

Some remarks on the geometry of Kropina spaces

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Abstract. Using the navigation data (h, W) of a Kropina space $(M, \alpha^2/\beta)$, we characterize weakly Berwald Kropina spaces and Berwald Kropina spaces by means of the Killing vector field W and the parallel vector field W , respectively.

Moreover, we show that the local 1-parameter group of local Finslerian isometries on $(M, \alpha^2/\beta)$ coincides with the local 1-parameter group of local Riemannian isometries on (M, h) .

1. Introduction

Finsler metrics generalize Riemannian metrics. The most natural Finsler structures are those obtained by deformations of Riemannian metrics. Randers metrics are the most famous metrics of this type because of their relation with the Zermelo's navigation problem. On the other hand, recently it was shown that Kropina metrics also give solutions to the Zermelo's navigation problem ([YS] and references therein). This suggests that Kropina spaces are Finsler spaces with rich geometrical properties and that their geometry is worth more investigations.

In the present paper we contribute with three remarks to the geometry of Kropina spaces.

The first remark concerns the relation between weakly Berwald, Berwald and Kropina spaces. We prove that the set of weakly Berwald Kropina metrics coincides with the set of strong Kropina metrics defined in the paper (Theorem 4.2). Moreover, a Kropina space is a Berwald space if and only if the wind vector field

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W is parallel with respect to the Riemannian metric h (Theorem 4.3). Nevertheless, in the Kropina case, Landsberg and Berwald spaces coincide.

The second remark is about the characterization of Kropina metrics of p -scalar curvature, i.e., Kropina metrics whose scalar flag curvature is a function of position alone. We give a characterization of these spaces in terms of navigation data (Theorem 5.1). Moreover, we show that a Kropina space of p -scalar curvature is a Berwald space if and only if its flag curvature vanishes (Corollary 5.2).

The third remark is on the isometry group of a Kropina metric. In general, it is difficult to determine the isometry group of a Finsler structure. However, in the case of strong Kropina metrics we prove that the 1-parameter group of local Finslerian isometries on $(M, \alpha^2/\beta)$ coincides with the 1-parameter group of local Riemannian isometries on (M, h) .

2. Preliminaries

Throughout the paper, M will be an n -dimensional (smooth) manifold, where $n \geq 2$. The tangent bundle of M is $\tau : TM \rightarrow M$, and $\overset{\circ}{\tau} : \overset{\circ}{TM} \rightarrow M$ is the bundle of nonzero tangent vectors to M . The $C^\infty(M)$ -module of (smooth) vector fields on M is denoted by $\mathfrak{X}(M)$. The dual $\mathfrak{X}^*(M)$ of $\mathfrak{X}(M)$ is the module of 1-forms on M . Any one-form b on M induces a smooth function β on TM given by

$$\beta(v) = b_{\tau(v)}(v), \quad v \in TM. \tag{2.1}$$

If f is a smooth function on M and $f^c(v) := v(f)$ for all $v \in TM$, then f^c is a smooth function on TM , called the complete lift of f . The derivative of a smooth map $\varphi : M \rightarrow N$ is denoted by φ_* is a vector bundle homomorphism from TM to TN .

Suppose (for simplicity) that X is a complete vector field on M , and let $(\varphi_t)_{t \in \mathbb{R}}$ (or (φ_t) for short) be the one-parameter group generated by X . Then the derivatives $(\varphi_t)_*$ form a one-parameter group on TM , whose generator is called the complete lift of X and denoted by X^c .

For coordinate descriptions, we choose a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M , and consider the induced chart $(\tau^{-1}(\mathcal{U}), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ on TM , where

$$x^i := u^i \circ \tau, \quad y^i(v) := v(u^i) \quad (v \in \tau^{-1}(\mathcal{U})).$$

If $X \in \mathfrak{X}(M)$, $b \in \mathfrak{X}^*(M)$ and $X \underset{(\mathcal{U})}{=} X^i \frac{\partial}{\partial u^i}$, $b \underset{(\mathcal{U})}{=} b_i du^i$, then

$$X^c \underset{(\mathcal{U})}{=} (X^i \circ \tau) \frac{\partial}{\partial x^i} + y^j \left(\frac{\partial X^i}{\partial u^j} \circ \tau \right) \frac{\partial}{\partial y^j}, \tag{2.2}$$

$$\beta \underset{(u)}{=} (b_i \circ \tau) y^i. \tag{2.3}$$

The coordinate expression of the complete lift of a smooth function f on M is

$$f^c \underset{(u)}{=} \left(\frac{\partial f}{\partial u^i} \circ \tau \right) y^i. \tag{2.4}$$

(Here, and throughout the paper, Einstein's summation convention is applied.) Observe that $y^i \frac{\partial \beta}{\partial y^i} = \beta$, therefore the function β is positive-homogeneous of degree 1.

We note finally that

$$X^c(f \circ \tau) = (Xf) \circ \tau, \quad X^c f^c = (Xf)^c \tag{2.5}$$

for all $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$.

3. The navigation data of a Kropina space

Let a be a Riemannian metric on M , written locally as $a = a_{ij} du^i \otimes du^j$. We denote the musical isomorphisms $\mathfrak{X}(M) \rightarrow \mathfrak{X}^*(M)$ and $\mathfrak{X}^*(M) \rightarrow \mathfrak{X}(M)$ by b and \sharp , respectively. Then

$$X^\flat := b(X) \underset{(u)}{=} a_{ij} X^j du^i \quad \text{if} \quad X \underset{(u)}{=} X^i \frac{\partial}{\partial u^i}, \tag{3.1}$$

$$b^\sharp := \sharp(b) \underset{(u)}{=} a^{ij} b_j \frac{\partial}{\partial u^i}, \quad (a^{ij}) := (a_{ij})^{-1} \quad \text{if} \quad b \underset{(u)}{=} b_i du^i. \tag{3.2}$$

With the help of the metric tensor a , we define a smooth function α^2 on TM by $\alpha^2(v) := a_{\tau(v)}(v, v)$, $v \in TM$. In terms of the induced coordinates, $\alpha^2 \underset{(u)}{=} (a_{ij} \circ \tau) y^i y^j$. Obviously, $y^i \frac{\partial \alpha^2}{\partial y^i} = 2\alpha^2$, therefore α^2 is positive-homogeneous of degree 2.

Now let $b \in \mathfrak{X}^*(M)$, and suppose that

$$b_p \neq 0 \in T_p^*M \quad \text{for all } p \in M. \tag{3.3}$$

(Clearly, this condition imposes topological restrictions on the manifold M .)

Let $A := \{v \in TM \mid \beta(v) > 0\}$. Then A is a conic domain in TM (see [JS], Definition 3.1) and

$$F := \frac{\alpha^2}{\beta} : A \subset TM \rightarrow \mathbb{R}$$

is a conic Finsler metric, called a *Kropina metric* for M . A *Kropina space* is a manifold M together with a Kropina metric for M .

Define the function

$$b^2 := a(b^\sharp, b^\sharp). \tag{3.4}$$

Using (3.2) we find that

$$b^2 \underset{(U)}{=} a^{ij} b_i b_j \quad \text{if} \quad b \underset{(U)}{=} b_i du^i.$$

By condition (3.3), b^2 never vanishes, so we can introduce a smooth function κ on M by

$$e^\kappa := \exp \circ \kappa = \frac{4}{b^2}. \tag{3.5}$$

Then

$$h := (\exp \circ \kappa)a \tag{3.6}$$

is a new Riemannian metric on M , obtained by a conformal change of a . Let, finally,

$$W := \frac{1}{2} b^\sharp. \tag{3.7}$$

Then the pair (h, W) is called the *navigation data* of the Kropina metric $\frac{\alpha^2}{\beta}$. We note that W is a unit vector field with respect to the Riemannian metric h . Indeed,

$$h(W, W) \stackrel{(3.7)}{=} \frac{1}{4} h(b^\sharp, b^\sharp) \stackrel{(3.6)}{=} \frac{1}{4} (\exp \circ \kappa)a(b^\sharp, b^\sharp) \stackrel{(3.4)}{=} \frac{1}{4} (\exp \circ \kappa)b^2 \stackrel{(3.5)}{=} 1.$$

Concerning the flag curvature of a Kropina space, we have the following important result.

Theorem 3.1 ([YO2], [YO3]). *A Kropina space $(M, \frac{\alpha^2}{\beta})$ with navigation data (h, W) is of constant flag curvature K if and only if the following two conditions hold:*

- (a) W is a Killing vector field on (M, h) ,
- (b) the Riemannian space (M, h) is of constant sectional curvature K . △

This motivates the following definition.

Definition 3.1. A Kropina space $(M, \frac{\alpha^2}{\beta})$ with navigation data (h, W) is called a *strong Kropina space* if W is a Killing vector field on (M, h) .

4. Berwald and weakly Berwald Kropina spaces

In this section, we will consider Kropina spaces which are Berwald spaces or weakly Berwald spaces.

Following the general practice, we denote by G^i ($i \in \{1, \dots, n\}$) the coefficients of the canonical spray of any Finsler space (M, F) (with respect to the chosen chart). We put

$$G^i_j := \frac{\partial G^i}{\partial y^j}, \quad G_j^i_k := \frac{\partial G^i_j}{\partial y^k}, \quad G_j^i_{kl} := \frac{\partial G^i_j^k}{\partial y^l}.$$

We recall that the functions $G_j^i_k$ are the Christoffel symbols of the Berwald derivative, while the functions $G_j^i_{kl}$ are the components of the Berwald curvature \mathbf{B} of (M, F) . A Finsler space is a *Berwald space* if $\mathbf{B} = 0$, and it is a *weakly Berwald space*, if $\text{tr } \mathbf{B} = 0$, i.e.,

$$G_{ij} := G_i^r{}_{jr} = 0 \quad \text{for all } i, j \in \{1, \dots, n\}.$$

Thus every Berwald space is a weakly Berwald space as well. S. BÁC SÓ and the first author investigated the relation between the concept of weakly Berwald spaces and some other concepts in [BY]. Furthermore, K. Okubo, M. Matsumoto and the first author again have obtained a necessary and sufficient condition for a Kropina space to be a weakly Berwald space, resp. a Berwald space:

Theorem 4.1 ([YOM]). *With the same notation as above, let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space. Let*

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}), \quad s_i = a^{jk} b_j s_{ki},$$

where the semicolon ; denotes covariant derivative with respect to the Levi-Civita connection of (M, a) . Then

- (1) $(M, \frac{\alpha^2}{\beta})$ is a weakly Berwald space if and only if
 (wB) $r_{ij} = \gamma a_{ij}$, $\gamma \in C^\infty(M)$;
- (2) $(M, \frac{\alpha^2}{\beta})$ is a Berwald space if and only if
 (B) $r_{ij} = \gamma a_{ij}$ ($\gamma \in C^\infty(M)$) and $s_j b_i - s_i b_j = b^2 s_{ij}$. △

Remark 4.1. Let $\overset{a}{\nabla}$ denote the Levi-Civita connection of (M, a) . The functions r_{ij} are just the components of the symmetric part $\text{Sym } \overset{a}{\nabla} b$ of the covariant differential $\overset{a}{\nabla} b$ given by

$$\text{Sym } \overset{a}{\nabla} b(X, Y) := \frac{1}{2} \left((\overset{a}{\nabla}_X b)(Y) + (\overset{a}{\nabla}_Y b)(X) \right).$$

Similarly, the functions s_{ij} are the components of the skew-symmetric part $\text{Alt } \overset{a}{\nabla} b$ of $\overset{a}{\nabla} b$, defined analogously. Thus condition (wB) can be reformulated as follows:

$$\text{Sym } \overset{a}{\nabla} b = \gamma a, \quad \gamma \in C^\infty(M).$$

Next, we characterize the weakly Berwald and Berwald Kropina spaces by means of their navigation data.

Theorem 4.2. *Let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space with navigation data (h, W) . Then $(M, \frac{\alpha^2}{\beta})$ is a weakly Berwald space if and only if it is a strong Kropina space.*

PROOF. Let $h \underset{(\mathcal{U})}{=} h_{ij} du^i \otimes du^j$ and $(h^{ij}) := (h_{ij})^{-1}$. Denote by $\overset{h}{\nabla}$ the Levi-Civita connection of (M, h) , and let $(\overset{h}{\Gamma}_j{}^i{}_k)$ be the family of the Christoffel symbols of $\overset{h}{\nabla}$ with respect to the chart $(\mathcal{U}, (u^i)_{i=1}^n)$. In coordinate calculations below, we shall write \parallel for the covariant derivatives with respect to $\overset{h}{\nabla}$.

Suppose first that $(M, \frac{\alpha^2}{\beta})$ is a weakly Berwald space. Then from (wB) and (3.6) we obtain that

$$r_{ij} = \gamma e^{-\kappa} h_{ij}. \tag{4.1}$$

On the other hand, from Section 2 in [YO2] we have

$$r_{ij} = 2e^{-\kappa} \left(\mathbf{R}_{ij} - \frac{1}{2} W_r \bar{\kappa}^r h_{ij} \right), \tag{4.2}$$

where $W_r := W^i h_{ir}$, $\mathbf{R}_{ij} := \frac{1}{2}(W_{i\parallel j} + W_{j\parallel i})$, $\bar{\kappa}^r := h^{rs} \frac{\partial \kappa}{\partial u^s}$. Comparing (4.1) and (4.2), we get

$$\mathbf{R}_{ij} = \frac{1}{2}(\gamma + W_r \bar{\kappa}^r) h_{ij}. \tag{4.3}$$

Since $W \underset{(\mathcal{U})}{=} W^i \frac{\partial}{\partial u^i}$ is a unit vector field on (M, h) , we have

$$1 = \|W\|_h^2 = h(W, W) \underset{(\mathcal{U})}{=} W^i W^j h_{ij} = W^i W_i,$$

whence

$$W_{i\parallel k} W^i = 0 \tag{4.4}$$

for all $k \in \{1, \dots, n\}$. So, transvecting (4.3) with $W^i W^j$, we get

$$0 = \mathbf{R}_{ij} W^i W^j = \frac{1}{2}(\gamma + W_r \bar{\kappa}^r) \|W\|_h^2 = \frac{1}{2}(\gamma + W_r \bar{\kappa}^r),$$

and hence (4.3) reduces to

$$R_{ij} = 0. \tag{4.5}$$

Observe that the functions R_{ij} are just the components of the symmetric part of $\overset{h}{\nabla}W^b$ (where the flat operator b is taken with respect to the Riemannian metric h). Thus (4.5) implies (e.g., by Proposition 27 in Chapter 7 of [Pet]) that W is a Killing vector field on (M, h) .

Conversely, suppose that W is a Killing field. Then we have (4.5), and so from (4.2) we get

$$r_{ij} = -e^{-\kappa} W_r \bar{\kappa}^r h_{ij} \stackrel{(3.6)}{=} -W_r \bar{\kappa}^r a_{ij}.$$

Thus, by Theorem 4.1(1), $(M, \frac{\alpha^2}{\beta})$ is a weakly Berwald space. □

Remark 4.2. Suppose that $(M, \frac{\alpha^2}{\beta})$ is a *strong* Kropina space. Then, by Theorem 5 in [YO2], the spray coefficients G^i of the canonical spray of $(M, \frac{\alpha^2}{\beta})$ can be expressed as

$$2G^i = (\overset{h}{\Gamma}_j^i{}_k \circ \tau) y^j y^k - 2 \frac{\alpha^2}{\beta} (S^i{}_j \circ \tau) y^j, \tag{4.6}$$

where $S^i{}_j := h^{ir} S_{rj}$, and the functions S_{ij} are the components of the skew-symmetric part $\text{Alt}(\overset{h}{\nabla}W^b)$ of $\overset{h}{\nabla}W^b$, i.e.,

$$S_{ij} = \frac{1}{2}(W_{i||j} - W_{j||i}). \tag{4.7}$$

From (4.6), the Christoffel symbols of the Berwald derivative of $(M, \frac{\alpha^2}{\beta})$ are

$$\begin{aligned} G_j^i{}_k &= \overset{h}{\Gamma}_j^i{}_k \circ \tau - \frac{1}{W_0} (h_{jk} \circ \tau) S^i{}_0 + \frac{h_{0j}}{(W_0)^2} (W_k \circ \tau) S^i{}_0 + \frac{h_{0k}}{(W_0)^2} (W_j \circ \tau) S^i{}_0 \\ &\quad - \frac{h_{00}}{(W_0)^3} ((W_j W_k) \circ \tau) S^i{}_0 - \frac{h_{0j}}{W_0} (S^i{}_k \circ \tau) - \frac{h_{0k}}{W_0} (S^i{}_j \circ \tau) \\ &\quad + \frac{1}{2} \frac{h_{00}}{(W_0)^2} (W_j S^i{}_k) \circ \tau + \frac{1}{2} \frac{h_{00}}{(W_0)^2} (W_k S^i{}_j) \circ \tau, \end{aligned} \tag{4.8}$$

where we used the following abbreviations:

$$h_{0i} := (h_{ij} \circ \tau) y^j, \quad h_{00} := (h_{ij} \circ \tau) y^i y^j, \quad S^i{}_0 := (S^i{}_j \circ \tau) y^j, \quad W_0 := (W_i \circ \tau) y^i.$$

Putting here $i = k := r$, after some calculations we find that $G_j^r{}_r = \overset{h}{\Gamma}_j^r{}_r \circ \tau$, whence

$$G_{ij} := G_i^r{}_{jr} = \frac{\partial}{\partial y^i} (\overset{h}{\Gamma}_j^r{}_r \circ \tau) = 0.$$

Thus we proved again that a strong Kropina space is a weakly Berwald space.

Theorem 4.3. *Let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space with navigation data (h, W) . Then $(M, \frac{\alpha^2}{\beta})$ is a Berwald space if and only if the vector field W is parallel with respect to the Levi–Civita connection of (M, h) . In this case the Christoffel symbols of the Berwald derivative of $(M, \frac{\alpha^2}{\beta})$ are the vertical lifts of the Christoffel symbols of $\overset{h}{\nabla}$, i.e., $G_j^i{}_k = \overset{h}{\Gamma}_j^i{}_k \circ \tau$, for any indices i, j, k . In other words, the geodesics of $(M, \frac{\alpha^2}{\beta})$ coincide with the geodesics of (M, h) .*

PROOF. Suppose first that $(M, \frac{\alpha^2}{\beta})$ is a Berwald space. Then, by the second relation of condition (B) in Theorem 4.1, $s_j b_i - s_i b_j = b^2 s_{ij}$. This can be manipulated to obtain

$$S_{ij} = W_i S_j - W_j S_i, \tag{4.9}$$

where $S_i := W^r S_{ri} \stackrel{(4.7)}{=} \frac{1}{2} W^r (W_{r\parallel i} - W_{i\parallel r})$. Since, in particular, $(M, \frac{\alpha^2}{\beta})$ is a weakly Berwald space, W is a Killing vector field by Theorem 4.2. Thus

$$R_{ij} := \frac{1}{2} (W_{i\parallel j} + W_{j\parallel i}) = 0,$$

whence

$$S_{ij} := \frac{1}{2} (W_{i\parallel j} - W_{j\parallel i}) = W_{i\parallel j}.$$

So it follows that $S_i = W^r W_{r\parallel i} \stackrel{(4.4)}{=} 0$, and from (4.9) we obtain that $S_{ij} = 0$. Relations $R_{ij} = 0$ and $S_{ij} = 0$ imply that $W_{i\parallel j} = 0$, as we claimed.

Conversely, suppose that W is a parallel vector field on the Riemannian space (M, h) . Then $R_{ij} = 0$, so W is a Killing vector field, and from Theorem 4.2 we conclude that $(M, \frac{\alpha^2}{\beta})$ is a weakly Berwald space. Since (4.9) holds automatically, the second relation in condition (B) is also valid, therefore $(M, \frac{\alpha^2}{\beta})$ is actually a Berwald space. We have, in fact, $S_{ij} = 0$, whence $S^i{}_j = 0$. Thus (4.8) reduces to $G_j^i{}_k = \overset{h}{\Gamma}_j^i{}_k \circ \tau$, which concludes the proof. \square

Remark 4.3. Let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space with navigation data (h, W) . If $(M, \frac{\alpha^2}{\beta})$ is a Berwald space, then by the preceding theorem, $W_{i\parallel j} = 0$. This implies that $(M, \frac{\alpha^2}{\beta})$ is a strong Kropina space. Thus the set of Berwaldian Kropina spaces is contained in the set of strong Kropina spaces.

5. Kropina spaces of p -scalar flag curvature

In this section we consider Kropina spaces whose scalar flag curvature depends only on the position.

Definition 5.1. Let $A \subset TM$ be a conic domain and (M, F) a conic Finsler space, with fundamental function $F : A \rightarrow \mathbb{R}$, of scalar flag curvature $K : A \rightarrow \mathbb{R}$. If K ‘depends only on the position’, i.e., it is of the form $K = k \circ \tau \upharpoonright A$, $k \in C^\infty(M)$, then (M, F) , with fundamental function $F : A \rightarrow \mathbb{R}$, is called to be of *p-scalar flag curvature*.

It turns out from the proof of Theorem 4 in [YO2] that its conclusion remains true if the assumption that ‘ $(M, \frac{\alpha^2}{\beta})$ is of constant flag curvature K ’ is replaced by ‘ $(M, \frac{\alpha^2}{\beta})$ is of *p-scalar flag curvature* $K = k \circ \tau \upharpoonright A$ ’. Thus we obtain

Theorem 5.1. *A Kropina space $(M, \frac{\alpha^2}{\beta})$ with navigation data (h, W) is of p-scalar flag curvature $K = k \circ \tau \upharpoonright A$ if and only if the following two conditions hold:*

- (a) *The vector field W is a Killing field.*
- (b) *The Riemannian space (M, h) is of sectional curvature k .*

In this case, $k(p) \geq 0$ for all $p \in M$.

PROOF. We have only to show the last assertion. From (4.4) we get

$$W_{r\|i}W^r_{\|j} + W_{r\|i\|j}W^r = 0. \tag{5.1}$$

Since W is a Killing vector field, we have

$$W_{i\|j\|k} = W_r \overset{h}{R}_k{}^r{}_{ij}, \tag{5.2}$$

where the functions $\overset{h}{R}_k{}^r{}_{ij}$ are the components of the curvature tensor of (M, h) (see, e.g., [Ha], (10.2) or [YO2], Lemma 4). They can be written in the form

$$\overset{h}{R}_k{}^r{}_{ji} = k(h_{kj}\delta^r{}_i - h_{ki}\delta^r{}_j), \tag{5.3}$$

because (M, h) is of scalar sectional curvature k . From (5.1)–(5.3) we obtain

$$\begin{aligned} W_{r\|i}W^r_{\|j} &= -W_{r\|i\|j}W^r = -W_s \overset{h}{R}_j{}^s{}_{ri}W^r = -kW_s(h_{jr}\delta^s{}_i - h_{ji}\delta^s{}_r)W^r \\ &= k(h_{ji} - W_jW_i), \end{aligned}$$

whence

$$h_{rs}W^s_{\|i}W^r_{\|j} = k(h_{ij} - W_iW_j). \tag{5.4}$$

Composing both sides of (5.4) with τ and transvecting by $y^i y^j$, we find that

$$(h_{rs} \circ \tau)W^r_{\|0}W^s_{\|0} = (k \circ \tau)(h_{00} - (W_0)^2), \tag{5.5}$$

where $W^r_{\parallel 0} := (W^r_{\parallel i} \circ \tau)y^i$.

Let $p \in \mathcal{U}$ and $v \in T_pM \setminus \{0\}$. Then

$$\begin{aligned} ((h_{rs} \circ \tau)W^r_{\parallel 0}W^s_{\parallel 0})(v) &= h_{rs}(p)W^r_{\parallel 0}(v)W^s_{\parallel 0}(v) \\ &= h_p(\overset{h}{\nabla}_v W, \overset{h}{\nabla}_v W) = \|\overset{h}{\nabla}_v W\|_h^2 > 0 \end{aligned}$$

if $\overset{h}{\nabla}_v W \neq 0$. Similarly, if v and $W(p)$ are linearly independent,

$$\begin{aligned} h_{00}(v) &= ((h_{ij} \circ \tau)y^i y^j)(v) = h_{ij}(p)y^i(v)y^j(v) = h_p(v, v) = \|v\|_h^2, \\ (W_0)^2(v) &= ((W_i \circ \tau)(y^i))^2(v) = (W_i(p)y^i(v))^2 = (h_{ir}(p)y^i(v)W^r(p))^2 \\ &= (h_p(v, W(p)))^2 < \|v\|_h^2 \|W(p)\|_h^2 = \|v\|_h^2 = h_{00}(v), \end{aligned}$$

therefore the function $h_{00} - (W_0)^2$ is positive outside $\text{span}(W)$. If $(\overset{h}{\nabla}W)_p \neq 0$, its kernel is a proper subspace of T_pM , hence we can choose a vector $v \in T_pM$ so that $\|\overset{h}{\nabla}_v W\|$ and $(h_{00} - (W_0)^2)(v)$ are both positive. Thus it follows from (5.5) that

$$k = 0 \text{ if } \overset{h}{\nabla}W = 0; \quad k > 0 \text{ if } \overset{h}{\nabla}W \neq 0.$$

This concludes the proof. □

Corollary 5.2. *Let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space of p -scalar flag curvature K with navigation data (h, W) . Then $(M, \frac{\alpha^2}{\beta})$ is a Berwald space, i.e., $\overset{h}{\nabla}W = 0$, if and only if $K = 0$. □*

Remark 5.1. (a) If in Theorem 5.1 M is connected and $\dim M > 2$, then it follows from Schur’s lemma that the function k is constant.

(b) By a theorem of M. Berger, every Killing vector field on a compact even-dimensional Riemannian space of positive sectional curvature has a zero (see, e.g., [Pet], p. 169). From this and from Theorem 5.1 we conclude that there exists no Kropina metric of p -scalar flag curvature for an even-dimensional compact manifold of positive sectional curvature.

6. Isometries of a Kropina space

We continue to assume that $(M, \frac{\alpha^2}{\beta})$ is a Kropina space with navigation data (h, W) . In this section we consider the local one-parameter group (φ_t) of W and discuss the relations between the following properties:

- (1) The transformations φ_t are local Riemannian isometries on (M, h) .
- (2) The transformations φ_t are local Finslerian isometries on $(M, \frac{\alpha^2}{\beta})$.

First we recall what we mean by a Finslerian isometry.

Definition 6.1. Let (M, F) be a Finsler space. A smooth transformation φ of M is called a *local Finslerian isometry* if its derivative preserves the Finslerian norms of the tangent vectors, i.e., $F(v) = F((\varphi_*)_p(v))$ for all $p \in M, v \in T_pM$. A local Finslerian isometry is a *Finslerian isometry* (or an *isometry* for short) if it is a diffeomorphism.

Remark 6.1. It is easy to see that if φ is a local Finslerian isometry of (M, F) , then for each point p in M there is a neighborhood \mathcal{U} of p and a neighborhood \mathcal{V} of $\varphi(p)$ such that $\varphi \upharpoonright \mathcal{U} : \mathcal{U} \rightarrow \mathcal{V}$ is a Finslerian isometry. This justifies the term ‘local’.

Lemma 6.1. *Given a Finsler space (M, F) , consider the vector field X on M , and let (φ_t) be the local one-parameter group of X . Then the following assertions are equivalent:*

- (i) *The transformations φ_t are Finslerian isometries on their domains.*
- (ii) $X^c F = 0$.
- (iii) $\frac{\partial F}{\partial x^i}(X^i \circ \tau) + \frac{\partial F}{\partial y^i}(\frac{\partial X^i}{\partial u^j} \circ \tau)y^j = 0$.

PROOF. Since X^c is generated by $((\varphi_t)_*)$, we have

$$X^c F = \mathcal{L}_{X^c} F = \lim_{t \rightarrow 0} \frac{1}{t}(F \circ (\varphi_t)_* - F),$$

whence the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is clear from (2.2). □

Definition 6.2. Let (M, F) be a Finsler space. A vector field X on M is called a *Killing vector field* of (M, F) if it satisfies one (and hence all) of the conditions (i)–(iii) in Lemma 6.1.

Corollary 6.2. *Let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space with navigation data (h, W) . Then W is a Killing vector field of $(M, \frac{\alpha^2}{\beta})$ if and only if*

$$2\beta W^c \alpha - \alpha W^c \beta = 0. \tag{6.1}$$

PROOF. $W^c \frac{\alpha^2}{\beta} = \frac{2\alpha}{\beta} W^c \alpha - \frac{\alpha^2}{\beta^2} W^c \beta$, so $W^c \frac{\alpha^2}{\beta} = 0$ if and only if (6.1) is satisfied. □

Lemma 6.3 (cf. formulae (26) and (27) in [LCM]). *Let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space. Then for any vector field X on M ,*

$$X^c \alpha \stackrel{(u)}{=} \frac{1}{2\alpha} (X_{i;j} + X_{j;i}) \circ \tau y^i y^j, \quad X^c \beta \stackrel{(u)}{=} (b_{i;j} X^j + b^j X_{j;i}) \circ \tau y^i, \quad (6.2)$$

where the semicolon means covariant derivative with respect the Levi-Civita connection $\overset{a}{\nabla}$ of (M, a) , and

$$X_i = a_{ik} X^k \text{ if } X \upharpoonright \mathcal{U} = X^k \frac{\partial}{\partial u^k}; \quad b^i = a^{il} b_l \text{ if } b \upharpoonright \mathcal{U} = b_l du^l.$$

PROOF. We verify the second equality in (6.2), the first can be checked similarly. We have

$$\begin{aligned} X^c \beta \stackrel{(u)}{=} & \stackrel{(2.3)}{=} X^c ((b_i \circ \tau) y^i) \stackrel{(2.5)}{=} (X b_i) \circ \tau y^i + (b_i \circ \tau) (X u^i)^c \\ & = \left(\frac{\partial b_i}{\partial u^j} X^j \right) \circ \tau y^i + (b_j \circ \tau) (X^j)^c \stackrel{(2.4)}{=} \left(\frac{\partial b_i}{\partial u^j} X^j + b_j \frac{\partial X^j}{\partial u^i} \right) \circ \tau y^i. \end{aligned}$$

Now let $\overset{a}{\Gamma}_j^i k$ be the Christoffel symbols of $\overset{a}{\nabla}$. Then

$$\begin{aligned} \frac{\partial b_i}{\partial u^j} X^j &= b_{i;j} X^j + \overset{a}{\Gamma}_i^k j X^j b_k, \\ b_j \frac{\partial X^j}{\partial u^i} &= X^j_{;i} b_j - \overset{a}{\Gamma}_i^j l X^l b_j = (a^{jk} X_k)_{;i} b_j - \overset{a}{\Gamma}_i^j l X^l b_j \\ &= a^{jk} X_{k;i} b_j - \overset{a}{\Gamma}_i^j l X^l b_j = X_{k;i} b^k - \overset{a}{\Gamma}_i^j l X^l b_j = b^j X_{j;i} - \overset{a}{\Gamma}_i^k j X^j b_k, \end{aligned}$$

so we obtain the desired equality. □

Proposition 6.4. *Let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space with navigation data (h, W) . Then W is a Killing vector field of $(M, \frac{\alpha^2}{\beta})$ if and only if*

$$\text{Sym } \overset{a}{\nabla} b = \frac{1}{2} \lambda a, \quad \lambda \in C^\infty(M).$$

PROOF. Let, as above,

$$a \stackrel{(u)}{=} a_{ij} du^i \otimes du^j, \quad W \stackrel{(u)}{=} W^i \frac{\partial}{\partial u^i}, \quad b \stackrel{(u)}{=} b_i du^i.$$

Since $W = \frac{1}{2} b^\sharp$, we have

$$W^i = \frac{1}{2} b^i = \frac{1}{2} a^{ij} b_j, \quad W_i = \frac{1}{2} b_i = \frac{1}{2} a_{ij} b^j.$$

Thus, by (6.1) and (6.2), it follows that W is a Killing vector field of $(M, \frac{\alpha^2}{\beta})$ if and only if

$$\beta(b_{i;j} + b_{j;i}) \circ \tau y^i y^j - \alpha^2(b_{i;j} b^j + b^j b_{j;i}) \circ \tau y^i = 0. \tag{6.3}$$

The function α^2 comes from the positive definite metric tensor a , so there is no one form $c \in \mathfrak{X}^*(M)$ with induced function $\gamma \in C^\infty(TM)$ such that $\alpha^2 = \beta\gamma$. Hence (6.3) holds if and only if there exists a smooth function λ on M such that

$$(b_{i;j} b^j + b^j b_{j;i}) \circ \tau y^i = \lambda\beta \quad \text{and} \quad (b_{i;j} + b_{j;i}) \circ \tau y^i y^j = (\lambda \circ \tau)\alpha^2.$$

Equivalently,

$$b_{i;j} b^j + b^j b_{j;i} = \lambda b_i \quad \text{and} \quad b_{i;j} + b_{j;i} = \lambda a_{ij}. \tag{6.4}$$

Now it can be readily seen that if $b = \underset{(U)}{b_i} du^i$ satisfies the second equation in (6.4) then it satisfies the first one. Since the functions $b_{i;j} + b_{j;i}$ are the components of $2\text{Sym} \overset{a}{\nabla} b$, this concludes the proof. \square

From Theorem 4.1, Theorem 4.2 and Proposition 6.4 we obtain the following result:

Theorem 6.5. *Let $(M, \frac{\alpha^2}{\beta})$ be a Kropina space with navigation data (h, W) . Then W is a Killing vector field of $(M, \frac{\alpha^2}{\beta})$ if and only if $(M, \frac{\alpha^2}{\beta})$ is a strong Kropina space.* \square

Remark 6.2. Let $(M, \frac{\alpha^2}{\beta})$ be a strong Kropina space. If M is a 3-dimensional compact manifold or a compact Lie-group, then the vector field W is complete, therefore the local Finslerian isometries of $(M, \frac{\alpha^2}{\beta})$ are actually Finslerian isometries. However, as we have already indicated, the existence of a strong Kropina structure on a compact manifold M implies topological restrictions on M ; for details we refer to [YS].

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