bar{W})$ is the navigation data of $\bar{F}$ given by* (1.4)–(1.5). *Let V be a vector field on R^n given by one of the following*

(i) $V = xQ$, *where Q and v are those in* (1.5) *with $vQ = 0$ $(v \neq 0)$;*

(ii) $V = 2\epsilon\sqrt{1+\mu|x|^2}\, x + xQ + \mu\langle x, d\rangle x + d$, *where ϵ is a constant with $\epsilon\mu = \epsilon\delta = 0$, and Q and d are those in* (1.5) *with $\delta d = \epsilon Q = 0$.*

If there is a function $f = f(x)$ on M such that $\eta = df \neq 0$, which satisfies

$$V^i f_{x^i} - 2\epsilon f = k, \tag{1.7}$$

where k is a constant, then V is a homothetic vector field of F with dilation ϵ. Conversely, if V is a homothetic vector field of F with dilation ϵ and $\eta = df \neq 0$ satisfying (1.7), *then V must be given by* (i) *or* (ii) *above.*

In particular, if $\bar{\beta} = 0$ in Theorem 1.1, then $\eta = \beta$ is closed and $\bar{F} = \alpha$ is of isotropic sectional curvature (=constant if $\dim M \geq 3$) from Theorem 1.2 in [SY]. Thus $F = \alpha + \eta$ is projectively flat. Since $\bar{F} = \bar{h} = \alpha$ is a Riemannian metric and V is conformal with respect to F, (1.3) holds by (2.4) in §2 and V is conformal with respect to α by Lemma 3.1. Consequently, V can be expressed in the form (1.5). On the other hand, by Theorem 1.1 and Lemma 3.1, V is homothetic with respect to α. Thus, $v = 0$ and $\delta\mu = 0$ by (1.6). Notice that the conformal factor $c(x)$ in this paper is minus one times the conformal factor in [SX] and [SY]. One obtains the following

Corollary 1.1. *Let $F = \alpha + \beta (\beta \neq 0)$ be a locally projectively flat Randers metric on an n-dimensional manifold M. Suppose V is a conformal vector field on (M, F) and $\dim M \geq 3$. Then V must be homothetic. In this case,*

$$V = 2\delta\sqrt{1+\mu|x|^2}\, x + xQ + \mu\langle d, x\rangle x + d,$$

where δ, μ are constants with $\delta\mu = 0$, Q is a constant skew symmetric matrix and d is a constant vector in R^n.

Consider a special Randers metric $F = \alpha + \beta$ on R^n

$$\alpha = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2)}}{1+\mu|x|^2}, \tag{1.8}$$

$$\beta = df \neq 0, \tag{1.9}$$

where $f = f(x)$ is a scalar function. F is projectively flat. Let

$$\Phi := f_{i|j} y^i y^j, \quad \Psi := f_{i|j|k} y^i y^j y^k. \tag{1.10}$$

where $f_{i|j}$ and $f_{i|j|k}$ are the coefficients of the covariant derivatives of f with respect to α. Then the flag curvature of F is given by

$$\mathbf{K}_F(x, y) = \frac{\mu\alpha^2}{F^2} - \frac{\Psi}{2F^3} + \frac{3\Phi^2}{4F^4} \tag{1.11}$$

(cf. see (6.10) in [CS]). Let V be a conformal vector field on (M, F) with conformal factor c. Then, V is homothetic and is given by (1.1). By [MH1], $\tilde{F}$ generated from (F, V) by solving (1.2) is of scalar flag curvature $K_{\tilde{F}}(x, y) = K_F(x, y - \tilde{F}V) - c^2$. Thus one obtains a series of Randers metric of scalar flag curvature. In general, such $\tilde{F}$ is not locally projectively flat, not of weakly isotropic flag curvature and not the metric constructed by Chen–Zhao ([CZ]). To see this, we consider a more special case.

Example 1.1. Let $\alpha = |y|$, $f = |x|^2/2$, and $V = xQ$ with $|xQ| < 1$, where Q is a skew symmetric matrix. Then (1.7) holds for $c = 0$ and $k = 0$ and V is a Killing vector field with respect to $F = \alpha + df$. By solving (1.2), we obtain a new Randers metric $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are given by

$$\tilde{\alpha} = \frac{\sqrt{(1 - |xQ|^2)(|y|^2 - \langle x, y\rangle^2) + (\langle xQ, y\rangle - \langle x, y\rangle)^2}}{1 - |xQ|^2} \tag{1.12}$$

$$\tilde{\beta} = -\frac{\langle xQ, y\rangle - \langle x, y\rangle}{1 - |xQ|^2}. \tag{1.13}$$

$\tilde{F}$ is of scalar flag curvature with

$$\mathbf{K}_{\tilde{F}} = \frac{3}{4}\left(\frac{|y - \tilde{F}V|}{|y - \tilde{F}V| + \langle x, y - \tilde{F}V\rangle}\right)^4. \tag{1.14}$$

From (1.13), one can see that $\tilde{\beta}$ is not closed. Thus $\tilde{F}$ is not locally projectively flat. From (1.14), one can see that $\tilde{F}$ is not of weakly isotropic flag curvature. Note that the metrics constructed by Chen–Zhao have the following special properties: α is of constant Ricci curvature (Einstein). Since $\tilde{\alpha}$ is not Einstein, $\tilde{F}$ is not the metric constructed by Chen–Zhao.

2. Preliminaries

In this section, we shall review the navigation problem and some results on conformal vector fields on a Randers manifold $(M, F=\alpha+\beta)$.

Consider a Riemannian metric $h=\sqrt{h_{ij}y^iy^j}$ and a vector field $W=W^i\frac{\partial}{\partial x^i}$ on a manifold M. The Zermelo navigation problem is to determine the shortest-time paths on (M,h) for an object driven by a constant interval force and under the influence of the external force field W (cf. [Zer]). It is shown that the shortest-time paths are the geodesics of the Randers metric $F=F(x,y)$ defined by the following equation:

$$h\left(x, \frac{y}{F(x,y)}-V_x\right)=1, \quad y\in T_x(M) \tag{2.1}$$

Solving (2.1), one obtains a Randers metric $F=\alpha+\beta$ as (1.1). In fact, every Randers metric $F=\alpha+\beta$ on a manifold M can be expressed in terms of a Riemannian metric $h=\sqrt{h_{ij}(x)y^iy^j}$ and a vector field $W=W^i(x)\frac{\partial}{\partial x^i}$ with $\|W_x\|_h<1$ by (1.1) (cf. [BRS]). We call (h,W) *a navigation data* of F. More generally, given a Finsler metric F and a vector field V with $F(x,-V)<1$ on a manifold M, one obtains a new Finsler metric $\tilde{F}$, which is defined by (1.2). See [Sh] in detail.

A vector field V on a Finsler manifold (M,F) is called a *conformal vector field* with a conformal factor $c=c(x)$ if the 1-parameter transformation φ_t generated by V is a conformal transformation on (M,F). In local coordinates, conformal vector fields V are characterized by

$$V_{i,j}+V_{j,i}+2C^p_{ij}V_{p,q}y^q=4cg_{ij}, \tag{2.2}$$

where C_{ijp} are the coefficient of Cartan torsion C of F, $C^p_{ij}=g^{pq}C_{ijq}$, $V_i=g_{ij}V^j$ and "," is the horizontal covariant derivative with respect to the Chern connection of F. See [ShX1]. When $F=\alpha+\beta$ be a Randers metric, conformal vector fields V are characterized by

$$V_{i|j}+V_{j|i}=4ca_{ij}; \tag{2.3}$$

$$V^jb_{i|j}+b^jV_{j|i}=2cb_i \tag{2.4}$$

where we use the Riemannian metric tensor a_{ij} to raise and lower the indices of V or b and "|" is the covariant derivative with respect to α ([Kang]). We can also express $F=\alpha+\beta$ in terms of the navigation data (h,W) by (2.1). It has been shown that V is conformal with respect to F if and only if

$$V_{i;j}+V_{j;i}=4ch_{ij} \tag{2.5}$$

$$V^jW_{i;j}+W^jV_{j;i}=2cW_i \tag{2.6}$$

where we use the Riemannian metric tensor h_{ij} to raise and lower the indices of V or W and ";" is the covariant derivative with respect to h ([ShX1]).

Consider a Randers metric $F = \alpha + \beta$ with the navigation data (h, W). If F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \zeta$, where θ is a 1-form on M and $\zeta = \zeta(x)$ is a scalar function on M, then at any point, there is a local coordinate system $(U, (x^i))$, in which h is of isotropic sectional curvature $\mu(x)(\mu =$ constant when $n \geq 3)$ and W is conformal with conformal factor $\sigma(x)$ with respect to h according to [SY]. If $\dim M \geq 3$, we can express h in the following projective form:

$$h = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1+\mu|x|^2}, \quad y \in T_x R^n \tag{2.7}$$

and W is given by the following

$$\begin{aligned} W = -2\left\{\left(\delta\sqrt{1+\mu|x|^2} + \langle v, x\rangle\right)x - \frac{|x|^2 v}{1+\sqrt{1+\mu|x|^2}}\right\} \\ + xQ + d + \mu\langle d, x\rangle x, \end{aligned} \tag{2.8}$$

where δ and μ are constants, $Q = (q^i{}_j)$ is a skew symmetric matrix independent of x and $v, d \in R^n$ are constant vectors (cf. [SX]). In this case,

$$\sigma = \frac{\delta + \langle v, x\rangle}{\sqrt{1+\mu|x|^2}}. \tag{2.9}$$

We call the above expressions of h and W *the local standard expression* of F. With this, we can determine all conformal vector fields V with conformal factor $c(x)$ on a Randers manifold (M, F) of weakly isotropic flag curvature when $\dim M \geq 3$ ([ShX1]). In fact, in the same local coordinates for the local standard expression of F, V is given by one of the following

(a) $V = xQ$, where Q and v are those in (2.8) with $vQ = 0$ $(v \neq 0)$. In this case, $c = 0$.

(b)
$$V = 2\left(\epsilon\sqrt{1+\mu|x|^2} + \langle a, x\rangle\right)x - \frac{2|x|^2 a}{1+\sqrt{1+\mu|x|^2}}, \tag{2.10}$$

where ϵ is a constant and a is a nonzero constant vector in R^n. In this case, $c = \frac{\epsilon+\langle a,x\rangle}{\sqrt{1+\mu|x|^2}}$.

(c)
$$V = 2\epsilon\sqrt{1+\mu|x|^2}x + xQ + d + \mu\langle x, d\rangle x, \tag{2.11}$$

where ϵ is a constant with $\delta\epsilon = 0$, Q and d are those in (2.8) with $\delta d = \mu\epsilon d = \epsilon Q = 0$. In this case, $c = \frac{\epsilon}{\sqrt{1+\mu|x|^2}}$.

3. Proofs of Theorem 1.1 and 1.2

In this section, we will study conformal vector fields on a Randers manifold of projectively isotropic flag curvature and prove Theorem 1.1 and 1.2. First, we need the following Lemmas.

Lemma 3.1 ([ShX2])**.** *Let $\bar{F} = \alpha + \bar{\beta}$ and $F = \alpha + \beta$ be Randers metrics on a manifold M. Let $\eta := \beta - \bar{\beta}$ and $(\bar{h}, \bar{W})$ be the navigation data for $\bar{F}$. Assume V is a vector field on M. Then each two of the following imply the third one.*

(1) *V is a conformal vector field on (M, F) with conformal factor $c(x)$;*

(2) *V is a conformal vector field on $(M, \bar{F})$ with conformal factor $c(x)$;*

(3) *$\eta = (\eta_i)$ satisfies*

$$V^i \eta_{j;i} + \eta^i V_{i;j} = 2c\eta_j, \tag{3.1}$$

where we use $\bar{h}_{ij}$ to raise and lower the indices of V, and η, and ";" is the covariant derivative with respect to the Levi–Civita connection of $\bar{h}$.

Lemma 3.2. *Let $\bar{F} = \alpha + \bar{\beta}$ be a Randers metric on an n-dimensional manifold M with the navigation data $(\bar{h}, \bar{W})$. Suppose that $\eta = \eta_i y^i$ is a closed 1-form on M and $V = (V^i)$ is a vector field on M satisfying (3.1). Then either $c(x) =$ constant or $\eta = \nu c_0$, where $\nu = \nu(x)$ is a scalar function on M with $\nu_j c_i = \nu_i c_j$ and $c_0 = c_i y^i$ is a 1-form on M, here $c_i := c_{x^i}$ and $\nu_i := \nu_{x^i}$.*

Proof. By assumption and (3.1), we have $\eta_{i;j} = \eta_{j;i}$ and $V = (V^i)$ satisfies

$$V^j \eta_{j;i} + \eta^j V_{j;i} = 2c\eta_i. \tag{3.2}$$

Taking the covariant derivative on the both sides of (3.2), we get

$$V^j{}_{;k}\eta_{j;i} + V^j \eta_{j;i;k} + \eta^j{}_{;k} V_{j;i} + \eta^j V_{j;i;k} = 2c\eta_{i;k} + 2c_k \eta_i. \tag{3.3}$$

Exchanging the indices i, k in (3.3) yields

$$V^j{}_{;i}\eta_{j;k} + V^j \eta_{j;k;i} + \eta^j{}_{;i} V_{j;k} + \eta^j V_{j;k;i} = 2c\eta_{k;i} + 2c_i \eta_k. \tag{3.4}$$

Observe that

$$V^j \eta_l \bar{R}_j{}^l{}_{ik} + V_l \eta^j \bar{R}_j{}^l{}_{ik} = V^j \eta^l \bar{R}_{jlik} + V^l \eta^j \bar{R}_{jlik} = 0, \tag{3.5}$$

where $\bar{R}_{jilk}$ is a Riemannian curvature tensor of $\bar{h}$. Subtracting (3.3) from (3.4) yields

$$c_k \eta_i = c_i \eta_k. \tag{3.6}$$

Here we have used the Ricci identity and (3.5).

Assume that $dc \neq 0$. It follows from (3.6) that there is a scalar function $\nu = \nu(x)$ such that

$$\eta_i = \nu c_i.$$

Further, $d\eta = 0$ implies that

$$\nu_j c_i = \nu_i c_j. \qquad \square$$

PROOF OF THEOREM 1.1. We prove the theorem by contradiction. Assume that V is a conformal vector field on (M, F) with a non-constant conformal factor $c = c(x)$. By Lemma 3.1, V is also a conformal vector field $(M, \bar{F})$ with the same conformal factor $c = c(x)$.

Since $\bar{F} = \alpha + \bar{\beta}$ is of weakly isotropic flag curvature and $n \geq 3$, at any point, there is a local coordinate system $(U, (x^i))$, in which $\bar{h} = \sqrt{h_{ij} y^i y^j}$ and $\bar{W} = \bar{W}^i \frac{\partial}{\partial x^i}$ are given by (1.4)–(1.5). We have

$$\bar{h}_{ij} = \frac{\delta_{ij}}{1 + \mu|x|^2} - \frac{\mu x^i x^j}{(1 + \mu|x|^2)^2}. \tag{3.7}$$

Its inverse $(\bar{h}^{ij})$ and the connection coefficients $\bar{\Gamma}^k_{ij}$ are respectively given by

$$\bar{h}^{ij} = (1 + \mu|x|^2)(\delta_{ij} + \mu x^i x^j), \quad \bar{\Gamma}^k_{ij} = -\frac{\mu(x_i \delta^k_j + x_j \delta^k_i)}{1 + \mu|x|^2}. \tag{3.8}$$

Assume that $c(x) \neq$ constant on U. By (2.10)–(2.11), V is given by one of the following (A1) $V = 2\big(\epsilon\sqrt{1 + \mu|x|^2} + \langle a, x\rangle\big)x - \frac{2|x|^2 a}{1+\sqrt{1+\mu|x|^2}}$ $(a \neq 0)$. In this case, $c = \frac{\epsilon + \langle a, x\rangle}{\sqrt{1+\mu|x|^2}}$.

(A2) $V = 2\epsilon\sqrt{1 + \mu|x|^2}\, x$ $(\mu \neq 0, \epsilon \neq 0)$. In this case, $c = \frac{\epsilon}{\sqrt{1+\mu|x|^2}}$. Moreover, by Lemma 3.2, there is a function ν on M such that $\eta_i = \nu c_i \neq 0$ with $\nu_j c_i = \nu_i c_j$. Consequently, there is a function $\sigma(x)$ on M such that $\nu_i = \sigma c_i$. Noting that $c_{;i;j} = -\mu c \bar{h}_{ij}$ from Lemma 2.2 in [SX] when $n \geq 3$. We have

$$\begin{aligned} \eta^i &= \bar{h}^{ij} \nu c_j = \nu\sqrt{1 + \mu|x|^2}(a^i - \mu\epsilon x^i), \\ \eta_{i;j} &= \nu_j c_i - \mu c \nu \bar{h}_{ij} = \sigma c_i c_j - \mu c \nu \bar{h}_{ij}. \end{aligned} \tag{3.9}$$

Case I: If V is given by (A1), then

$$V_i = \bar{h}_{ij} V^j = \frac{2 c x_i}{1 + \mu|x|^2} - \frac{2|x|^2 a_i}{(1 + \mu|x|^2)(1 + \sqrt{1 + \mu|x|^2}\,)}, \tag{3.10}$$

where the indices of a and x are raised and lowered by δ_{ij}. From (3.10) and (3.8), we have

$$V_{j;i} = 2c\bar{h}_{ji} - \frac{2(a_j x_i - a_i x_j)}{(1+\mu|x|^2)^2}, \tag{3.11}$$

Plugging (3.9)–(3.11) into (3.1) yields

$$Ax + (1+\tau)^{-1}Ba = 0, \tag{3.12}$$

where

$$A := -\mu c\tau(\sigma D + 2c\nu) - 2\nu|a|^2 + 2\mu\epsilon\nu\langle x, a\rangle, \tag{3.13}$$

$$B := \sigma\tau^2(1+\tau)D + 2\mu c\nu\tau|x|^2 + 2\nu(1+\tau)\left(\langle x, a\rangle - \mu\epsilon|x|^2\right), \tag{3.14}$$

$$D := V^j c_j = 2\epsilon\langle a, x\rangle + \frac{2\langle a, x\rangle^2}{\tau} - \frac{2|a|^2|x|^2}{\tau(1+\tau)} - \frac{2c\mu(\epsilon + c)}{1+\tau}|x|^2, \tag{3.15}$$

where $\tau := \sqrt{1+\mu|x|^2}$. From (3.12) and $a \neq 0$, we get $A = 0$ and $B = 0$. If $\mu = 0$, then $A = 0$ implies $\nu = 0$, i.e. $\eta = 0$. This is impossible by assumption. Hence $\mu \neq 0$. Multiplying $\tau(1+\tau)$ on the both sides of $A = 0$ and $c\mu$ on the both sides of $B = 0$, and adding these two identities yield

$$\begin{aligned} &-\tau^2(1+\tau)\mu c^2 - \tau(1+\tau)|a|^2 + \tau(1+\tau)\mu\epsilon\langle x, a\rangle \\ &\qquad + c^2\mu^2\tau|x|^2 + (1+\tau)c\mu\left(\langle x, a\rangle - \mu\epsilon|x|^2\right) = 0. \end{aligned} \tag{3.16}$$

By the irrationality of τ, (3.16) is decomposed as

$$\begin{cases} \tau^2\left(-\mu c^2 - |a|^2 + \mu\epsilon\langle x, a\rangle\right) + c\mu\left(\langle x, a\rangle - \mu\epsilon|x|^2\right) = 0, \\ -\mu c^2\tau^2 - |a|^2 + \mu\epsilon\langle x, a\rangle + c^2\mu^2|x|^2 + c\mu\left(\langle x, a\rangle - \mu\epsilon|x|^2\right) = 0. \end{cases} \tag{3.17}$$

$(3.17)_1$–$(3.17)_2$ gives

$$\mu\epsilon\langle a, x\rangle = |a|^2 + c^2\mu. \tag{3.18}$$

Plugging (3.18) into $(3.17)_1$ and using $\mu \neq 0$, $c \neq$ constant yield $\langle x, a\rangle - \mu\epsilon|x|^2 = 0$. By replacing x by $-x$ and adding these two identities, we get $\langle x, a\rangle = \mu\epsilon|x|^2 = 0$, which means $a = 0$. This is a contradiction with $a \neq 0$. Consequently, c is a constant.

Case II: If V is given by (A2), then

$$V_i = \frac{2cx_i}{1+\mu|x|^2}, \quad V_{i;j} = 2c\bar{h}_{ij}. \tag{3.19}$$

In the same way as Case I, we get $\bar{A}x = 0$, which means $\bar{A} = 0$, where

$$\bar{A} := \sigma c^2\mu^2(\epsilon + c)|x|^2 - (1+\tau)c^2\mu\nu. \tag{3.20}$$

Since $\mu \neq 0$ and c is not constant, $\bar{A} = 0$ means $\nu = 0$ by the irrationality of τ, which is impossible because of $\eta \neq 0$. This completes the proof. □

Moreover, in [ShX2], we have shown the following Lemma.

Lemma 3.3 ([ShX2]). *Assume that the vector field V in Theorem 1.1 is homothetic, i.e., $c =$ constant and $\eta = df$ for some scalar function $f = f(x)$. Then* (3.1) *is equivalent to the following equation*

$$V^j f_{x^j} - 2cf = k, \tag{3.21}$$

where $k =$ constant.

Proof of Theorem 1.2. By the assumption and (2.10)–(2.11), V is homothetic with respect to $\bar{F}$ with dilation ϵ. Thus V is homothetic with respect to F with dilation ϵ from Lemma 3.3 and Lemma 3.1. Conversely, if V is homothetic with respect to F with dilation ϵ and η satisfies (1.7), then by Lemma 3.3 and Lemma 3.1, V is also homothetic with respect to $\bar{F}$ with dilation ϵ and (i) or (ii) in Theorem 1.2 follows from (2.10)–(2.11) directly. □

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References

[BRS] D. Bao, C. Robles and Z. Shen, Zermelo navigation on Riemannian manifolds, *J. Diff. Geom.* **66**, no. 3 (2003), 377–435.

[CS] S. S. Chern and Z. Shen, Riemannian–Finsler Geometry, *World Scientific Publisher, Singapore*, 2005.

[CZ] B. Chen and L. Zhao, A note on Randers metrics of scalar flag curvature, *Canad. Math. Bull.* **55**, no. 3 (2012), 474–486.

[JB] P. Joharinad and B. Bidabad, Conformal vector fields on Finsler space, *Diff. Geom. Appl.* **31**, no. 1 (2013).

[Kang] L. Kang, On conformal fields of (α, β)-space, 2011 (*in print*).

[MH1] X. Mo and L. Huang, On curvature decreasing property of a class of navigation problem, *Publ. Math. Debrecen* **71** (2007), 141–163.

[MH2] L.Huang and X.Mo, On the flag curvature of a class of navigation problems on Finsler manifolds, 2012 (*in print*).

[Sh] Z.Shen, Finsler metrics with $K = 0$ and $S = 0$, *Canad. J. Math.* **55**, no. 1 (2003), 112–132.

[ShX1] Z. Shen and Q. Xia, On conformal vector fields on Randers manifolds, *Sci. China Math.* **55**, no. 9 (2012), 1869–1882.

[ShX2] Z. Shen and Q. Xia, A class of Randers metrics of scalar flag curvature, *Internat. J. Math.* **24 7** (2013), 13500559 (9 pages).

[SX] Z. Shen and H. Xing, On Randers metrics with isotropic S-curvature, *Acta Math. Sinica, English Series* **24**, no. 5 (2008), 789–796.

[SY] Z. Shen and G. C. Yildirim, A Characterization of Randers metrics of scalar flag curvature, Vol. 23, Surveys in Geomitric Analysis and Relativity, ALM, 330–343.

[Zer] E. Zermelo, Über das Navigtions problem bei ruhender oder veränderlicher Windverteilung, *Z. Angrew. Math. Mech.* **11** (1931), 114–124.

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