Publ. Math. Debrecen
85/1-2 (2014), 15–30
DOI: 10.5486/PMD.2013.5745

On Einstein Matsumoto metrics

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Abstract. This paper contributes to the study of Matsumoto metrics $F = \frac{\alpha^2}{\alpha-\beta}$ with β of constant length related to α, where the α is a Riemannian metric and the β is a one form. It is shown that such a Matsumoto metric F is an Einstein metric if and only if α is Ricci-flat and β is parallel with respect to α. A nontrivial example of Ricci flat Matsumoto metrics is given.

1. Introduction

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. F is called an Einstein metric with Einstein scalar σ if its Ricci curvature Ric satisfies

$$\mathrm{Ric} = \sigma F^2, \tag{1.1}$$

where $\sigma = \sigma(x)$ is a scalar function on M. In particular, F is said to be Ricci constant (resp. Ricci flat) if $\sigma = \text{const.}$ (resp. $\sigma = 0$) in (1.1). ([2], [4]).

An important class of Finsler metrics is so called (α, β)-metrics, which are iteratively appearing in physical studies, and are expressed in the form of $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. (α, β)-metrics with $\phi = 1 + s$ are called Randers metrics. D. Bao and C. Robles have characterized Einstein Randers metrics, and shown that every Einstein Randers metric is necessarily Ricci constant in dimension $n \geq 3$. When $n = 3$, a Randers metric is Einstein if and only if it is of constant flag curvature, see [4].

Mathematics Subject Classification: 53B40, 53C60.
Key words and phrases: Einstein metric, Matsumoto metric, Ricci flat, S-curvature.
Supported partially by NNSFC (No. 11171297).

For a non-Randers (α,β)-metric F with a polynomial function $\phi(s)$ of degree greater than 2, it was proved that F is an Einstein metric if and only if it is Ricci-flat ([5]). An (α,β)-metric with $\phi = s^{-1}$ is called a Kropina metric. It was shown that a Kropina metric $F = \frac{\alpha^2}{\beta}$ is an Einstein metric if and only if h is an Einstein metric and W is a unit Killing form with respect to h, where (h, W) is the navigation data of F ([13]).

The Matsumoto metric is an interesting (α,β)-metric with $\phi = 1/(1-s)$, introduced by using gradient of slope, speed and gravity in [7]. This metric formulates the model of a Finsler space. Many authors ([1], [7], [8], etc.) have studied this metric by different perspectives. M. Rafie-Rad, etc., discussed Einstein Matsumoto metrics recently. However they treated $b := \|\beta_x\|_\alpha$, in a_i for $i = 0, \dots, 14$, as constant (see [10]). In [11] B. Rezaei, etc. discussed the Einstein Matsumoto metric under the assumption that β is a constant Killing form with respect to α, i.e., β satisfies the Killing equation and has constant length with respect to α. While the results of [11] are not true. Here we generalize their study.

The purpose of the present paper is to study Einstein Matsumoto metrics $F = \frac{\alpha^2}{\alpha-\beta}$, where β has constant length with respect to α. And main results are as follows.

Theorem 1.1. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \geq 3$. Suppose that the length of β with respect to α is constant. Then F is an Einstein metric if and only if α is Ricci-flat and β is parallel with respect to α. In this case, F is Ricci-flat.*

Theorem 1.2. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \geq 3$. Suppose that $\beta^\sharp$ dual to β is a homothetic vector field related to α, i.e., $r_{00} = c\alpha^2$, where $c =$ constant. Then F is an Einstein metric if and only if α is Ricci-flat and β is parallel with respect to α. In this case, F is Ricci-flat.*

For the S-curvature of the Matsumoto metric with respect to the Busemann–Hausdorff volume form ([3]), we have following

Theorem 1.3. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \geq 2$. Then S-curvature vanishes if and only if β is a constant Killing form.*

From above theorems, we can easily get the following

Corollary 1.1. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \geq 3$. Suppose F is an Einstein metric. Then*

S-curvature vanishes if and only if α is Ricci-flat and β is parallel with respect to α. In this case, F is Ricci-flat.

The content of this paper is arranged as follows. In §2 we introduce notations and conventions, and give the formula of the spray coefficients of Matsumoto metrics. The necessary conditions for Matsumoto metrics to be Einstein are given in §3. In §4, we first give the necessary and sufficient conditions for Matsumoto metrics to be Einstein under the hypothesis that β is a constant Killing form with respect to α. Then, by using it, Theorem 1.1 and Theorem 1.2 are proved. A nontrivial example of Ricci flat Matsumoto metrics is shown. By the way, we characterize Matsumoto metrics F with constant Killing form β, which are of constant flag curvature. In §5 we investigate the S-curvature of Matsumoto metrics and Theorem 1.3 is proved.

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M and G^i the geodesic coefficients of F, which are defined by

$$G^i := \frac{1}{4} g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}.$$

For any $x \in M$ and $y \in T_x M \setminus \{0\}$, the Riemann curvature $\mathbf{R}_y := R^i{}_k \frac{\partial}{\partial x^i} \bigotimes dx^k$ is defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

Ricci curvature is the trace of the Riemann curvature, which is defined by

$$\mathrm{Ric} := R^k{}_k. \tag{2.2}$$

A Finsler metric F is called an Einstein metric with Einstein scalar σ if

$$\mathrm{Ric} = \sigma F^2, \tag{2.3}$$

where $\sigma = \sigma(x)$ is a scalar function on M. In particular, F is said to be Ricci constant (resp. Ricci flat) if F satisfies (2.3) where $\sigma = \text{const.}$ (resp. $\sigma = 0$).

By definition, an (α, β)-metric on M is expressed in the form $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a positive definite Riemannian metric and $\beta =$

$b_i(x)y^i$ is a 1-form. It is known that (α,β)-metric with $b := \|\beta_x\|_\alpha < b_0$ is a Finsler metric if and only if $\phi = \phi(s)$ is a positive smooth function on an open interval $(-b_0, b_0)$ satisfying the following condition

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad \forall |s| \le b < b_0.$$

Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \tag{2.4}$$

where "$|$" denotes the horizontal covariant derivative with respect to α. Denote

$$\begin{aligned}
&r^i{}_j := a^{ik} r_{kj}, \quad r_j := b^i r_{ij}, \qquad r := r_{ij} b^i b^j = b^j r_j, \quad r^i := a^{ij} r_j \\
&s^i{}_j := a^{ik} s_{kj}, \quad s_j := b^i s_{ij}, \qquad s^i := a^{ij} s_j, \\
&r_{i0} := r_{ij} y^j, \qquad r_{00} := r_{ij} y^i y^j, \quad r_0 := r_i y^i, \\
&s_{i0} := s_{ij} y^j, \qquad s^i{}_0 := s^i{}_j y^j, \qquad s_0 := s_i y^i,
\end{aligned} \tag{2.5}$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij} b_j$.

Let G^i and $\bar{G}^i$ be the geodesic coefficients of F and α, respectively. Then we have the following

Lemma 2.1 ([5]). *For an (α,β)-metric $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, the geodesic coefficients G^i are given by*

$$G^i = \bar{G}^i + \alpha Q s^i{}_0 + \Psi(r_{00} - 2\alpha Q s_0) b^i + \frac{1}{\alpha}\Theta(r_{00} - 2\alpha Q s_0) y^i, \tag{2.6}$$

where

$$\begin{aligned}
Q &:= \frac{\phi'}{\phi - s\phi'}, \\
\Psi &:= \frac{\phi''}{2[\phi - s\phi' + (b^2 - s^2)\phi'']}, \\
\Theta &:= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']}.
\end{aligned}$$

From now on, we consider a special kind of (α,β)-metrics which is called Matsumoto-metrics with the form

$$F = \alpha\phi(s), \quad \phi(s) := \frac{1}{1-s}, \quad s = \frac{\beta}{\alpha}.$$

Let b_0 be the largest number such that for any s with $|s| \le b < b_0$. From Lemma 3.1 in [6], we have known that F is a Finsler metric if and only if $b < b_0 = \frac{1}{2}$. So we always assume that ϕ satisfies this condition.

Now we get the spray coefficients of Matsumoto metrics by using Lemma 2.1.

Proposition 2.1. *For the Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$, its geodesic coefficients are*

$$G^i = \bar{G}^i - \frac{\alpha}{2s-1}s^i{}_0 - \frac{1}{3s-2b^2-1}\left(\frac{2\alpha}{2s-1}s_0 + r_{00}\right)b^i + \frac{4s-1}{2(3s-2b^2-1)}\left(\frac{2\alpha}{2s-1}s_0 + r_{00}\right)\frac{y^i}{\alpha}. \tag{2.7}$$

Proof. For $\phi(s) = \frac{1}{1-s}$ and by a direct computation, we can obtain (2.7) from (2.6). □

For an (α, β)-metric, the form β is said to be a Killing (resp. closed) form if $r_{ij} = 0$ (resp. $s_{ij} = 0$). β is said to be a constant Killing form if it is a Killing form and has constant length with respect to α, equivalently $r_{ij} = 0$ and $s_i = 0$.

3. Einstein Matsumoto metrics

By Proposition 2.1, we can obtain following proposition.

Proposition 3.1. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \geq 2$. If F is an Einstein metric, then the followings hold*

1) *α is an Einstein metric, i.e., $\overline{\mathrm{Ric}} = \lambda\alpha^2$,*

2) *β is a conformal form with respect to α, i.e., $r_{00} = c\alpha^2$,*

where $\lambda = \lambda(x)$ and $c = c(x)$ are functions on M. And in this case,

$$T^i = -\frac{\alpha}{2s-1}s^i{}_0 - \frac{1}{3s-2b^2-1}\left(\frac{2\alpha}{2s-1}s_0 + c\alpha^2\right)b^i + \frac{4s-1}{2(3s-2b^2-1)}\left(\frac{2\alpha}{2s-1}s_0 + c\alpha^2\right)\frac{y^i}{\alpha}.$$

Proof. Let

$$G^i = \bar{G}^i + T^i,$$

where

$$T^i = -\frac{\alpha}{2s-1}s^i{}_0 - \frac{1}{3s-2b^2-1}(\frac{2\alpha}{2s-1}s_0 + r_{00})b^i + \frac{4s-1}{2(3s-2b^2-1)}\left(\frac{2\alpha}{2s-1}s_0 + r_{00}\right)\frac{y^i}{\alpha}.$$

Thus by (2.1), (2.2) and (2.6), the Ricci curvature of F is related to the Ricci curvature of α by

$$\mathrm{Ric} = \overline{\mathrm{Ric}} + 2T^k{}_{|k} - y^j T^k{}_{.\,k|j} + 2T^j T^k{}_{.\,j\,.\,k} - T^k{}_{.\,j} T^j{}_{.\,k}, \tag{3.1}$$

where $\overline{\mathrm{Ric}}$ denotes the Ricci curvature of α, "$|$" and "." denote the horizontal covariant derivative and vertical covariant derivative with respect to α, respectively(see Lemma 8.1.4 in [12]).

So the necessary and sufficient condition for the Matsumoto metric to be an Einstein metric is

$$\begin{aligned} 0 &= \mathrm{Ric} - \sigma(x)F^2 \\ &= \overline{\mathrm{Ric}} + 2T^k{}_{|k} - y^j T^k{}_{.\,k|j} + 2T^j T^k{}_{.\,j\,.\,k} - T^k{}_{.\,j} T^j{}_{.\,k} - \sigma(x)\frac{\alpha^2}{(1-s)^2}. \end{aligned} \tag{3.2}$$

Multiplying both sides of (3.2) by $\alpha^{12}(s-1)^2(2s-1)^4(3s-2b^2-1)^4$ and by a quite long computational procedure using Maple program, we obtain

$$0 = \sum_{i=0}^{14} t_i\, \alpha^i, \tag{3.3}$$

where

$$t_0 = 144(8n-11)\beta^{10} r_{00}^2,$$

$$t_1 = -96\{61n - 82 + (20n-26)b^2\}\beta^9 r_{00}^2 - 432(2n-3)\beta^{10} r_{00|0},$$

$$\begin{aligned} t_2 = {} & 12\{1085n - 1439 + (792n-1032)b^2 + 64(n-1)b^4\}\beta^8 r_{00}^2 + 1296\beta^{10}\overline{\mathrm{Ric}} \\ & - 288(8n-14)\beta^9 r_0 r_{00} + 864\beta^9 s_0 r_{00} + 72\{63n - 91 + (24n-32)b^2\}\beta^9 r_{00|0}, \end{aligned}$$

$$\begin{aligned} t_3 = {} & -864(2n-1)\beta^9 r_{0k} s^k{}_0 - 24\{697n - 926 + (852n-1144)b^2 + (152n-144)b^4\}\beta^7 r_{00}^2 \\ & - 3456(2+b^2)\beta^9\overline{\mathrm{Ric}} + 96\{118n - 205 + (32n-44)b^2\}\beta^8 r_{00} r_0 - 864\beta^9 r_{00} r^k{}_k \\ & - 48\{-16n + 97 + 16(n-1)b^2\}\beta^8 r_{00} s_0 - 864\beta^9 b^k r_{00|k} \\ & - 24\{435n - 602 + (354n-440)b^2 + (48n-56)b^4\}\beta^8 r_{00|0} \\ & + 864\beta^9 r_{0|0} - 432(2n-5)\beta^9 s_{0|0}, \end{aligned}$$

$$\begin{aligned} t_4 = {} & 144\{57n - 22 + (24n-8)b^2\}\beta^8 r_{0k} s^k{}_0 + 864(5+2b^2)\beta^8 (r_{00} r^k{}_k + b^k r_{00|k}) \\ & - 576\beta^8 r r_{00} + 3\{4606n - 6255 + (8400n - 12080)b^2 + (2480n-2272)b^4\}\beta^6 r_{00}^2 \\ & + 216(15+4b^2)(5+4b^2)\beta^8\overline{\mathrm{Ric}} - 32\{752n - 1301 + (440n-566)b^2 \\ & + 32(n-1)b^4\}\beta^7 r_{00} r_0 + 8\{-413n + 1322 + (376n-664)b^2 + 64(n-1)b^4\}\beta^7 r_{00} s_0 \end{aligned}$$

$$+ 4\{3473n - 4583 + (4512n - 5136)b^2 + (1320n - 1368)b^4 + 64(n-1)b^6\}\beta^7 r_{00|0}$$
$$- 864(5 + 2b^2)\beta^8 r_{0|0} + 576\beta^8 r_0^2 - 1152(2n-3)\beta^8 r_0 s_0$$
$$+ 72\{57n - 142 + (24n - 56)b^2\}\beta^8 s_{0|0} - 144(8n-21)\beta^8 s_0^2 - 1296\beta^9 s^k{}_{0|k}$$

and other coefficients of α are tedious, listed in [14].

If we replace y by $-y$, then $t_{2i}(-y) = t_{2i}(y)$ and $t_{2\bar{j}+1}(-y) = -t_{2\bar{j}+1}(y)$ for $i = 0, \dots, 7$ and $\bar{j} = 0, \dots, 6$. Hence (3.3) is equivalent to the following

$$\begin{cases} 0 = t_0 + t_2\alpha^2 + t_4\alpha^4 + t_6\alpha^6 + t_8\alpha^8 + t_{10}\alpha^{10} + t_{12}\alpha^{12} + t_{14}\alpha^{14}, \\ 0 = t_1 + t_3\alpha^2 + t_5\alpha^4 + t_7\alpha^6 + t_9\alpha^8 + t_{11}\alpha^{10} + t_{13}\alpha^{12}. \end{cases} \tag{3.4}$$

From the first equation of (3.4), we know that α^2 divides t_0. Since α^2 is an irreducible polynomial of y and β^{10} factors into ten linear terms, it must be the case that α^2 divides r_{00}^2. Thus $r_{00} = c\alpha^2$ for some function $c = c(x)$, i.e., β is a conformal form with respect to α. So it is easy to get

$$\begin{cases} r_{00} = c\alpha^2, \quad r_{ij} = ca_{ij}, \quad r_{0j} = cy_j, \quad r_i = cb_i, \quad r = cb^2, \quad r^i{}_j = c\delta^i{}_j, \\ r_{0k}s^k{}_0 = 0, \quad r_{0k}s^k = cs_0, \quad r_0 = c\beta, \quad s^k{}_0 r_k = cs_0, \\ r_{00|k} = c_k\alpha^2, \quad r_{00|0} = c_0\alpha^2, \quad r^k{}_k = nc, \quad r_{0|0} = c_0\beta + c^2\alpha^2, \end{cases} \tag{3.5}$$

where $y_i := a_{ij}y^j$, $c_k := \frac{\partial c}{\partial x^k}$ and $c_0 := c_k y^k$.

Plugging (3.5) into the first equation of (3.4) and removing the common factor α^2, we obtain

$$0 = \bar{t}_0 + \bar{t}_2\alpha^2 + \cdots + \bar{t}_{12}\alpha^{12},$$

where

$$\begin{cases} \bar{t}_0 = 1296\overline{\mathrm{Ric}}\beta^{10}, \\ \bar{t}_2 = 72(225 + 240b^2 + 48b^4)\beta^8\overline{\mathrm{Ric}} + 72(-151 - 56b^2 + 63n + 24nb^2)\beta^9 c_0 \\ \qquad -72(142 + 56b^2 - 57n - 24nb^2)\beta^8 s_{0|0} - 144(8n-21)\beta^8 s_0^2 \\ \qquad -288(8n-15)\beta^9 s_0 c - 1296\beta^9 s^k{}_{0|k} - 144(-21 + 8n)\beta^{10}c^2. \end{cases}$$

Due to the irreducibility of α, we have α^2 divides $\overline{\mathrm{Ric}}$, i.e., there exists some function $\lambda = \lambda(x)$ such that

$$\overline{\mathrm{Ric}} = \lambda\alpha^2. \tag{3.6}$$

It implies that α is an Einstein metric. It completes the proof of Proposition 3.1. □

Remark. For Riemann curvature and Ricci curvature of (α, β)-metrics, L. Zhou gave some formulas in [15]. However, Cheng has corrected some errors of his formulas in [5]. To avoid making such mistakes, we use the definitions (2.1) and (2.2) of Riemann curvature and Ricci curvatures to compute them.

4. The proofs of Theorem 1.1 and Theorem 1.2

Lemma 4.1. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \geq 3$. Suppose β is a constant Killing form, i.e., $r_{ij} = 0$, $s_i = 0$. Then F is an Einstein metric if and only if α is Ricci-flat and β is parallel with respect to α. In this case, F is Ricci-flat.*

Proof. If F is an Einstein metric, then (3.3) holds by Theorem 3.1. Removing the common factor $\alpha^2(\alpha - 2\beta)(3\beta - 2b^2\alpha - \alpha)^4$ from (3.3), we obtain

$$\begin{aligned}
0 = &-8\overline{\mathrm{Ric}}\beta^5 + 28\overline{\mathrm{Ric}}\beta^4\alpha + 2(-19\overline{\mathrm{Ric}} + 4{s^k}_{0|k}\beta)\beta^3\alpha^2 \\
&+ (-24{s^k}_{0|k}\beta + 25\overline{\mathrm{Ric}} + 2s_{0k}{s^k}_0)\beta^2\alpha^3 \\
&+ 2(-4\overline{\mathrm{Ric}} + 13{s^k}_{0|k}\beta - 2s_{0k}{s^k}_0 + {s^j}_k{s^k}_j\beta^2 + 4\sigma\beta^2)\beta\alpha^4 \\
&+ (\overline{\mathrm{Ric}} - 12{s^k}_{0|k}\beta + 2s_{0k}{s^k}_0 - 5{s^j}_k{s^k}_j\beta^2 - 12\sigma\beta^2)\alpha^5 \\
&+ 2(2{s^j}_k{s^k}_j\beta + {s^k}_{0|k} + 3\sigma\beta)\alpha^6 - ({s^j}_k{s^k}_j + \sigma)\alpha^7.
\end{aligned}$$

Obviously, the equation above is equivalent to

$$\begin{cases}
0 = -4\overline{\mathrm{Ric}}\beta^5 + (-19\overline{\mathrm{Ric}} + 4{s^k}_{0|k}\beta)\beta^3\alpha^2 \\
\quad +(-4\overline{\mathrm{Ric}} + 13{s^k}_{0|k}\beta - 2s_{0k}{s^k}_0 + {s^j}_k{s^k}_j\beta^2 + 4\sigma\beta^2)\beta\alpha^4 \\
\quad +(2{s^j}_k{s^k}_j\beta + {s^k}_{0|k} + 3\sigma\beta)\alpha^6, \\
0 = 28\overline{\mathrm{Ric}}\beta^4 + (-24{s^k}_{0|k}\beta + 25\overline{\mathrm{Ric}} + 2s_{0k}{s^k}_0)\beta^2\alpha^2 + (\overline{\mathrm{Ric}} \\
\quad -12{s^k}_{0|k}\beta + 2s_{0k}{s^k}_0 - 5{s^j}_k{s^k}_j\beta^2 - 12\sigma\beta^2)\alpha^4 - ({s^j}_k{s^k}_j + \sigma)\alpha^6.
\end{cases} \tag{4.1}$$

From the first equation of (4.1), we have $\overline{\mathrm{Ric}} = \lambda\alpha^2$ for some function $\lambda = \lambda(x)$ on M. Using the Bianchi identity, i.e., $b_{j|k|l} - b_{j|l|k} = b^s\bar{R}_{jskl}$, we obtain

$$ {s^l}_{k|l} = \lambda b_k. \tag{4.2}$$

Contracting both sides of (4.2) with b^k and y^k, respectively, we have

$$\begin{cases}
{s^k}_j{s^j}_k = -\lambda b^2, \\
{s^k}_{0|k} = \lambda\beta.
\end{cases} \tag{4.3}$$

Substituting (4.3) into (4.1) yields

$$0 = (-4\sigma\beta^2 + \lambda b^2\beta^2 + 6\lambda\beta^2 + 2s_{0k}{s^k}_0) + (-3\sigma + 3\lambda + 2\lambda b^2)\alpha^2, \tag{4.4}$$

and

$$0 = 2(2\lambda\beta^2 + s_{0k}s^k{}_0)\beta^2 + (-12\sigma\beta^2 + 2s_{0k}s^k{}_0 + 5\lambda b^2\beta^2 + 13\lambda\beta^2)\alpha^2 + \{-\sigma + (1+b^2)\lambda\}\alpha^4. \quad (4.5)$$

$3 \times (4.5) - (4.4) \times \alpha^2$ yields

$$0 = 6(2\lambda\beta^2 + s_{0k}s^k{}_0)\beta^2 + (-32\sigma\beta^2 + 4s_{0k}s^k{}_0 + 14\lambda b^2\beta^2 + 33\lambda\beta^2)\alpha^2 + \lambda b^2\alpha^4. \quad (4.6)$$

Since α^2 is irreducible polynomial of y, we assume that

$$2\lambda\beta^2 + s_{0k}s^k{}_0 = h\alpha^2 \quad (4.7)$$

holds for some function $h = h(x)$ on M. Differentiating both sides of (4.7) with respect to y^iy^j yields $4\lambda b_ib_j + s_{ik}s^k{}_j + s_{jk}s^k{}_i = 2ha_{ij}$. Then contracting it with b^ib^j gives $h = 2\lambda b^2$. Thus $s_{0k}s^k{}_0 = h\alpha^2 - 2\lambda\beta^2 = 2\lambda(b^2\alpha^2 - \beta^2)$. Plugging it into (4.6), we get

$$0 = (-32\sigma + 28\lambda b^2 + 25\lambda)\beta^2 + 9\lambda b^2\alpha^2. \quad (4.8)$$

Hence (4.8) is equivalent to

$$\begin{cases} 0 = -32\sigma + 28\lambda b^2 + 25\lambda, \\ 0 = 9\lambda b^2. \end{cases} \quad (4.9)$$

From the second equation of (4.9), we have $\lambda = 0$. Plugging it into the first equation of (4.9) gives $\sigma = 0$, i.e., F is Ricci-flat.

Moreover, substituting $\lambda = 0$ into (4.3) yields $s_{ij} = 0$. Together with $r_{ij} = 0$, we have $b_{i|j} = 0$, i.e., β is parallel with respect to α.

Converse is obvious. It completes the proof of Lemma 4.1. □

It is found that if β satisfies $s_i = 0$ or $r_{00} = c\alpha^2$, where $c =$ constant, then β is a constant Killing form when F is Einstein. Firstly, we prove the following

Theorem 4.1. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \geq 3$. Suppose β satisfies $s_i = 0$. Then F is an Einstein metric if and only if α is Ricci-flat and β is parallel with respect to α. In this case, F is Ricci-flat.*

Proof. If F is an Einstein metric, then $r_{00} = c\alpha^2$ and $\overline{\mathrm{Ric}} = \lambda\alpha^2$ hold by Theorem 3.1. Plugging $s_i = 0, r_{00} = c\alpha^2$ and $\overline{\mathrm{Ric}} = \lambda\alpha^2$ into the second equation of (3.4) yields

$$0 = 432(5-2n)\beta^{10}c_0 + \{96[48n - 123 + (12n-18)b^2]c^2\beta^9 - 3456(2+b^2)\lambda\beta^9 - 864b^kc_k\beta^9$$

$$- 24[435n - 602 + (354n - 440)b^2 + (48n - 56)b^4]c_0\beta^8\}\alpha^2 + \dots \quad (4.10)$$

From (4.10), we have that α^2 divides $\beta^9 c_0$. Since α^2 is irreducible polynomial of y, we have $c_0 = 0$, i.e., $c =$ constant Plugging it into the first equation of (3.4) yields

$$\begin{aligned}0 = {} & 144(21 - 8n)\beta^{10}c^2 \\ & - 8\beta^6(4832\beta^2c^2 - 81\beta^2 s^j{}_k s^k{}_j - 568nb^2\beta^2c^2 + 702\lambda\beta^2 + 128b^4\beta^2c^2 \\ & - 1177n\beta^2c^2 - 324\sigma\beta^2 - 64nb^4\beta^2c^2 + 432\lambda b^2\beta^2 + 1376b^2\beta^2c^2 \\ & + 270s_{0k}s^k{}_0 + 216b^2 s_{0k}s^k{}_0)\alpha^2 + \dots \end{aligned} \quad (4.11)$$

From (4.11), we get $c = 0$ for the division reason again. So β is a constant Killing form. Thus by Lemma 4.1, we get the necessary conditions.

Sufficiency is obvious. It completes the proof of Theorem 4.1. □

Proof of Theorem 1.1. If F is an Einstein metric, then $r_{00} = c\alpha^2$ by Theorem 3.1. Thus $r_k = cb_k$. Since the length of β with respect to α is constant, we have $0 = b^2{}_{|k} = 2(r_k + s_k)$, i.e., $r_k + s_k = 0$. Hence we get $cb_k + s_k = 0$. Contracting both sides of it with b^k yields that $c = 0$. Above all, $r_{00} = 0$ and $s_k = 0$, i.e., β is a constant Killing form. Thus by Lemma 4.1, we obtain that α is Ricci-flat and β is parallel with respect to α.

Conversely, if α is Ricci-flat and β is parallel with respect to α, then the length of β with respect to α is constant. Hence by Lemma 4.1, we get F is Einstein. It completes the proof of Theorem 1.1. □

Note that the condition that $s_k = 0$ in Theorem 4.1 is weaker than one that the length of β is constant (with respect to α) in Theorem 1.1.

Proof of Theorem 1.2. Assume that F is an Einstein metric and β is a homothetic form, i.e., $r_{00} = c\alpha^2$, where $c =$ constant. Then (3.4) holds, i.e.,

$$\begin{cases} 0 = t_0 + t_2\alpha^2 + t_4\alpha^4 + t_6\alpha^6 + t_8\alpha^8 + t_{10}\alpha^{10} + t_{12}\alpha^{12} + t_{14}\alpha^{14}, \\ 0 = t_1 + t_3\alpha^2 + t_5\alpha^4 + t_7\alpha^6 + t_9\alpha^8 + t_{11}\alpha^{10} + t_{13}\alpha^{12}, \end{cases} \quad (4.12)$$

where

$$t_0 = 144(8n - 11)c^2\beta^{10}\alpha^4,$$

and

$$\begin{aligned} t_2 = {} & 12\{1085n - 1439 + (792n - 1032)b^2 + 64(n-1)b^4\}c^2\beta^8\alpha^4 \\ & + 1296\lambda\beta^{10}\alpha^2 - 288(8n - 14)c^2\beta^{10}\alpha^2 + 864c\beta^9 s_0\alpha^2. \end{aligned}$$

For division reason again, we have α^2 can divide $\beta(f\beta + gs_0)$, where $f := 144(8n-11)c^2 + 1296\lambda - 288(8n-14)c^2, g := 864c$. So we have $f\beta + gs_0 = 0$. Differentiating both sides of it by y^i and contracting it with b^i yields $f = 0$. So $g = 0$ or $s_0 = 0$.

$g = 0$ implies that $c = 0$. Plugging it into $f = 0$ yields $\lambda = 0$ and ${s^k}_{0|k} = 0$. Substituting all these into (4.12) yields

$$t_1 = 0, t_3 = -432(2n-5)\beta^9 s_{0|0}, \tag{4.13}$$

and

$$t_0 = t_2 = 0, t_4 = 72\{57n - 142 + (24n-56)b^2\}\beta^8 s_{0|0} - 144(8n-21)\beta^8 s_0^2. \tag{4.14}$$

From (4.13), we know that α^2 can divide $s_{0|0}$. Plugging it into (4.14) yields that α^2 can divide s_0^2. That is $s_0^2 = k(x)\alpha^2$, which is a contraction unless $k(x) = 0$, i.e., $s_0 = 0$.

Above all, $s_0 = 0$. This is the just case in Theorem 4.1. It completes the proof of Theorem 1.2. □

Example. Let (M, α) be a 5-dimensional Riemanian manifold. Consider the Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$, $(1 \le i, j \le 5)$, which, in local coordinate (x^i), can be described as follows

$$(a_{ij}) = \begin{pmatrix} (x^4)^2 & 0 & 0 & 0 & 0 \\ 0 & (x^4)^2 & 0 & 0 & 0 \\ 0 & 0 & (x^4)^{-1} & 0 & 0 \\ 0 & 0 & 0 & x^4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x^4 > 0$. A direct computation shows that α is a non-Euclidean Ricci flat metric. And let $\beta = cy^5$, where c is a nonzero constant and $c^2 < \frac{1}{2}$. It is easy to check that such a β is parallel with respect to α, i.e., $b_{i|j} = 0$. Define $F = \frac{\alpha^2}{\alpha - \beta}$. Thus by Theorem 1.1, we conclude that $F = \frac{\alpha^2}{\alpha-\beta}$ is a Ricci-flat Matsumoto metric.

Theorem 4.2. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \ge 3$. Suppose the length of β with respect to α is constant. Then F is of constant flag curvature K if and only if the following conditions hold:*

(1) *α is a flat metric;*

(2) *β is parallel with respect to α.*

In this case, $K = 0$ and F is locally Minkowskian.

Proof. Suppose that F is of constant flag curvature K, i.e.,

$$R^i{}_k = K(F^2\delta^i{}_k - g_{ij}y^jy^k).$$

Then we have

$$\mathrm{Ric} = \sigma F^2, \qquad \sigma := (n-1)K, \tag{4.15}$$

which means that F is Einstein. Since the length of β with respect to α is constant, by Theorem 1.1, we get α is Ricci-flat and β is parallel with respect to α. In this case, F is Ricci-flat, which means that $K = 0$. So $G^i = \bar{G}^i$ and $R^i{}_k = \bar{R}^i{}_k = 0$, i.e., α is flat.

Conversely, if α is flat and β is parallel with respect to α, then $R^i{}_k = 0$, i.e., $K = 0$. It completes the proof of Theorem 4.2. □

Remark. In literature [9], the proof of Theorem 1 depends on Theorem 3, of which the proof includes the hypothesis that the length of β with respect to α is constant, see the step 1 in the proof of Theorem 3 (A and A_i (i= 0,1,2,...) are some constants) in [9]. So, Theorem 4.2 here is the correct version of Theorem 1 in [9]. We do not know what happened to the case that the above hypothesis is canceled?

5. *S*-curvature of Matsumoto metrics

The S-curvature is an important geometric quantity. In this section, we investigate the S-curvature of Matsumoto metrics.

For a Finsler metric F and a volume form $dV = \sigma_F(x)\,dx$ on an n-dimensional manifold M, the S-curvature S is given by

$$S(x,y) = \frac{\partial G^i}{\partial y^i} - y^i\frac{\partial \ln \sigma_F}{\partial x^i}. \tag{5.1}$$

The volume form can be the Busemann–Hausdorff volume form $dV_{BH} = \sigma_{BH}dx$ or the Holmes–Thompson volume form $dV_{HT} = \sigma_{HT}dx$.

To compute the S-curvature, one should first find a formula for the Busemann–Hausdorff volume form dV_{BH} and the Holmes–Thompson dV_{HT}.

Proposition 5.1 (Proposion 4.1 in [3]). *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an (α, β)-metric on an $n-$dimensional manifold M. Denote*

$$f(b) := \begin{cases} \dfrac{\int_0^\pi \sin^{n-2}(t)dt}{\int_0^\pi \frac{\sin^{n-2}(t)}{\phi(b\cos t)^n}dt} & \text{if } dV = dV_{BH}, \\ \dfrac{\int_0^\pi \sin^{n-2}(t)T(b\cos t)dt}{\int_0^\pi \sin^{n-2}(t)dt} & \text{if } dV = dV_{HT}. \end{cases} \tag{5.2}$$

Then the volume form dV is given by $dV = f(b)dV_\alpha$, where $dV_\alpha = \sqrt{det(a_{ij})}dx$ denotes the Riemannian volume form of α, $T(s) := \phi(\phi - s\phi')^{n-2}[\phi - s\phi' + (b^2 - s^2)\phi'']$.

By Proposition 2.1 and Proposition 5.1, we have

$$\begin{aligned} \frac{\partial G^i}{\partial y^i} &= \frac{\partial \bar{G}^i}{\partial y^i} + \frac{2s_0}{(2s-1)^2} + \frac{6(b^2-s^2)}{(2s-1)(3s-2b^2-1)^2}s_0 - \frac{2s}{(2s-1)(3s-2b^2-1)}s_0 \\ &+ \frac{4(b^2-s^2)}{(2s-1)^2(3s-2b^2-1)}s_0 + (n+1)\frac{4s-1}{(2s-1)(3s-2b^2-1)}s_0 \\ &+ \frac{3(b^2-s^2)}{\alpha(3s-2b^2-1)^2}r_{00} + (n+1)\frac{4s-1}{2\alpha(3s-2b^2-1)}r_{00} - \frac{2}{(3s-2b^2-1)^2}r_0; \end{aligned} \tag{5.3}$$

and

$$y^i\frac{\partial \ln \sigma_F}{\partial x^i} = y^i\frac{\partial \ln \sigma_\alpha}{\partial x^i} + \Lambda(r_0 + s_0), \tag{5.4}$$

where $\Lambda := \frac{f'(b)}{bf(b)}$.

Plugging (5.3) and (5.4) into (5.1), we obtain

$$\begin{aligned} S &= \frac{2s_0}{(2s-1)^2} + \frac{6(b^2-s^2)}{(2s-1)(3s-2b^2-1)^2}s_0 - \frac{2s}{(2s-1)(3s-2b^2-1)}s_0 \\ &+ \frac{4(b^2-s^2)}{(2s-1)^2(3s-2b^2-1)}s_0 + (n+1)\frac{4s-1}{(2s-1)(3s-2b^2-1)}s_0 \\ &+ \frac{3(b^2-s^2)}{\alpha(3s-2b^2-1)^2}r_{00} + (n+1)\frac{4s-1}{2\alpha(3s-2b^2-1)}r_{00} - \frac{2}{(3s-2b^2-1)^2}r_0 \\ &+ \Lambda(r_0+s_0). \end{aligned} \tag{5.5}$$

Proof of Theorem 1.3. Assume that $S = 0$. Multiplying both sides of (5.5) by $2\alpha^5(2s-1)^2(3s-2b^2-1)^2$, we obtain

$$0 = 24(2n+1)\beta^4 r_{00} + \{-4(13+8b^2+19n+8nb^2)\beta^3 r_{00} + 72\Lambda\beta^4 r_0 + 72\Lambda\beta^4 s_0\}\alpha$$

$$
\begin{aligned}
&+\{2(19+32b^2+22n+20nb^2)\beta^2 r_{00}-2(60+48b^2)\Lambda\beta^3 r_0\\
&-24(1-2n+5\Lambda+4\Lambda b^2)\beta^3 s_0\}\alpha^2\\
&+\{-(11+40b^2+11n+16nb^2)\beta r_{00}+2(-8+37\Lambda+64\Lambda b^2+16\Lambda b^4)\beta^2 r_0\\
&+2(12-26n-16nb^2+37\Lambda+64\Lambda b^2+16\Lambda b^4)\beta^2 s_0\}\alpha^3\\
&+\{(1+8b^2+n+2nb^2)r_{00}+4(4-5\Lambda-14\Lambda b^2-8\Lambda b^4)\beta r_0\\
&-2(5-8b^2-9n-12nb^2+10\Lambda+28\Lambda b^2+16\Lambda b^4)\beta s_0\}\alpha^4\\
&+\{-(4-2\Lambda-8\Lambda b^2-8\Lambda b^4)r_0\\
&+2(1-4b^2-n-2nb^2+\Lambda+4\Lambda b^2+4\Lambda b^4)s_0\}\alpha^5.
\end{aligned}
\tag{5.6}
$$

(5.6) is equivalent to the following

$$
\begin{cases}
0=24(2n+1)\beta^4 r_{00}+\{2(19+32b^2+22n+20nb^2)\beta^2 r_{00}\\
\qquad -2(60+48b^2)\Lambda\beta^3 r_0-24(1-2n+5\Lambda+4\Lambda b^2)\beta^3 s_0\}\alpha^2\\
\qquad +\{(1+8b^2+n+2nb^2)r_{00}+4(4-5\Lambda-14\Lambda b^2-8\Lambda b^4)\beta r_0\\
\qquad -2(5-8b^2-9n-12nb^2+10\Lambda+28\Lambda b^2+16\Lambda b^4)\beta s_0\}\alpha^4,\\
0=\{-4(13+8b^2+19n+8nb^2)\beta^3 r_{00}+72\Lambda\beta^4 r_0+72\Lambda\beta^4 s_0\}\\
\qquad +\{-(11+40b^2+11n+16nb^2)\beta r_{00}\\
\qquad +2(-8+37\Lambda+64\Lambda b^2+16\Lambda b^4)\beta^2 r_0\\
\qquad +2(12-26n-16nb^2+37\Lambda+64\Lambda b^2+16\Lambda b^4)\beta^2 s_0\}\alpha^2\\
\qquad +\{-(4-2\Lambda-8\Lambda b^2-8\Lambda b^4)r_0\\
\qquad +2(1-4b^2-n-2nb^2+\Lambda+4\Lambda b^2+4\Lambda b^4)s_0\}\alpha^4.
\end{cases}
\tag{5.7}
$$

From the first equation of (5.7), we have

$$
r_{00}=c\alpha^2 \tag{5.8}
$$

for some function $c=c(x)$ on M. So $r_0=c\beta$.

Plugging (5.8) and $r_0=c\beta$ into (5.7), we obtain

$$
\begin{cases}
0=24c(1+2n-5\Lambda-4\Lambda b^2)\beta^4-24(1-2n+5\Lambda+4\Lambda b^2)\beta^3 s_0\\
\qquad +\{2c(27+32b^2+22n+20nb^2-10\Lambda-28\Lambda b^2-16\Lambda b^4)\beta^2\\
\qquad -2(5-8b^2-9n-12nb^2+10\Lambda+28\Lambda b^2+16\Lambda b^4)\beta s_0\}\alpha^2\\
\qquad +c(1+8b^2+n+2nb^2)\alpha^4,\\
0=72\Lambda\beta^4(c\beta+s_0).
\end{cases}
\tag{5.9}
$$

From the second equation of (5.9), we have $c\beta + s_0 = 0$ for $n \geq 2$. Differentiating both sides of it with respect to y^i yields $cb_i + s_i = 0$. Contracting it with b^i gives $cb^2 = 0$. So $c = 0$ and $s_0 = 0$. Thus $r_{00} = 0, s_0 = 0$, i.e., β is a constant Killing form.

Conversely, if β is a constant Killing form, then $S = 0$ by (5.5). Thus we have completed the proof of Theorem 1.3. □

By Theorem 1.3 and Theorem 1.1, we can directly get Corollary 1.1.

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(Received January 7, 2013; revised March 22, 2013)