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On the number of solutions of the generalized Ramanujan–Nagell equation $x^2 - D = p^n$

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Abstract. Let D be a positive integer, and let p be an odd prime with $p \nmid D$. In this paper, by using Baker's method, we prove that if $\max(D,p) > 10^{65}$, then the equation $x^2 - D = p^n$ has at most three positive integer solutions (x,n).

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. Let $D \in \mathbb{N}$, and let p be an odd prime with $p \nmid D$. Further let $N(D,P)$ denote the number of solutions (x,n) of the equation

$$(1) \qquad x^2 - D = p^n, \quad x,n \in \mathbb{N}.$$

In [1], BEUKERS proved that $N(D,p) \le 4$. Simultaneously, he suspected that $N(D,p) \le 3$. Recently, the author [4] proved that if $\max(D,p) \ge 10^{240}$, then $N(D,p) \le 3$. In this paper we shall improve the above result.

If D,p satisfy

$$(2) \qquad (p,D) = \begin{cases} \left(3, \left(\dfrac{3^m+1}{4}\right)^2 - 3^m\right), \ 2 \nmid m, \\ \left(4a^2+1, \left(\dfrac{p^m-1}{4a}\right)^2 - p^m\right), \end{cases} \qquad a,m \in \mathbb{N},\ m > 1,$$

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then the pair (D,p) is called exceptional. BEUKERS [1] showed that if (D,p) is exceptional, then (1) has at least three solutions

$$(x_1,n_1)=\begin{cases}\left(\dfrac{3^m-7}{4},1\right),\\[2mm] \left(\dfrac{p^m-1}{4a}-2a,1\right)\end{cases}\qquad (x_2,n_2)=\begin{cases}\left(\dfrac{3^m+1}{4},m\right),\\[2mm] \left(\dfrac{p^m-1}{4a},m\right),\end{cases}$$

(3)

$$(x_3,n_3)=\begin{cases}\left(2\cdot 3^m-\dfrac{3^m+1}{4},\ 2m+1\right), & \text{if } p=3,\\[2mm] \left(2ap^m+\dfrac{p^m-1}{4a},\ 2m+1\right), & \text{if } p\neq 3.\end{cases}$$

In this paper we prove the following result.

Theorem. *If*

$$\max(D,p)>\begin{cases}3478, & \textit{if } p=3 \textit{ and } (D,p) \textit{ is exceptional},\\ 2\cdot 10^{19}, & \textit{if } p\neq 3 \textit{ and } (D,p) \textit{ is exceptional},\\ 10^{65}, & \textit{otherwise},\end{cases}$$

then $N(D,p)\le 3$.

2. Auxiliary Lemmas

Lemma 1 ([4, Lemma 3]). *For $D\in\mathbb{N}$ which is not a square, let $u_1+v_1\sqrt{D}$ be the fundamental solution of the equation*

(4) $$u^2-Dv^2=1$$

If the equation

(5) $$X^2-DY^2=p^z,\quad \gcd(X,Y)=1,\quad Z>0$$

has solutions (X,Y,Z), then (5) has a unique positive solution (X_1,Y_1,Z_1) which satisfies

$$Z_1\le Z,\quad 1<\frac{X_1+Y_1\sqrt{D}}{X_1-Y_1\sqrt{D}}<(u_1+v_1\sqrt{D}\,)^2,$$

where Z runs over all solutions of (5). Such (X_1,Y_1,Z_1) is called the least solution of (5). Then every solution (X,Y,Z) of (5) can be expressed as

$$Z=Z_1t,\quad X+Y\sqrt{D}=(X_1\pm Y_1\sqrt{D}\,)^t(u+v\sqrt{D}\,),$$

where $t\in\mathbb{N}$, (u,v) is a solution of (4).

Lemma 2 ([2, Theorem 10·8·2]). *Let $k \in \mathbb{Z}$ with* $\gcd(k, D) = 1$. *If* $|k| < \sqrt{D}$ *and* (X', Y') *is a positive solution of the equation*

$$X'^2 - DY'^2 = k, \quad \gcd(X', Y') = 1, \tag{6}$$

then X'/Y' is a convergent of $\sqrt{D}$.

It is a well known fact that the simple continued fraction of $\sqrt{D}$ can be expressed as $[a_0, \dot{a}_1, \ldots, \dot{a}_s]$, where $a_0 = [\sqrt{D}\,]$, $a_s = 2a_0$ and $a_i < 2a_0$ for $i = 1, \cdots, s-1$.

Lemma 3. *For any $m \in \mathbb{Z}$ with $m \geq 0$, let p_m/q_m, r_m denote the mth convergent and complete quotient of $\sqrt{D}$ respectively. Further let $k_m = (-1)^{m-1}(p_m^2 - Dq_m^2)$. Then we have:*

(i) *$k_m > 0$ and $a_{m+1} = [(\Delta_m + \sqrt{D}\,)/k_m]$ for a suitable $\Delta_m \in \mathbb{N}$.*

(ii) *Let*

$$s' = \begin{cases} s-1, & \text{if } 2 \mid s, \\ 2s-1, & \text{if } 2 \nmid s. \end{cases}$$

Then $p_{s'} + q_{s'}\sqrt{D}$ is the fundamental solution of (5).

(ii) *If $1 < k < \sqrt{D}$, $k \in \mathbb{N}$, $2D \not\equiv 0 \pmod{k}$ and* (6) *has solution (X', Y'), then* (6) *has at least two positive solutions (p_j, q_j) and $(p_{s'-j-1}, q_{s'-j-1})$, where $j \in \mathbb{Z}$ with $0 \leq j \leq s'-1$.*

Proof. The lemma follows from Satz 10 and Satz 18 of [6, Chapter III] and from various results scattered in [6, Section 26].

Lemma 4. *Let (X_1, Y_1, Z_1) be the least solution of* (5). *If $p^{z_1 r} < \sqrt{D}$ for some $r \in \mathbb{N}$, then $u_1 + v_1\sqrt{D} > D^{r/2}$.*

Proof. Under the assumption, by Lemma 1, there exists $X_i, Y_i \in \mathbb{Z}$ $(i = 1, \ldots, r)$ such that

$$X_i^2 - DY_i^2 = p^{z_1 i}, \quad \gcd(X_i, Y_i) = 1,\ i = 1, \cdots, r\ .$$

Since $p^{z_1 r} < \sqrt{D}$, by Lemma 2 and (iii) of Lemma 3, $\sqrt{D}$ has $2r$ convergents p_{m_i}/q_{m_i}, $p_{m_i'}/q_{m_i'}$ $(i = 1, \cdots, r)$ such that

$$k_{m_i} = k_{m_i'} = p^{z_1 i}, \quad 2 \nmid m_i m_i', \quad 0 < m_i, m_i' < s', i = 1, \ldots, r,$$

where s' was defined as in (ii) of Lemma 3. Therefore, by (i)

$$\begin{aligned} a_{m_i+1} &= \left[\frac{\Delta_{m_i} + \sqrt{D}}{k_{m_i}}\right] > \frac{\sqrt{D}}{p^{z_1 i}} - 1, \\ a_{m_i'+1} &= \left[\frac{\Delta_{m_i'} + \sqrt{D}}{k_{m_i'}}\right] > \frac{\sqrt{D}}{p^{z_1 i}} - 1,\ i = 1, \cdots, r. \end{aligned} \tag{7}$$

Notice that $p_0 = a_0 = [\sqrt{D}]$, $p_1 = a_0 a_1 + 1$ and $p_{m+2} = a_{m+2} p_{m+1} + p_m$ for $m \geq 0$. By (ii) of Lemma 3, we get from (7) that

$$u_1 + v_1\sqrt{D} = P_{s'} + q_{s'}\sqrt{D} \geq P_{s'} + \sqrt{D} \geq$$
$$\geq \left(a_0 \prod_{j=0}^{(s'-3)/2} (a_{2j+1} + a_{2j+2}) - a_0 \right) + \sqrt{D} > a_0 \prod_{j=0}^{(s'-3)/2} (a_{2j+1} + 1) \geq$$
$$\geq a_0 \prod_{i=1}^{r} (a_{m_i} + 1)(a_{m'_i} + 1) > a_0 \left(\prod_{i=1}^{r} \frac{\sqrt{D}}{p^{z_1 i}} \right)^2 = \frac{a_0 D^r}{p^{z_1 r(r+1)}} > D^{r/2},$$

since $a_0 = [\sqrt{D}]$. The lemma is proved.

Lemma 5 ([5, Formula 3·76]). *For any $m \in \mathbb{N}$ and any complex numbers α, β, we have*

$$\alpha^m + \beta^m = \sum_{i=0}^{[m/2]} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} (\alpha + \beta)^{m-2i} (\alpha\beta)^i,$$

where

$$\begin{bmatrix} m \\ i \end{bmatrix} = \frac{(m-i-1)!\, m}{(m-2i)!\, i!} \in \mathbb{N}, \quad i = 0, \cdots, [m/2].$$

Lemma 6 ([2, Theorem 6·10·3]). *Let a/b, a'/b', $a''/b'' \in \mathbb{Q}$ be positive with $ab' - a'b = \pm 1$. If a''/b'' lies in the interval $\xi = (a/b, a'/b')$, then there exist $k, k' \in \mathbb{N}$ such that $a'' = ak + a'k'$ and $b'' = bk + b'k'$.*

Let α be an algebraic number of degree d with the minimal polynomial

$$a_0 z^d + \cdots + a_{d-1} z + a_d = a_0 \prod_{i=1}^{d} (z - \sigma_i \alpha), \quad a_0 > 0,$$

where $\sigma_1\alpha, \cdots, \sigma_d\alpha$ are all conjugates of α. Then

$$h(\alpha) = \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \log \max(1, |\sigma_i d|) \right)$$

is called the logarithmic absolute height of α.

Lemma 7. *Let α_1, α_2 be real algebraic numbers with $\alpha_1 > 1$ and $a_2 > 1$, and let r denote the degree of $\mathbb{Q}(\alpha_1, \alpha_2)$. Let b_1, $b_2 \in \mathbb{N}$, and let $b = b_1/rh(\alpha_2) + b_2/rh(\alpha_1)$. For any $T \geq 1$, if $0.52 + \log b \geq T$ and $\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2 \neq 0$, then*

$$\log|\Lambda| > -70 \left(1 + \frac{0.1137}{T} \right)^2 r^4 h(\alpha_1) h(\alpha_2) (0.52 + \log b)^2.$$

Proof. Let $B = \log(5c_4/c_1) + \log b$, $K = [c_1 r^3 B h(\alpha_1) h(a_2)]$, $L = [c_2 r B]$, $R_1 = [c_3 r^{3/2} B^{1/2} h(\alpha_2)] + 1$, $S_1 = [c_3 r^{3/2} B^{1/2} h(\alpha_1)] + 1$, $R_2 = [c_4 r^2 B h(\alpha_2)]$, $S_2 = [c_4 r^2 B h(\alpha_1)]$, $R = R_1 + R_2 - 1$, $S = S_1 + S_2 - 1$, where c_1, c_2, c_3, c_4 are positive constants. Notice that $(u - 1/T)v < [uv] \leq uv$ for any real numbers u, v with $u \geq 0$ and $v \geq T$. By the proof of [3, Theorems 1 and 3], if $B \geq T$,

$$\sqrt{c_1} = \frac{\varrho + 1}{(\log \varrho)^{3/2}} + \sqrt{\frac{(\varrho + 1)^2}{(\log \varrho)^3} + \frac{\varrho + 1}{T \log \varrho}}, \qquad c_2 > \frac{2}{\log \varrho},$$
$$c_3 = \max(\sqrt{c_1}, \sqrt{c_2}\,), \qquad c_4 = \sqrt{2 c_1 c_2} + \frac{1}{T}, \tag{8}$$

then

$$\log |\Lambda| > -(c_1 c_2 \log \varrho + 1) r^4 h(\alpha_1) h(\alpha) B^2, \tag{9}$$

where ϱ is a positive constant with $\varrho > 1$. Set $\varrho = 5.803$. We can choose c_1, c_2, c_3, c_4 such that (8) holds and such that

$$c_1 c_2 \log \varrho + 1 < 70 \left(1 + \frac{0.1137}{T}\right)^2, \quad B < 0.52 + \log b. \tag{10}$$

Substituting (10) into (9), the lemma is proved.

Lemma 8 ([7, Theorem I·2]). *Let $a, k, \ell, q, r, s \in \mathbb{N}$ be such that $2 \nmid k\ell$ and q is not a square. If there exist $X, \Delta \in \mathbb{Z}$ such that*

$$X^2 + \Delta = a^2 q^k, \quad a^2 q^k \geq 4^{1+s/r} |\Delta|^{2+s/r},$$

then

$$\left| \frac{Y}{a q^{\ell/2}} - 1 \right| > \frac{2^3 (3^4 a^2 q^k / 4)^{1/s}}{3^7 a^5 q^{(3+\nu/2)k}} q^{-(1+\nu)\ell/2},$$

for any $Y \in \mathbb{N}$, where ν satisfies $q^{k\nu} = 9a^2 (3^4 a^2 q^k/4)^{r/s}$.

3. Proof of theorem for exceptional cases

Throughout this section we assume that (D, p) is exceptional. Let (X_1, Y_1, Z_1) and $u_1 + v_1\sqrt{D}$ be the least solution of (5) and the fundamental solution of (4) respectively, and let

$$\varepsilon = X_1 + Y_1 \sqrt{D}, \quad \bar{\varepsilon} = X_1 - Y_1 \sqrt{D}, \tag{11}$$

$$\varrho = u_1 + v_1 \sqrt{D}, \quad \bar{\varrho} = u_1 - v_1 \sqrt{D}. \tag{12}$$

Now we suppose that $N(D,p) > 3$. Then (1) has four solutions (x_i, n_i) $(i = 1, \cdots, 4)$, where (x_j, n_j) $(j = 1, 2, 3)$ satisfy (3). By the proof of [4, Theorem 1], we have $n_4 > n_3$, $2 \nmid n_4$ and

$$\begin{aligned} x_i + \delta_i \sqrt{D} = \varepsilon^{n_i} \bar{\varrho}^{s_i},\ \delta_i \in \{-1, 1\},\ s_i \in \mathbb{Z},\ 0 \le s_i \le n_i, \\ \gcd(n_i, s_i) = 1,\ i = 1, \cdots, 4. \end{aligned} \tag{13}$$

Since $p < D$ by (2), we find from $n_1 = 1$, and $1 < (x_1 + \sqrt{D}\,)/(x_1 - \sqrt{D}\,) < 4D < \varrho^2$ that $(X_1, Y_1, Z_1) = (x_1, 1, 1)$ by Lemma 1. Together with (13) this implies that $\delta_1 = 1$ and $s_1 = 0$.

Assertion 1. $\delta_2 = -1$ *and* $s_2 = 1$.

Proof. Let $X + Y\sqrt{D} = \varepsilon^m = \varepsilon^{n_2}$, $u + v\sqrt{D} = \varrho^{s_2}$. From (2) and (13) we get

$$x_2 = Xu - DYv,\ \delta_2 = Yu - Xu,\ X, Y, u, v \in \mathbb{Z}. \tag{14}$$

Recalling that $(X_1, Y_1, Z_1) = (x_1, 1, 1)$, we have $X \equiv 2^{m-1} x_1^m \pmod{p}$ and $Y \equiv 2^{m-1} x_1^{m-1} \pmod{p}$. From (14), we get $x_2 \equiv 2^{m-1} x_1^m (u - x_1 u)$ $\pmod{p}$ and $\delta_2 \equiv 2^{m-1} x_1^{m-1} (u - x_1 v) \pmod{p}$, since $x_1^2 \equiv D \pmod{p}$. Hence, $\delta_2 \equiv x_2/x_1 \equiv -1 \pmod{p}$ by (2). Since $p \ge 3$ and $\delta_2 \in \{-1, 1\}$, we get $\delta_1 = -1$.

Since $m > 1$, by Lemma 3 of [1], we see from (13) that $s_2 \ne 0$. If $m = 2$, then $s_2 = 1$ by (13). If $(D, p, m) = (22, 3, 3)$, then from $x_2 - \sqrt{D} = 7 - \sqrt{22} = (5 + \sqrt{22})^3 (197 - 42\sqrt{22}) = \varepsilon^3 \bar{\varrho}$, we get $s_2 = 1$. If $m \ge 3$ and $s_2 > 1$, then from (13) we have

$$(x_1 + \sqrt{D}\,)^m > \frac{1}{p^m} (x_1 + \sqrt{D}\,)^m (x_2 + \sqrt{D}\,) = \frac{\varepsilon^m}{x_2 - \sqrt{D}} = \varrho^{s_2} \ge \varrho^2. \tag{15}$$

On the other hand, by (2),

$$\sqrt{D} > \begin{cases} 3^{m-2}, & \text{if } p = 3, \\ p^{m-1}, & \text{if } p \ne 3 \text{ and } m \ge 3. \end{cases}$$

Therefore, by Lemma 4,

$$\varrho^2 > \begin{cases} D^{m-2}, & \text{if } p = 3, \\ D^{m-1}, & \text{if } p \ne 3 \text{ and } m \ge 3. \end{cases} \tag{16}$$

Since $x_1 + \sqrt{D} < 2.05\sqrt{D}$, the combination of (15) and (16) yields

$$(2.05)^m > \begin{cases} D^{m/2-2}, & \text{if } p = 3, \\ D^{m/2-1}, & \text{if } p \ne 3 \text{ and } m \ge 3. \end{cases}$$

This is impossible except $(D,p,m)=(22,3,3)$. Thus $s_2=1$ the assertion is proved.

Assertion 2. *There exists some* $k,k'\in\mathbb{N}$ *such that*

$$n_4=\begin{cases} mk+(2m+1)k', \\ (m+1)k+(2m+1)k', \end{cases}$$

$$x_4+\delta_4\sqrt{D}=\begin{cases} (x_2-\sqrt{D}\,)^k(x_3+\sqrt{D}\,)k', & \textit{if } p=3, \\ \left(\dfrac{p^m+1}{2}+2a\sqrt{D}\right)^k(x_3-\sqrt{D}\,)^{k'}, & \textit{if } p\neq 3. \end{cases}$$

Proof. By (2) and (13) we have

$$x_3+\sqrt{D}=\begin{cases} (x_1+\sqrt{D}\,)(x_2-\sqrt{D}\,)^2, & \text{if } p=3, \\ (x_1-\sqrt{D}\,)(x_2+\sqrt{D}\,)^2, & \text{if } p\neq 3. \end{cases}$$

Recalling that $(\delta_1,s_1)=(1,0)$ and $(\delta_2,s_2)=(-1,1)$ by Assertion 1, we get

$$x_3+\sqrt{D}=\begin{cases} \varepsilon^{2m+1}\bar{\varrho}^2, & \text{if } p=3, \\ \bar{\varepsilon}^{2m+1}\varrho^2, & \text{if } p\neq 3. \end{cases} \tag{17}$$

For any solution (x,n) of (1), let

$$\Lambda(x,n)=\log\frac{x+\sqrt{D}}{x-\sqrt{D}},$$

and let $\alpha=(\log\varepsilon/\bar{\varepsilon}))/\log\varrho^2$. By Lemma 5 of [1], $n_4\geq 2n_3+n_2=5m+2$. From (2), $m\geq 3$ for $p=3$ and $m\geq 2$ for $p\neq 3$. So we have

$$\Lambda(x_2,n_2)>\left\{\begin{array}{ll} \log\dfrac{3^{2m}-14\cdot 3^m+1}{4\cdot 3^m}>\log\dfrac{4}{3}, & \text{if } p=3 \\ \log\dfrac{p^{2m}-2(2p-1)p^m+1}{(p-1)p^m}>\log\dfrac{4(p-1)}{p}, & \text{if } p\neq 3 \end{array}\right\}> \tag{18}$$

$$>\Lambda(x_3,n_3)>\Lambda(x_4,n_4).$$

When $p=3$, by (17) and Assertion 1, we have

$$\frac{1}{m}-\alpha=\frac{\Lambda(x_2,n_2)}{m\log\varrho^2}>0,\quad \alpha-\frac{2}{2m+1}=\frac{\Lambda(x_3,n_3)}{(2m+1)\log\varrho^2}>0. \tag{19}$$

Since

$$\left|\alpha - \frac{s_4}{n_4}\right| = \frac{\Lambda(x_4, n_4)}{n_4 \log \varrho^2},$$

we see from (18) and (19) that s_4/n_4 lies in the interval $\xi = (1/m, 2/2m+1))$. Therefore, by Lemma 6, we get

$$s_4 = k + 2k', \quad n_4 = mk + (2m+1)k', \quad k, k' \in \mathbb{N}. \tag{20}$$

When $p \neq 3$, by (2) and Assertion 1, we have

$$\frac{p^m+1}{2} + 2a\sqrt{D} = (x_1 + \sqrt{D})(x_2 - \sqrt{D}) = \varepsilon^{m+1}\bar{\varrho}. \tag{21}$$

Since

$$\begin{aligned} \log \frac{(p^m+1)/2 + 2a\sqrt{D}}{(p^m+1)/2 - 2a\sqrt{D}} > \log \frac{p^{2m} - 2(2p-1)p^m + 1}{p^{m+1}} > \\ > \Lambda(x_3, n_3) > \Lambda(x_4, n_4) \end{aligned} \tag{22}$$

by (18), we see from

$$\begin{aligned} &\alpha - \frac{1}{m+1} = \left(\log \frac{(p^m+1)/2 + 2a\sqrt{D}}{(p^m+1)/2 - 2a\sqrt{D}}\right) \Big/ (m+1)\log \varrho^2 > 0, \\ &\frac{2}{2m+1} - \alpha = \frac{\Lambda(x_3, n_3)}{(2m+1)\log \varrho^2} > 0 \end{aligned}$$

that s_4/n_4 lies in the interval $\xi = (1/(m+1), 2/(2m+1))$. Hence, by Lemma 6, we get

$$s_4 = k + 2k', \; n_4 = (m+1)k + (2m+1)k', \; k, k' \in \mathbb{N}. \tag{23}$$

Thus, the assertion follows immediately from (13), (17), (20), (23) and Assertion 1.

Assertion 3. *If $p = 3$, then $k + k' - 1 \geq 2 \cdot 3^{m-1}$.*

Proof. Let $\varepsilon_2 = x_2 + \sqrt{D}$, $\bar{\varepsilon}_2 = x_2 - \sqrt{D}$, $\varepsilon_3 = x_3 + \sqrt{D}$, $\bar{\varepsilon}_3 = x_3 - \sqrt{D}$, and let

$$X + Y\sqrt{D} = \varepsilon_2^k, \quad X' + Y'\sqrt{D} = \varepsilon_3^{k'}. \tag{24}$$

Then, by Lemma 5, $X, Y, X', Y' \in \mathbb{Z}$ satisfy

$$
(25)\quad
\begin{aligned}
X &= \frac{1}{2}\left(\varepsilon_2^k + \bar{\varepsilon}_2^k\right) = \frac{1}{2}\sum_{i=0}^{[k/2]}(-1)^i \begin{bmatrix} k \\ i \end{bmatrix}(\varepsilon_2 - \bar{\varepsilon}_2)^{k-2i}(\varepsilon_2\bar{\varepsilon}_2)^i = \\
&= \frac{1}{2}\sum_{i=0}^{[k/2]}(-1)^i \begin{bmatrix} k \\ i \end{bmatrix}(2x_2)^{k-2i}3^{mi} \equiv 2^{k-1}x_2^k \equiv \frac{1}{2^{k+1}} \pmod{3^m},\\
Y &= \frac{1}{2\sqrt{D}}(\varepsilon_2^k - \bar{\varepsilon}_2^k) = \frac{\varepsilon_2^k - \bar{\varepsilon}_2^k}{\varepsilon_2 - \bar{\varepsilon}_2} \equiv \varepsilon_2^{k-1} + \bar{\varepsilon}_2^{k-1} \equiv \\
&\equiv (2x_2)^{k-1} \equiv \frac{1}{2^{k-1}} \pmod{3^m},\\
X' &= \frac{1}{2}(\varepsilon_3^{k'} + \bar{\varepsilon}_3^{k'}) \equiv 2^{k'-1}x_3^{k'} \equiv \frac{(-1)^{k'}}{2^{k'+1}} \pmod{3^{2m+1}},\\
Y' &= \frac{1}{2\sqrt{D}}(\varepsilon_3^{k'} - \bar{\varepsilon}_3^{k'}) = \frac{\varepsilon_3^{k'} - \bar{\varepsilon}_3^{k'}}{\varepsilon_3 - \bar{\varepsilon}_3} \equiv \frac{(-1)^{k'-1}}{2^{k'-1}} \pmod{3^{2m+1}}.
\end{aligned}
$$

By Assertion 2, we get from (24) and (25) that

$$
(26)\qquad \delta_4 = XY' - X'Y \equiv \frac{(-1)^{k'-1}}{2^{k+k'-1}} \pmod{3^m}.
$$

Since $2 \nmid mn_4$, we see from (20) that $k + k' - 1 \equiv 0 \pmod 2$. Further, by (26), we get $2^{k+k'-1} \equiv \pm 1 \pmod{3^m}$. Therefore $k + k' - 1 \equiv 0 \pmod{2 \cdot 3^{m-1}}$. Notice that $k, k' \in \mathbb{N}$ and $k+k'-1 > 0$. Thus $k+k'-1 \geq 2\cdot 3^{m-1}$. The assertion is proved.

Assertion 4. *If $p \neq 3$ and $p^{m-1} \geq 20$, then $k' - 1 \geq 2p^{m-1}$.*

Proof. Let $\varepsilon_2' = (p^m+1)/2 + 2a\sqrt{D}$, $\bar{\varepsilon}_2' = (p^m+1)/2 - 2a\sqrt{D}$, and let

$$
(27)\qquad X + Y\sqrt{D} = {\varepsilon_2'}^k, \quad X' + Y'\sqrt{D} = \varepsilon_3^{k'}.
$$

According to the analysis for (25), X, Y, X', Y' satisfy

$$
(28)\quad
\begin{aligned}
X &\equiv \frac{1}{2} \pmod{p^m}, & Y &\equiv 2a \pmod{p^m},\\
X' &\equiv \frac{(-1)^{k'}}{2^{k'+1}a^{k'}} \pmod{p^m}, & Y' &\equiv \frac{(-1)^{k'-1}}{(2a)^{k'-1}} \pmod{p^m}.
\end{aligned}
$$

By Assertion 2, we get from (27) and (28) that

$$
\delta_4 = X'Y - XY' \equiv \frac{(-1)^{k'}}{(2a)^{k'-1}} \pmod{p^m}.
$$

This implies that

$$(2a)^{k'-1} \equiv \pm 1 \pmod{p^m}. \tag{29}$$

Since $p = 4a^2 + 1$, we see from (29) that

$$k' - 1 \equiv 0 \pmod{2p^{m-1}}. \tag{30}$$

Since $p^{n_4} > D^2$, we have

$$\begin{aligned} \log \frac{x_4 + \sqrt{D}}{x_4 - \sqrt{D}} &= \log \left(1 + \frac{2\sqrt{D}}{x_4 - \sqrt{D}}\right) = \\ &= \frac{2\sqrt{D}}{x_4} \sum_{j=0}^{\infty} \frac{1}{2j+1} \left(\frac{D}{x_4^2}\right)^j < \frac{4\sqrt{D}}{x_4} < \frac{4}{\sqrt{D}}. \end{aligned} \tag{31}$$

By Assertion 2 and (22), if $p^{m-1} \geq 20$ and $k \geq k'$, then

$$\begin{aligned} \log \frac{x_4 + \sqrt{D}}{x_4 - \sqrt{D}} &= \left| k \log \frac{\varepsilon_2'}{\bar{\varepsilon}_2'} - k' \log \frac{\varepsilon_3}{\bar{\varepsilon}_3} \right| = \\ &= (k - k') \log \frac{\varepsilon_2'}{\bar{\varepsilon}_2'} + k' \left(\log \frac{\varepsilon_2'}{\bar{\varepsilon}_2} - \log \frac{\varepsilon_3}{\bar{\varepsilon}_3} \right) > \\ &> (k - k') \log(p^m - 4) + k' \left(\log(p^{m-1} - 4) - \log\left(4 - \frac{1}{p}\right) \right) > 1, \end{aligned}$$

which contradicts (31). Thus $k' > k$, and $k' - 1 \equiv 0 \pmod{2p^{m-1}}$ by (30). The assertion is proved.

Assertion 5. *If (D,3) is special, and $D \neq 22$, 3478, then $N(D,3) = 3$.*

Proof. Notice that $3788^2 + 37 = 3^{15}$. by the definitions as in Lemma 8, we may put $X = 3788$, $\Delta = 37$, $a = 1$, $q = 3$, $k = 15$, $r = 2$, $s = 3$ and $\nu = 0.9217$. Then we have

$$\left| \frac{Y}{3^{\ell/2}} - 1 \right| > 3^{-51 - 0.96085\ell} \tag{32}$$

for any $\ell, Y \in \mathbb{N}$ with $2 \nmid \ell$. If $N(D,3) > 3$, then from (32) we get

$$\left| \frac{x_4}{3^{n_4/2}} - 1 \right| > 3^{-51 - 0.96085 n_4}, \tag{33}$$

since $2 \nmid n_4$. We see from (2) that $D < 3^{2m}$, hence

$$\frac{x_4}{3^{n_4/2}} - 1 = \frac{D}{3^{n_4/2}(x_4 + 3^{n_4/2})} < \frac{3^{2m}}{2 \cdot 3^{n_4}}. \tag{34}$$

The combination of (33) with (34) yields

$$50.4 + 2m > 0.03915 n_4. \tag{35}$$

On the other hand, by Assertions 2 and 3, we have

$$n_4 = mk + (2m+1)k' \geq m(k+k'-1) + 2m+1 \geq 2\cdot 3^{m-1}m + 2m + 1. \tag{36}$$

From (35) and (36),

$$50.4 + 2m > 0.03915(2\cdot 3^{m-1}m + 2m + 1),$$

whence we conclude that $m \leq 5$, since $2 \nmid m$. The assertion is proved.

Assertion 6. *If (D,p) is special, $p \neq 3$ and $\max(D,p) > 2\cdot 10^9$, then $N(D,p) = 3$.*

Proof. Let

$$\alpha_1 = \frac{(p^m+1)/2 + 2a\sqrt{D}}{(p^m+1)/2 - 2a\sqrt{D}}, \quad \alpha_2 = \frac{x_3 + \sqrt{D}}{x_3 - \sqrt{D}}.$$

By Assertion 2, we get

$$\log \frac{x_4 + \sqrt{D}}{x_4 - \sqrt{D}} = |k \log \alpha_1 - k' \log \alpha_2| > 0. \tag{37}$$

Since α_1 α_2 satisfy

$$p^{m+1}\alpha_1^2 - 2\left(\left(\frac{p^m+1}{2}\right)^2 + 4a^2 D\right)\alpha_1 + p^{m+1} = 0,$$
$$p^{2m+1}\alpha_2^2 - 2(x_3^2 + D)\alpha_2 + p^{2m+1} = 0$$

respectively, we have

$$h(\alpha_1) = \log\left(\frac{p^m+1}{2} + 2a\sqrt{D}\right), \quad h(\alpha_2) = \log(x_3 + \sqrt{D}). \tag{38}$$

By Lemma 7, we have

$$|k\log\alpha_1 - k'\log\alpha_2| > \tag{39}$$
$$> \exp\left(-70\left(1 + \frac{0.1137}{T}\right)^2 2^4 h(\alpha_1)h(\alpha_2)(0.52 + \log b)^2\right)$$

for any $T \geq 1$, where

$$b = \frac{k}{2h(\alpha_2)} + \frac{k'}{2h(\alpha_1)}\ , \tag{40}$$

which satisfies $0.52 + \log b > T$. We may choose $T = 10$ and then from (39) we get

$$|k \log \alpha_1 - k' \log \alpha_2| > \exp(-1146 h(\alpha_1) h(\alpha_2)(0.52 + \log b)^2). \tag{41}$$

By Assertions 2 and 4, the combination of (41) with (31) yields

$$\begin{aligned} &\log 4\sqrt{D} + 1146 h(\alpha_1) h(\alpha_2)(0.52 + \log b)^2 > \log x_4 > \log p^{n_4/2} = \\ &= \frac{1}{2}\big((m+1)k + (2m+1)k'\big) \log p > k' \log p^{m+1/2}. \end{aligned} \tag{42}$$

When $m = 2$, we get $h(\alpha_1) < \log(p^2 + 1)$ and $h(\alpha_2) < \log 2x_3 < \log 3p^{2+1/2}$ by (38). Since $k' > k$ by Assertion 4, we obtain from (40) and (42) that

$$\frac{\log 4\sqrt{D}}{h(\alpha_1)h(\alpha_2)} + 1146(0.52 + \log b)^2 > \frac{k'}{h(\alpha_1)} \left(1 + \frac{\log 3}{\log p^{2+1/2}}\right) > b,$$

whence we conclude that $b < 160000$. Further, since $k'/2h(\alpha_1) < b$ by (40) and $k' - 1 \geq 2p$ by Assertion 4, we get

$$2p + 1 \leq k' < 320000\, h(\alpha_1) < 320000 \log(p^2 + 1).$$

It implies that $p < 4200000$ and $D < 2 \cdot 10^{19}$ by (2).

When $m \geq 3$, we get $h(\alpha_1) < \log(p^m + 1)$ and $h(\alpha_2) < \log 3p^{m+1/2}$ by (38). Then, by (42) we also obtain that $b < 160000$. Since $p \leq p^{m/3}$ in this case, we can conclude $p^m < 9 \cdot 10^9$ and $D < 10^{16}$ by the same way. Thus the assertion is proved.

By Assertion 5 and 6, the theorem holds for the exceptional cases.

4. Proof of theorem for the non-exceptional cases

Throughout this section we assume that the pair (D, p) is not exceptional.

Lemma 9. *Let (x, n), (x', n'), (x'', n'') be solutions of* (1) *with $n < n' < n''$. Then $2 \nmid n''$ and either $n'' \geq 2n' + \max(3, n)$ or $p^{n''} > 4p^{8n'/3}/9$.*

Proof. By Lemma 5 of [1], we have $2 \nmid n''$, $n'' \geq 2n' + \max(3, n)$ and $p^{n'} < 2(p^{(n''-2n')/2} + 1)^3$. Since $p^{n''-2n'} \geq 3^3$, we get $p^{n''} > 4p^{an'/3}/9$. The lemma is proved.

Lemma 10 ([1, Theorem 1])**.** *Let (x, n), (x', n') be two solutions of* (1) *with $n' > n$. Then $p^n \leq \max(2 \cdot 10^6, 600D^2)$.*

Lemma 11. *Let* (x, n), (x', n') *be two solutions of* (1) *with* $n' > n$. *Then* $p^{n'} > 4\sqrt{D}$.

Proof. Since ${x'}^2 - x^2 = p^n(p^{n'-n} - 1)$, we have $x' - \zeta x = 2ap^n$, where $\zeta \in \{-1, 1\}$, $a \in \mathbb{N}$. If $\zeta = 1$, then

$$p^{n'} = p^n + 4ap^n x + 4a^2 p^{2n} > 4ap^n\sqrt{D} \geq 4p^n\sqrt{D},$$

since $x > \sqrt{D}$. If $\zeta = -1$, then

$$p^{n'} = p^n(1 + 4a(ap^n - x)) \tag{43}$$

It follows that $a > x/p^n > \sqrt{D}/p^n$. Hence, from (43), we get $p^{n'} > 4\sqrt{D}$. The lemma is proved.

Lemma 12. *Let* (x, n), (x', n') *be two solutions of* (1) *with* $p^n > p^{n'} < D$. *If* $D \geq 25000$, *then* $\log \varrho < 1.1(\log D)^2$.

Proof. Under the assuimptions, by Lemma 4 of [4],

$$\begin{gathered} n = Z_1 t,\ n' = Z_1 t',\ x + \delta\sqrt{D} = \varepsilon^t \bar{\varrho}^s,\ x' + \delta'\sqrt{D} = \varepsilon^{t'} \bar{\varrho}^{s'}, \\ \delta, \delta' \in \{-1, 1\}, \end{gathered} \tag{44}$$

where $s, t, s', t' \in \mathbb{Z}$ such that

$$0 \leq s \leq t,\ 0 \leq s' \leq t',\ 1 \leq t \leq t',\ \gcd(s, t) = \gcd(s', t') = 1.$$

If $st' = s't$, then there exists $k \in \mathbb{N}$ such that $s' = sk$ and $t' = tk$. Since $t' > t$, we get $k > 1$ and $x' + \delta'\sqrt{D} = (x + \delta\sqrt{D})^k$ by (44). This is impossible by Lemma 3 of [1]. Hence $st' \neq s't$, and by (44),

$$\left| t' \log(x + \delta\sqrt{D}) - t \log(x' + \delta'\sqrt{D}) \right| = |s't - st'| \log \varrho \geq \log \varrho. \tag{45}$$

Since $D > p^{n'} > p^n$, we have $(1+\sqrt{2})\sqrt{D} > x'+\sqrt{D} > x+\sqrt{D} > 2\sqrt{D}$ and $\log D / \log p^{z_1} > t' > t$. Therefore

$$\begin{aligned} \left| t' \log(x + \delta\sqrt{D}) - t \log(x' + \delta'\sqrt{D}) \right| &< t' \log(x+\sqrt{D}) + t \log(x'+\sqrt{D}) \\ &< \frac{\log D}{\log p^{Z_1}}(2\log(1\sqrt{2}) + \log D) < 1.1(\log D)^2, \end{aligned}$$

since $p^{Z_1} \geq 3$ and $D \geq 25000$. On combining this with (45) yields the lemma.

Assertion 7. *If* $\max(D,p) > 10^{65}$, *then* $N(D,p) \le 3$.

PROOF. By the proof of [1, Theorem 2], it suffices to prove that the assertion holds for $D > 25000$, $D > 40p^2$ and D is not a sqare. This implies that $\max(D,p) = D$.

Suppose that $N(D,p) > 3$. Then (1) has four solutions (x_i, n_i) $(i = 1,\dots,4)$ with $n_1 < n_2 < n_3 < n_4$. By Lemma 4 of [4], we have

$$(46) \qquad n_i = Z_1 t_i,\ x_i + \delta_i\sqrt{D} = \varepsilon^{t_i}\bar{\varrho}^{s_i},\ \delta_i \in \{-1,1\},\ i = 1,\dots,4,$$

where the s_i, t_i are integers such that

$$(47) \qquad 0 \le s_i \le t_i, \quad \gcd(s_i,t_i) = 1, \quad i = 1,\dots,4,$$

If $p^{n_2} > D$, by Lemmas 9 and 10, we get

$$600D^2 = \max(2\cdot 10^6, 600D^2) > p^{p_3} > 4p^{8n_2/3} \,/\, 9 > 4D^{8/3} \,/\, 9 > 600D^2,$$

a contradiction. Hence $p^{n_2} < D$.

By Lemma 11, we have $p^{n_2} > 4\sqrt{D}$. Further, by Lemma 9,

$$(48) \qquad p^{n_3} > 4p^{8n_2/3} \,/\, 9 > 18D^{4/3}.$$

Furthermore, we see from the proof of Theorem 2 of [4] that if $D > 10^{30}$, then

$$(49) \qquad t_3 + t_4 > \frac{x_3 \log\varrho}{4\sqrt{D}} = \frac{1}{4}\left(1 + \frac{p^{n_3}}{D}\right)^{1/2} \log\varrho > D^{1/6}\log\varrho$$

by (48).

Let $\alpha_1 = \varepsilon/\bar{\varepsilon}$, $\alpha_2 = \varrho$. By (11) and (12), α_1 and α_2 satisfy $p^{Z_1}\alpha_1^2 - 2(X_1^2 + DY_1^2)\alpha_1 + p^{Z_1} = 0$ and $\alpha_2^2 - 2u_1\alpha_2 + 1 = 0$ respectively. So we have $h(\alpha_1) = \log\varepsilon$ and $h(\alpha_2) = \frac{1}{2}\log\varrho$. Notice that $1 < \varepsilon/\bar{\varepsilon} < \varrho^2$ by Lemma 1. We get $\varepsilon^2 < \varepsilon\bar{\varepsilon}\varrho^2 = p^{Z_i}\varrho^2$. So we have $\varepsilon < p^{Z_1/2}\varrho$ and $h(\alpha_1) < \log p^{Z_1/2}\varrho$.

By Lemma 7, we get

$$(50) \qquad \begin{aligned} \bigl|t_4\log\alpha_1 - 2s_4\log\alpha_2\bigr| &> \exp\bigl(-1146h(\alpha_1)h(\alpha_2)(0.52+\log b)^2\bigr) > \\ &> \exp\bigl(-1146(\log p^{Z_1/2}\varrho)(\log\varrho)(0.52+\log b)^2\bigr), \end{aligned}$$

where

$$(51) \qquad b = \frac{t_4}{2h(\alpha_2)} + \frac{s_4}{h(\alpha_1)} \le t_4\left(\frac{1}{2\log\varrho} + \frac{1}{\log p^{Z_1/2}\varrho}\right).$$

On the other hand, by (31) and (46),

$$(52) \qquad |t_4\log\alpha_1 - 2s_4\log\varrho| = \log\frac{x_4+\sqrt{D}}{x_4-\sqrt{D}} < \frac{4\sqrt{D}}{x_4},$$

since $p^{n_4} > p^{n_3} > D^{4/3}$. The combination of (50) and (52) yields

$$\log 4\sqrt{D} + 1146(\log p^{Z_1/2}\varrho)(\log\varrho)(0.52+\log b)^2 > \log x_4 > \frac{t_4}{2}\log p^{Z_1},$$

whence we get

$$\begin{aligned} 1+1146\left(\frac{1}{2}+\frac{\log\varrho}{\log p^{Z_1}}\right) > \\ \frac{\log 4\sqrt{D}}{(\log\varrho)(\log p^{Z_1})} + 1146\left(\frac{1}{2}+\frac{\log\varrho}{\log p^{Z_1}}\right)(0.52+\log b)^2 > \frac{b}{2}. \end{aligned} \tag{53}$$

We conclude from (53) that

$$b < 20000(\log\varrho)(\log\log\varrho)^2. \tag{54}$$

Since $b > t_4/2\log\varrho$ by (51), we get from (54) that

$$t_4 < 40000(\log\varrho)^2(\log\log\varrho)^2. \tag{55}$$

Notice that $t_3 = \log p^{n_3}/\log p^{Z_1} < \log 600D^2$ by Lemma 10. From (49) and (55), we get

$$\log 600D^2 + 40000(\log\varrho)^2(\log\log\varrho)^2 > D^{1/6}\log\varrho. \tag{56}$$

From (56),

$$5 + 40000(\log\varrho)\log\log\varrho)^2 > D^{1/6}, \tag{57}$$

since $\varrho > \sqrt{D}$. By Lemma 12, we have $\log\varrho < 1.1(\log D)^2$, since $p^{n_2} < D$. On applying this together with (57), we obtain $D < 10^{65}$. Thus, the assertion is proved.

The combination of Assertions 5, 6 and 7 yields the theorem.

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