Publ. Math. Debrecen
85/1-2 (2014), 31–46
DOI: 10.5486/PMD.2014.5756

Explicit description of three-dimensional homogeneous Ricci solitons

By ERNANI RIBEIRO JR. (Fortaleza) and JOÃO FILHO SILVA (Acarape)

Abstract. The purpose of this article is to describe explicitly the three-dimensional homogeneous Ricci soliton with isometry group of dimension 4. That with the work of Baird and Danielo [3] with isometry group of dimension 3 complete the description of all simply connected three-dimensional homogeneous Ricci solitons. We describe the family of vectors field in $\mathbb{S}^2\times\mathbb{R}$, $\mathbb{H}^2\times\mathbb{R}$ and Nil^3 that provides a Ricci soliton structure.

1. Introduction

Ricci solitons appeared in the seminal work of Richard Hamilton [9] on the Ricci flow. They model the formation of singularities in the Ricci flow and correspond to self-similar solutions, i.e. they are stationary points of this flow in the space of metrics modulo diffeomorphisms and scalings. Thus, classifying Ricci solitons or understanding their geometry is definitely an important issue.

Definition 1. A Ricci soliton is a Riemannian manifold (M^n,g), $n\geq 2$, endowed with a vector field X satisfying

$$\mathrm{Ric}+\frac{1}{2}\mathcal{L}_X g=\lambda g, \tag{1.1}$$

where λ is a constant and $\mathcal{L}$ stands for the Lie derivative.

Mathematics Subject Classification: Primary: 53C25, 53C30, 53C21; Secondary: 53C44.
Key words and phrases: Ricci soliton, Ricci flow, homogeneous manifolds, isometry group.
The first partially supported by grants from PJP-FUNCAP/Brazil and Universal-CNPq/Brazil.
The second partially supported by CNPq/Brazil.

If X is the gradient vector field of a function f on M^n such a manifold is called a gradient Ricci soliton. In this case, (1.1) becomes

$$\mathrm{Ric} + \nabla^2 f = \lambda g, \tag{1.2}$$

where $\nabla^2 f$ stands for the Hessian of f.

The Ricci soliton (M^n, g, X, λ) will be called expanding, steady or shrinking if $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively. Moreover, when either the vector field X is trivial, or the potential f is constant, the Ricci soliton will be called *trivial*, otherwise it will be a *nontrivial* Ricci soliton.

Perelman [11] proved that compact Ricci solitons are always of gradient type. More precisely, if (M^n, g, X, λ) is a Ricci soliton with M^n compact then there exists a smooth $f \in C^\infty(M)$, called the Perelman potential, such that $X = \nabla f$. On the other hand, a smooth vector field X in a compact manifold can be decomposed as a sum $X = Y + \nabla h$, where Y is a divergence free smooth vector field and $h \in C^\infty(M)$, called Hodge-de Rham potential. In [1], Aquino, Barros and Ribeiro shed light on the Perelman potential of a compact Ricci soliton, showing that the Perelman potential is, (up to a multiplication by a constant factor), the Hodge-de Rham potential. Homogeneous gradient Ricci solitons have been studied in [13], where it is shown that any gradient Ricci soliton is *rigid*, more precisely, it is a flat bundle $N \times_\Gamma \mathbb{R}^k$, where N is Einstein and the potential function reduces to a function on $\mathbb{R}^k$ as in the Gaussian soliton.

The classification of simply connected homogeneous spaces in dimension three is well known, see [14]. They are divided in three groups in accord to the dimension of their isometry groups. Precisely, they have isometry groups of dimension 3, 4, and 6. Those with isometry groups of dimension 6 are the simply connected space forms $\mathbb{S}^3(\kappa^2)$, $\mathbb{R}^3$, $\mathbb{H}^3(-\kappa^2)$. Thus their metrics are Einstein and the Ricci soliton structures are well known, see details in the survey paper [4].

The simply connected three dimensional homogeneous spaces with isometry groups of dimension 3 have the geometry of the Lie group Sol_3. Baird and Danielo [3], described all the Ricci soliton structures of Sol_3. They showed that Ricci soliton structures of Sol_3 are non-gradient Ricci solitons $(\mathrm{Sol}_3, X, \lambda)$, where X is a vector field in special class of vector fields and $\lambda < 0$. More precisely, they wrote:

> *"up to addition of a Killing vector field, the soliton flows on* Nil *and on Sol are unique; the soliton flows on* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$ *are unique; the only soliton flows on* $\mathbb{S}^3$ *and* $\mathbb{H}^3$ *are given by Killing vector fields; the only non-Killing solitons on* $\mathbb{R}^3$ *are the well-know Gaussian solitons".* [see page 27.]

Therefore, Baird and Danielo have proved the uniqueness of three-dimensional homogeneous Ricci solitons up to Killing vector fields. Thus, we have a big family of Ricci solitons structures in this kind. We point out that two vector fields X_1 and X_2 on a Riemannian manifold (M^n, g) satisfying Ricci solitons equation differ in a homothetic vector field and they differ in a Killing vector field if and only if $\lambda_1 = \lambda_2$. It is important to observe that the existence of non-Killing homothetic vector fields is a very restrictive condition. Indeed, if a three-dimensional homogeneous Riemannian manifold admits a non-Killing homothetic vector field then it is locally conformally flat; for more details see [6]. Hence the only three-dimensional homogeneous spaces which may admit two-distinct Ricci soliton constants are locally conformally flat ones.

In [2], Baird showed an explicit description of Ricci solitons structures on Nil^4 by using ODE technics. Inspired on ideas developed in Baird's work we shall describe explicitly the Ricci soliton structures of simply connected three-dimensional homogeneous spaces with isometry group of dimension 4. In fact, a simply connected homogeneous space with isometry group of dimension 4 is Riemannian fibration over a space form $\mathbb{N}^2_\kappa$ of dimension 2 having constant sectional curvature κ. In other words, there is a Riemannian submersion $\pi : E^3(\kappa, \tau) \to \mathbb{N}^2_\kappa$ with fibers diffeomorphic either to $\mathbb{S}^1$ or to $\mathbb{R}$ depending whether $\mathbb{N}^2_\kappa$ is compact or not. The vector field E_3 tangent to the fibers is a Killing vector field such that $\nabla_X E_3 = \tau X \times E_3$ for all $X \in \mathfrak{X}(M)$, where τ is a constant called curvature of the bundle. So, it is well-known the following classification

$$M = M_{\kappa,\tau} = \begin{cases} \mathbb{S}^2_\kappa \times \mathbb{R}, \ \kappa > 0 \text{ and } \tau = 0; \\ \mathbb{H}^2_\kappa \times \mathbb{R}, \ \kappa < 0 \text{ and } \tau = 0; \\ \mathrm{Nil}_3(\kappa, \tau)(\text{Heisenberg space}), \ \kappa = 0 \text{ and } \tau \neq 0; \\ \widetilde{PSl}_2(\kappa, \tau), \ \kappa < 0 \text{ and } \tau \neq 0; \\ \mathbb{S}^3_\tau \text{ (Berger sphere)}, \ \kappa > 0 \text{ and } \tau \neq 0. \end{cases}$$

We now show examples of Ricci solitons structures on $\mathbb{S}^2_\kappa \times \mathbb{R}$, $\mathbb{H}^2_\kappa \times \mathbb{R}$ and $\mathrm{Nil}^3(k, \tau)$.

Example 1. Let (M^3, g) be either $\mathbb{S}^2_\kappa \times \mathbb{R}$ or $\mathbb{H}^2_\kappa \times \mathbb{R}$ and let $\varphi : M \to \mathbb{R}$ be a function given by $\varphi(x, y, t) = \kappa t$. Taking into account local coordinates $\{E_1, E_2, E_3\}$, which will be defined in the next section, we consider $X \in \mathfrak{X}(M)$ given by $X = \varphi E_3$. Under these conditions, (M, g, X, λ) is a Ricci soliton with $\lambda = \kappa$. Moreover $X = \nabla f$, where $f : M \to \mathbb{R}$ is given by $f(x, y, t) = \kappa t^2/2$.

The existence of Ricci soliton structure on Nil^3 is well-known. Since Nil^3 is not locally conformally flat any Ricci soliton structure differs from a given one by a Killing vector field, for more details see [7]. In [3] appeared an example of Ricci soliton structure on Nil^3, here we shall show a different example.

Example 2. Let $(M^3, g) = \mathrm{Nil}^3(k, \tau)$ be and let $\varphi_1, \varphi_2, \varphi_3 : M^3 \to \mathbb{R}$ be functions given by $\varphi_1(x, y, t) = -4\tau^2 x$, $\varphi_2(x, y, t) = -4\tau^2 y$ and $\varphi_3(x, y, t) = -8\tau^2 t$. Moreover, let $X \in \mathfrak{X}(M)$ given by $X = \sum_{i=1}^{3} \varphi_i E_i$, where $\{E_1, E_2, E_3\}$ will be defined in the next section. Then (M, g, X, λ) is a non gradient expanding Ricci soliton and $\lambda = -6\tau^2$.

In our first result we shall describe the family of vector fields on $\mathbb{S}^2_\kappa \times \mathbb{R}$ and $\mathbb{H}^2_\kappa \times \mathbb{R}$ that gives a Ricci soliton structure. To this end, we consider local coordinates $\{E_1, E_2, E_3\}$ according to next section. We now announce our first result.

Theorem 1. *Let (M^3, g, X, λ) be a three-dimensional homogeneous Ricci soliton. We assume that (M^3, g) is either $\mathbb{S}^2_\kappa \times \mathbb{R}$ or $\mathbb{H}^2_\kappa \times \mathbb{R}$. Then $\lambda = \kappa$ and*

$$X = \varphi_1 E_1 + \varphi_2 E_2 + \varphi_3 E_3,$$

where φ_1, φ_2 and φ_3 are smooth functions on M^3 given by
(1) $\varphi_1(x, y, t) = \{3\kappa a(x^2 - y^2) - 2\kappa bxy + 2cy + 3a\}\rho$,
(2) $\varphi_2(x, y, t) = \{\kappa b(x^2 - y^2) + 6\kappa axy - 2cx - b\}\rho$,
(3) $\varphi_3(x, y, t) = \kappa t + d$,
$a, b, c, d \in \mathbb{R}$ and $\rho = \frac{2}{1+\kappa(x^2+y^2)}$.

It is worth pointing out that if $a = b = c = d = 0$ in the previous theorem we obtain Example 1. In the sequel we construct the family of vector fields on Nil^3 that provides a Ricci soliton structure. To do this, we recall that Nil^3 admits global coordinates and then we choose the orthonormal frame $\{E_1, E_2, E_3\}$ as in the next section. With these settings we have the following theorem.

Theorem 2. *Let $(\mathrm{Nil}^3, X, \lambda)$ be a Ricci soliton. Then $\lambda = -6\tau^2$ and*

$$X = \varphi_1 E_1 + \varphi_2 E_2 + \varphi_3 E_3,$$

where φ_1, φ_2 and φ_3 are smooth functions on M^3 given by
(1) $\varphi_1(x, y, t) = -4\tau^2 x + ay + b$,
(2) $\varphi_2(x, y, t) = -ax - 4\tau^2 y + c$,
(3) $\varphi_3(x, y, t) = \tau\{a(x^2 + y^2) - 2(cx - by) - 8\tau t + d\}$
and $a, b, c, d \in \mathbb{R}$.

Moreover, taking $a = b = c = d = 0$ in the previous theorem we obtain Example 2. On the other hand, it is very important to emphasize the non-existence of Ricci soliton structures on $\widetilde{PSl}_2$, which was proved by Baird and Danielo in [3]. In fact, they proved the following result.

Theorem 3 ([3]). *$\widetilde{PSl}_2$ does not admit any Ricci soliton structure.*

We now deduce a characterization for gradient three-dimensional homogeneous Ricci solitons. More exactly, we have the following theorem.

Theorem 4. *Let $(M^3, g, \nabla f, \lambda)$ be a non flat gradient three-dimensional homogeneous Ricci soliton. Then (M^3, g) is isometric to either $\mathbb{S}^2_\kappa \times \mathbb{R}$ or $\mathbb{H}^2_\kappa \times \mathbb{R}$. Moreover, $\lambda = \kappa$ and $\nabla f = \varphi E_3$, where $\varphi(x, y, t) = \kappa t + c$ and $c \in \mathbb{R}$.*

It was recently proved in [5] that a three-dimensional complete noncompact non-flat shrinking gradient soliton is a quotient of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$ by a subgroup of the $\mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$. Replacing the assumption of non-flat shrinking gradient soliton by homogeneous shrinking Ricci soliton we may combine Theorem 1 with Theorem 4 to deduce the following corollary.

Corollary 1. *Let (M^3, g, X, λ) be a non-flat three-dimensional homogeneous shrinking Ricci soliton. Then (M^3, g) is isometric to $\mathbb{S}^2_\kappa \times \mathbb{R}$.*

Finally, we recall that Nil^3 admits a non gradient Ricci soliton structure, for more details see Example 2, see also [8] and [7]. In particular, Theorem 4 shows that every Ricci soliton structure on Nil^3 is non gradient. This fact appeared in [3] by different argument.

2. Preliminares

In this section we shall develop a few tools concerning three-dimensional homogeneous manifolds in order to prove our results. For comprehensive references on such a theory, we indicate for instance [14].

2.1. Non compact homogeneous Riemannian manifold. Observe that the projection $\pi : M^3_{\kappa,\tau} \to \mathbb{M}^2_\kappa$, given by $\pi(x, y, t) = (x, y)$ is a Killing submersion, where $\mathbb{M}^2_\kappa$ is endowed with the metric $ds^2 = \rho^2(dx^2 + dy^2)$ and $\rho = \frac{2}{1+\kappa(x^2+y^2)}$ if $\kappa \neq 0$ and $\rho = 1$ if $\kappa = 0$. The natural orthonormal frame on N^2_κ is given by $\{e_1 = \rho^{-1}\partial_x, e_2 = \rho^{-1}\partial_y\}$. Moreover, translations along of the fibers are isometries, therefore E_3 is a Killing vector field. Since $\{e_1, e_2\}$ is an orthonormal referential on N^2_κ, we may take a horizontal lifting of $\{e_1, e_2\}$ to $\{E_1, E_2\}$ jointly with E_3 to obtain an orthonormal referential $\{E_1, E_2, E_3\}$ on $M^3_{\kappa,\tau}$. Moreover, since $\{\partial_x, \partial_y\}$ is a natural referential for $\mathbb{M}^2_\kappa$, then a natural referential for $M^3_{\kappa,\tau}$ is $\{\partial_x, \partial_y, \partial_t\}$, where ∂_t is tangent to the fibers, thus, $E_3 = \partial_t$. Using this referential we have the following lemma for non compact three-dimensional homogeneous manifold which can be found in [14].

Lemma 1. *Rewriting the referential $\{E_1, E_2, E_3\}$ in terms of $\{\partial_x, \partial_y, \partial_t\}$, we have:*

(1) *If $\kappa \neq 0$, then $E_1 = \frac{1}{\rho}\partial_x + 2\kappa\tau y\partial_t$, $E_2 = \frac{1}{\rho}\partial_y - 2\kappa\tau x\partial_t$ and $E_3 = \partial_t$.*

(2) *If $\kappa = 0$, then $E_1 = \partial_x - \tau y\partial_t$, $E_2 = \partial_y + \tau x\partial_t$ and $E_3 = \partial_t$.*

Moreover, considering $M^3_{\kappa,\tau}$ endowed with the metric

$$g = \begin{cases} dx^2 + dy^2 + [\tau(ydx - xdy) + dt]^2, & \text{if } \kappa = 0 \\ \rho^2(dx^2 + dy^2) + [2\kappa\tau\rho\,(xdy - ydx) + dt]^2, & \text{if } \kappa \neq 0, \end{cases}$$

we have the following identities for the Riemannian connection

$$\begin{cases} \nabla_{E_1}E_1 = \kappa y E_2 & \nabla_{E_1}E_2 = -\kappa y E_1 + \tau E_3 & \nabla_{E_1}E_3 = -\tau E_2 \\ \nabla_{E_2}E_1 = -\kappa x E_2 - \tau E_3 & \nabla_{E_2}E_2 = \kappa x E_1 & \nabla_{E_2}E_3 = \tau E_1 \\ \nabla_{E_3}E_1 = -\tau E_2 & \nabla_{E_3}E_2 = \tau E_1 & \nabla_{E_3}E_3 = 0. \end{cases} \tag{2.1}$$

In particular, we obtain from the above identities the following relations for the Lie brackets:

$$[E_1, E_2] = -\kappa y E_1 + \kappa x E_2 + 2\tau E_3 \tag{2.2}$$

and

$$[E_2, E_3] = [E_1, E_3] = 0. \tag{2.3}$$

We may assume without loss of generalities that $\kappa = -1, 0$ or 1.

As a consequence of the previous lemma we obtain the following result for the Ricci tensor of a three-dimensional non compact homogeneous Riemannian manifold with isometry group of dimension 4.

Proposition 1. *Let $(M^3_{\kappa,\tau}, g)$ be a simply connected non compact three-dimensional homogeneous Riemannian manifold with four-dimensional isometry group. Then the referential $\{E_1, E_2, E_3\}$ diagonalizes the Ricci tensor. More precisely, we have*

$$\mathrm{Ric} = (\kappa - 2\tau^2)g - (\kappa - 4\tau^2)E_3^\flat \otimes E_3^\flat, \tag{2.4}$$

where $E_3^\flat$ stands for the 1-form associated to E_3.

Proof. Since $\{E_1, E_2, E_3\}$ is an orthonormal frame, we can write the Ricci tensor as follows

$$\mathrm{Ric}(X,Y) = \sum_{j,k=1}^{3} \langle X, E_j\rangle\langle Y, E_k\rangle \,\mathrm{Ric}(E_j, E_k). \tag{2.5}$$

In order to compute $\mathrm{Ric}(E_j, E_k)$ first we consider $\mathrm{Ric}(E_j, E_j)$. The first term is

$$\begin{aligned}
\mathrm{Ric}(E_1, E_1) &= \langle \nabla_{E_2}\nabla_{E_1}E_1 - \nabla_{E_1}\nabla_{E_2}E_1 + \nabla_{[E_1,E_2]}E_1, E_2\rangle \\
&\quad + \langle \nabla_{E_3}\nabla_{E_1}E_1 - \nabla_{E_1}\nabla_{E_3}E_1 + \nabla_{[E_1,E_3]}E_1, E_3\rangle \\
&= \langle \nabla_{E_2}\big(\kappa y E_2\big) + \nabla_{E_1}\big(\kappa x E_2 + \tau E_3\big) - \kappa y \nabla_{E_1}E_1 + \kappa x \nabla_{E_2}E_1 + 2\tau\nabla_{E_3}E_1, E_2\rangle \\
&\quad + \langle \nabla_{E_3}\big(\kappa y E_2\big) + \nabla_{E_1}\big(\tau E_2\big), E_3\rangle = \frac{2\kappa}{\rho} - \kappa^2(x^2+y^2) - 2\tau^2 = \kappa - 2\tau^2.
\end{aligned}$$

In a similar way we obtain

$$\mathrm{Ric}(E_2, E_2) = \kappa - 2\tau^2$$

and

$$\mathrm{Ric}(E_3, E_3) = 2\tau^2.$$

We now claim that $\mathrm{Ric}(E_j, E_k) = 0$ for $j \neq k$. Indeed, let us compute only $\mathrm{Ric}(E_1, E_2)$.

$$\begin{aligned}
\mathrm{Ric}(E_1, E_2) &= \langle \nabla_{E_3}\nabla_{E_1}E_2 - \nabla_{E_1}\nabla_{E_3}E_2 + \nabla_{[E_1,E_3]}E_2, E_3\rangle \\
&= \langle \nabla_{E_3}\big(- \kappa y E_1 + \tau E_3\big) - \nabla_{E_1}\big(\tau E_1\big), E_3\rangle \\
&= -\kappa y \langle \nabla_{E_3}E_1, E_3\rangle - \tau\langle \nabla_{E_1}E_1, E_3\rangle = 0.
\end{aligned}$$

which finishes our claim. Therefore, using (2.5) we deduce that

$$\mathrm{Ric}(X,Y) = \sum_{j=1}^{3}\langle X, E_j\rangle\langle Y, E_j\rangle \,\mathrm{Ric}(E_j, E_j). \tag{2.6}$$

This completes the proof. □

3. Key lemmas

In this section we shall present some lemmas that will be crucial for our purposes. To this end, we shall state a corollary concerning mixed derivatives.

Corollary 2. *Under the choice of the given referential the following rules hold for all smooth function f on M.*

(1) *If $\tau = 0$, then $\partial^2_{xy} f = \partial^2_{yx} f, \partial^2_{xt} f = \partial^2_{tx} f$ and $\partial^2_{yt} f = \partial^2_{ty} f$.*

(2) *If $\kappa = 0$, then $\partial^2_{xy} f = \partial^2_{yx} f, \partial^2_{xt} f = \partial^2_{tx} f$ and $\partial^2_{yt} f = \partial^2_{ty} f$.*

(3) *If $\kappa \neq 0$ and $\tau \neq 0$, then $\partial^2_{xy} f = \partial^2_{yx} f, \partial^2_{xt} f = \partial^2_{tx} f$ and $\partial^2_{yt} f = \partial^2_{ty} f$.*

Proof. Using (2.2) and $\tau = 0$ we have

$$[E_1, E_2]f = -\frac{\kappa}{\rho} y \partial_x f + \frac{\kappa}{\rho} x \partial_y f. \tag{3.1}$$

On the other hand

$$[E_1, E_2]f = \frac{\partial_x}{\rho}\left(\frac{\partial_y}{\rho} f\right) - \frac{\partial_y}{\rho}\left(\frac{\partial_x}{\rho} f\right). \tag{3.2}$$

Now we compare (3.1) with (3.2) to deduce $\partial^2_{xy} f = \partial^2_{yx} f$. In a similar way we show that $\partial^2_{xt} f = \partial^2_{tx} f$ and $\partial^2_{yt} f = \partial^2_{ty} f$, which gives the first statement. Proceeding we have

$$\begin{aligned} [E_1, E_3]f &= (\partial_x - \tau y \partial_t)(\partial_t f) - \partial_t(\partial_x f - \tau y \partial_t f) \\ &= \partial^2_{xt} f - \tau y \partial^2_{tt} f - \partial^2_{tx} f + \tau y \partial^2_{tt} f = 0, \end{aligned} \tag{3.3}$$

which gives $\partial^2_{xt} f = \partial^2_{tx} f$. Using $[E_2, E_3] = 0$ we have $\partial^2_{yt} f = \partial^2_{ty} f$. We now use that $[E_1, E_2] = 2\tau E_3$ to deduce the last claim of the second item. The proof of the last item is analogous. $\square$

Lemma 2. *Let $\big(M^3, g, X, \lambda\big)$ be a three-dimensional homogeneous Ricci soliton. If E_1, E_2 and E_3 are given by Lemma 1, then the following assertions hold:*

$$E_1\langle X, E_1\rangle - \kappa y \langle X, E_2\rangle = \lambda - (\kappa - 2\tau^2). \tag{3.4}$$

$$E_2\langle X, E_2\rangle - \kappa x \langle X, E_1\rangle = \lambda - (\kappa - 2\tau^2). \tag{3.5}$$

$$E_3\langle X, E_3\rangle = \lambda - 2\tau^2. \tag{3.6}$$

$$E_2\langle X, E_1\rangle + E_1\langle X, E_2\rangle + \kappa(y\langle X, E_1\rangle + x\langle X, E_2\rangle) = 0. \tag{3.7}$$

$$E_3\langle X, E_1\rangle + E_1\langle X, E_3\rangle + 2\tau\langle X, E_2\rangle = 0. \tag{3.8}$$

$$E_3\langle X, E_2\rangle + E_2\langle X, E_3\rangle - 2\tau\langle X, E_1\rangle = 0. \tag{3.9}$$

Proof. By using equation (1.1) we can write

$$\mathcal{L}_X g(E_i, E_j) = 2\big(\lambda\delta_{ij} - \mathrm{Ric}(E_i, E_j)\big). \tag{3.10}$$

From definition of the Lie derivative and the compatibility of the metric we have

$$E_i\langle X, E_j\rangle + E_j\langle X, E_i\rangle - \langle X, \nabla_{E_i} E_j + \nabla_{E_j} E_i\rangle = 2\big(\lambda\delta_{ij} - R_{ij}\big). \tag{3.11}$$

Now, a straightforward computations using Lemma 1 and (3.11) gives the desired statement. For instance, if $i = j$ we have $E_i\langle X, E_i\rangle - \langle X, \nabla_{E_i} E_i\rangle = \big(\lambda - (\kappa - 2\tau^2) + (\kappa - 4\tau^2)\delta_{i3}\big)$. Since $\nabla_{E_1} E_1 = \kappa y E_2, \nabla_{E_2} E_2 = \kappa x E_1$ and $\nabla_{E_3} E_3 = 0$ we easily obtain the first three statements. The other ones are obtained by the same way and we left its check in for the reader. □

We now deduce a fundamental proposition.

Proposition 2. *Let $\big(M^n, g\big)$ a Riemannian manifold with non-null constant scalar curvature. We suppose that (M^n, g, X, λ) is a nontrivial structure of Ricci soliton. Then this structure is unique up to Killing vector field.*

Proof. We notice that if (M^n, g) admit two Ricci soliton structures (M^n, g, X, λ) and (M^n, g, Y, μ), then $\mathcal{L}_{(X-Y)} g = 2(\lambda - \mu)g$. Now, we may invoke [10] to conclude that $X - Y$ is a Killing vector field and $\lambda = \mu$, that was to be proved. □

Next, we consider the Heisenberg space with a Ricci soliton structure to deduce the following lemma.

Lemma 3. *Let $(\mathrm{Nil}^3, X, \lambda)$ be a Ricci soliton. Then $\langle X, E_1\rangle$ and $\langle X, E_2\rangle$ do not depend on the variable t. In particular, $\lambda = -6\tau^2$.*

Proof. First, we recall that on Nil^3 we have $\kappa = 0$. Moreover, from Proposition 2 and Example 2 we deduce that $\lambda = -6\tau^2$. Now, taking the derivative of (3.8) in the direction of ∂_x we have

$$E_3\partial_x\langle X, E_1\rangle = -\partial_x E_1\langle X, E_3\rangle - 2\tau\partial_x\langle X, E_2\rangle. \tag{3.12}$$

Moreover, by using (3.6), (3.8) and (3.9) we have

$$E_3E_3\langle X, E_1\rangle = -2\tau E_3\langle X, E_2\rangle \tag{3.13}$$

and

$$E_3E_3\langle X, E_2\rangle = 2\tau E_3\langle X, E_1\rangle. \tag{3.14}$$

Next, since $\kappa = 0$ we derive (3.4) in the direction of E_3 to deduce

$$E_3\partial_x\langle X, E_1\rangle - \tau y E_3E_3\langle X, E_1\rangle = 0. \tag{3.15}$$

A straightforward computation shows that combining (3.12), (3.13), (3.15) and (3.6) we get

$$2\tau E_1\langle X, E_2\rangle = -\partial^2_{xx}\langle X, E_3\rangle. \tag{3.16}$$

Whence, by using (3.7) we have

$$2\tau E_2\langle X, E_1\rangle = \partial^2_{xx}\langle X, E_3\rangle. \tag{3.17}$$

On the other hand, from (3.7) it is not difficult to check that

$$\partial^2_{xx}\langle X, E_3\rangle = \partial^2_{yy}\langle X, E_3\rangle. \tag{3.18}$$

In the same way we have $\partial^2_{xy}\langle X, E_3\rangle = 0$. So, we may use (3.6) to conclude that $\partial^2_{xx}\langle X, E_3\rangle$ is constant. Then we derive (3.17) with respect to E_3 to obtain

$$\partial^2_{yt}\langle X, E_1\rangle = -\tau x\partial^2_{tt}\langle X, E_1\rangle. \tag{3.19}$$

Proceeding, we derive (3.4) in the direction of ∂_y and substituting (3.19), we have

$$\partial^2_{xy}\langle X, E_1\rangle = \tau(\partial_t\langle X, E_1\rangle - \tau xy\partial^2_{tt}\langle X, E_1\rangle). \tag{3.20}$$

Now, taking the derivative of (3.17) in direction of ∂_x and using (3.15) it follows that

$$\partial^2_{xy}\langle X, E_1\rangle = -\tau(\partial_t\langle X, E_1\rangle + \tau xy\partial^2_{tt}\langle X, E_1\rangle).$$

Furthermore, we compare the last equality with (3.20) to obtain

$$E_3\langle X, E_1\rangle = 0. \tag{3.21}$$

Hence, we can compare with (3.13) to deduce

$$E_3\langle X, E_2\rangle = 0, \tag{3.22}$$

which finishes the proof of the lemma. □

4. Proof of the results

4.1. Proof of Theorem 1.

Proof. First, from Proposition 2 and Example 1 we deduce that $\lambda = \kappa$. On the other hand, we recall that $\tau = 0$ gives $E_1 = \frac{1}{\rho}\partial_x$ and $E_2 = \frac{1}{\rho}\partial_y$. Hence from (3.8) we have $\partial^2_{yt}\langle X, E_1\rangle + \kappa y\partial_x\langle X, E_3\rangle + \frac{1}{\rho}\partial^2_{yx}\langle X, E_3\rangle = 0$. Using once more (3.8) and Corollary 2 we deduce

$$\frac{1}{\rho}\partial^2_{xy}\langle X, E_3\rangle = \kappa y\rho\partial_t\langle X, E_1\rangle - \partial^2_{yt}\langle X, E_1\rangle.$$

Now we derive (3.9) in the direction of ∂_x and we compare with (3.9) to infer

$$\frac{1}{\rho}\partial^2_{xy}\langle X, E_3\rangle = \kappa x\rho\partial_t\langle X, E_2\rangle - \partial^2_{xt}\langle X, E_2\rangle,$$

then

$$\partial^2_{yt}\langle X, E_1\rangle - \partial^2_{xt}\langle X, E_2\rangle = \kappa\rho(y\partial_t\langle X, E_1\rangle - x\partial_t\langle X, E_2\rangle). \tag{4.1}$$

Now, taking the derivative of (3.7) in the direction of E_3 we have

$$\partial^2_{yt}\langle X, E_1\rangle + \partial^2_{xt}\langle X, E_2\rangle = -\kappa\rho(y\partial_t\langle X, E_1\rangle + x\partial_t\langle X, E_2\rangle), \tag{4.2}$$

hence, we combine (4.1) and (4.2) to obtain

$$\partial^2_{xt}\langle X, E_2\rangle = -\kappa y\rho\partial_t\langle X, E_1\rangle \tag{4.3}$$

and

$$\partial^2_{yt}\langle X, E_1\rangle = -\kappa x\rho\partial_t\langle X, E_2\rangle. \tag{4.4}$$

Next, taking the derivative of (3.4) and (3.5) in the direction of E_3, we also have

$$\partial^2_{xt}\langle X, E_1\rangle = \kappa y\rho\partial_t\langle X, E_2\rangle, \tag{4.5}$$

and

$$\partial^2_{yt}\langle X, E_2\rangle = \kappa x\rho\partial_t\langle X, E_1\rangle. \tag{4.6}$$

Computing the derivative of (4.3) with respect to ∂_y and using (4.4), we obtain

$$\partial^3_{xyt}\langle X, E_2\rangle = \kappa\rho[(\kappa y^2\rho - 1)\partial_t\langle X, E_1\rangle + \kappa xy\partial_t\langle X, E_2\rangle],$$

and deriving (4.6) with respect to ∂_x and using (4.5), we have

$$\partial^3_{xyt}\langle X, E_2\rangle = \kappa\rho[(1 - \kappa x^2\rho)\partial_t\langle X, E_1\rangle + \kappa xy\partial_t\langle X, E_2\rangle],$$

hence, $\partial_t\langle X, E_1\rangle = 0$ and it follows from (4.5) that $\partial_t\langle X, E_2\rangle = 0$. Then $\langle X, E_1\rangle$ and $\langle X, E_2\rangle$ do not depend on the variable t. On the other hand, since $\tau = 0$ using (3.6), (3.8) and (3.9) we also obtain that $\langle X, E_3\rangle$ depends only on t.

Letting $f = \frac{1}{\rho}\langle X, E_1\rangle$ and $h = \frac{1}{\rho}\langle X, E_2\rangle$ we use (3.4) and (3.5) to obtain

$$\partial_x f = \partial_y h. \tag{4.7}$$

Moreover, from (3.7) we also obtain

$$\partial_y f = -\partial_x h. \tag{4.8}$$

In particular, f and g satisfy Cauchy-Riemann. Next, using equation (3.4) we have

$$\frac{1}{\rho}\partial_x f = \kappa(xf + yh). \tag{4.9}$$

From this we have
$$\frac{1}{\rho}\partial^3_{yxx}f+\kappa y(\partial^2_{xx}f+\partial^2_{yy}f)=0. \tag{4.10}$$
Furthermore, we use Corollary 2, equation (4.10) and that f satisfies Cauchy–Riemann to conclude that
$$\partial^3_{yyy}f=-\partial^3_{yxx}f=0.$$
Since $\langle X,E_1\rangle$ does not depend on the variable t, we obtain
$$f=3Ay^2+2By+C, \tag{4.11}$$
where $A,B,C\in C^\infty(M)$ are functions that depend only on x. Moreover, it follows from identity (4.7) that
$$h=(\partial_xA)y^3+(\partial_xB)y^2+(\partial_xC)y+D, \tag{4.12}$$
where $D\in C^\infty(M)$ is a function depending only of x.

On the other hand, substituting (4.11) and (4.12) in (4.9) we get
$$(3\kappa\partial_xA)y^4+(2\kappa\partial_xB)y^3+[3(1+\kappa x^2)\partial_xA+\kappa\partial_xC]y^2+2(1+\kappa x^2)(\partial_xB)y$$
$$+(1+\kappa x^2)\partial_xC=\kappa[(2\partial_xA)y^4+(2\partial_xB)y^3+[6xA+2\partial_xC]y^2+(4xB+2D)y+2xC].$$
So, we obtain $\partial_xA=0$ and then we can write $A=-\kappa a$, where $a\in\mathbb{R}$. It follows that
$$(\kappa\partial_xC)y^2+2(1+\kappa x^2)(\partial_xB)y+(1+\kappa x^2)\partial_xC$$
$$=\kappa[(-6\kappa ax+2\partial_xC)y^2+(4xB+2D)y+2xC]$$
and comparing both sides we have
$$\begin{cases}\partial_xC=6a\kappa x\\(1+\kappa x^2)\partial_xB=\kappa(2xB+D)\\(1+\kappa x^2)\partial_xC=2\kappa xC\end{cases}. \tag{4.13}$$
We notice that the third equation of (4.13) gives
$$(1+\kappa x^2)\partial^2_{xx}C=2\kappa C.$$
Whence we may invoke the first equation of (4.13) to obtain
$$C=3a(\kappa x^2+1). \tag{4.14}$$
Now, using that $A=-\kappa a$ jointly with (4.14), (4.11) and (4.12) we obtain the

following relations

$$\frac{1}{\rho}\langle X, E_1\rangle = -3\kappa a y^2 + 2By + 3a(\kappa x^2 + 1) \tag{4.15}$$

and

$$\frac{1}{\rho}\langle X, E_2\rangle = (\partial_x B)y^2 + 6a\kappa xy + D, \tag{4.16}$$

which substituted in (4.8) yields $\partial^2_{xx}B = 0$.

Proceeding, we can write $B = -\kappa bx + c$, where $b, c \in \mathbb{R}$. Now, we compare with the second equation of (4.13) to obtain $D = \kappa bx^2 - 2cx - b$, then from (4.15) and (4.16) we deduce

$$\langle X, E_1\rangle = [3\kappa a(x^2 - y^2) - 2\kappa bxy + 2cy + 3a]\rho, \tag{4.17}$$

and

$$\langle X, E_2\rangle = [\kappa b(x^2 - y^2) + 6\kappa axy - 2cx - b]\rho. \tag{4.18}$$

Since $\langle X, E_3\rangle$ depends only on t we invoke (3.6) to conclude that $\langle X, E_3\rangle = \kappa t + d$, where $d \in \mathbb{R}$. Therefore, using this last identity, (4.17) and (4.18) we complete the proof of the theorem. □

4.2. Proof of Theorem 2.

Proof. First, from Lemma 3 it follows that $\lambda = -6\tau^2$. Moreover, by Lemma 3 we also have that $\langle X, E_1\rangle$ does not depend on the variable t. Taking into account the value of E_2 we may use (3.17) to obtain

$$\partial_y\langle X, E_1\rangle = a, \tag{4.19}$$

where $a = \partial^2_{xx}\langle X, E_3\rangle/2\tau$, that is constant. But, using (3.4) we deduce

$$\partial_x\langle X, E_1\rangle = -4\tau^2. \tag{4.20}$$

Next, from (4.19) and (4.20) we get

$$\langle X, E_1\rangle = -4\tau^2 x + ay + b, \tag{4.21}$$

where b is constant.

On the other hand, using again Lemma 3 we conclude that $\langle X, E_2\rangle$ does not depend on the variable t. Therefore, equation (3.16) yields

$$\partial_x\langle X, E_2\rangle = -a. \tag{4.22}$$

Whence using (3.5) we arrive at

$$\partial_y\langle X, E_1\rangle = -4\tau^2. \tag{4.23}$$

From (4.22) and (4.23) we have

$$\langle X, E_2\rangle = -ax - 4\tau^2 y + c, \tag{4.24}$$

where c is constant.

Substituting (4.24) in (3.8) and using (3.6) we obtain

$$\partial_x\langle X, E_3\rangle = 2\tau(ax - c). \tag{4.25}$$

Moreover, substituting (4.21) in (3.8) and using (3.6) we also obtain

$$\partial_y\langle X, E_3\rangle = 2\tau(ay + c). \tag{4.26}$$

So, it follows from (3.6), (4.25) and (4.26) that

$$\langle X, E_3\rangle = \tau[a(x^2 + y^2) - 2(cx - by) - 8\tau t + d],$$

which finishes the proof of the theorem. □

4.3. Proof of Theorem 4.

Proof. Let us suppose that there is a gradient Ricci soliton structure on Nil_3. Then using equation (3.6) we have

$$E_3E_3\langle\nabla f, E_1\rangle = 0. \tag{4.27}$$

Taking the derivative of (3.8) in the direction of E_3 and using (3.6) and (4.27) we deduce

$$E_3\langle\nabla f, E_2\rangle = 0. \tag{4.28}$$

In a similar way we also obtain

$$E_2\langle\nabla f, E_3\rangle = 0. \tag{4.29}$$

Then comparing (4.28) and (4.29) with (3.9) we obtain $\langle\nabla f, E_1\rangle = 0$, which is a contradiction by Theorem 2. Moreover, using Theorem 3 it follows that we can not have a Ricci soliton structure on $\widetilde{PSl}_2$. Therefore, if $(M^3, g, \nabla f, \lambda)$ is a 3-dimensional homogeneous gradient Ricci soliton, then (M^3, g) is either $\mathbb{S}^2_\kappa \times \mathbb{R}$ or $\mathbb{H}^2_\kappa \times \mathbb{R}$. From Proposition 1 we have $\mathrm{Ric} \geq 0$ and $\mathrm{Ric} \leq 0$, respectively. Therefore, we may apply Proposition 3 of [12] to conclude $\mathrm{Ric}(\nabla f, \nabla f) = 0$, which implies once more from Proposition 1 that

$$\kappa\big(\langle\nabla, E_1\rangle^2 + \langle\nabla, E_2\rangle^2\big) = 0.$$

From what it follows that $\langle\nabla f, E_1\rangle \equiv \langle\nabla f, E_2\rangle \equiv 0$. So, it is sufficient to use Theorem 1 to get the promised result. □

Acknowledgement. The authors want to thank the referees for their careful reading and helpful suggestions. Moreover, the first author would like to extend his special thank to A. Barros for helpful discussions during the elaboration of this article.

References

[1] C. Aquino, A. Barros and E. Ribeiro Jr., Some applications of the Hodge-de Rham decomposition to Ricci solitons, *Results in Math.* **60** (2011), 235–246.

[2] P. Baird, Explicit constructions of Ricci solitons, Variational Problems in Differential Geometry, London Math. Soc. Lecture Note Ser., 394, *Cambridge Univ. Press, Cambridge*, 2012.

[3] P. Baird and L. Danielo, Three-dimensional Ricci solitons which project to surface, *J. Reine Angew. Math.* **608** (2007), 65–71.

[4] H.-D. Cao, Recent progress on Ricci soliton, *Adv. Lect. Math. (ALM)* **11** (2009), 1–38.

[5] H.-D Cao, B.-L. Chen and X.-P. Zhu, Recent Developments on Hamilton's Ricci Flow, Surveys in Differential Geometry, Vol XII, 47-112, Surv. Differ. Geom., XII, int. Press, *Somerville, MA*, 2008.

[6] M. Capocci and G. Hall, Classification and conformal symmetry in three-dimensional space-times, *J. Math. Phys.* **40** (1999), 1466–1478.

[7] B. Chow, C. Sun-Chin, G. David, J. Isenberg, T. Ivey, D. Knopf, P. Lu, L. Feng and L. Ni, The Ricci Flow: Techniques and Applications, Part I. Geometric aspects. Mathematical Surveys and Monographs, 135, *American Mathematical Society, Providence, RI*, 2007.

[8] C. Guenter, J. Isenberg and D. Knopf, Linear stability of homogeneous Ricci solitons, *Int. Math. Res. Notices.* (2006), 1–30.

[9] R. S. Hamilton, The formation of singularities in the Ricci flow, Surveys in Differential Geometry, (Cambridge, MA, 1993), 2, 7–136, *International Press, Combridge, MA*, 1995.

[10] M. Obata and K. Yano, Conformal changes of Riemannian metrics, *J. Diff. Geom.* **4** (1970), 53–72.

[11] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, Prerpint, (2002). arXiv math/0211159.

[12] P. Petersen and W. Wylie, Rigidity of gradient Ricci solitons, *Pacific J. Math.* **241** (2009), 329–345.

[13] P. Petersen and W. Wylie, On gradient Ricci solitons with symmetry, *Proc. Amer. Math. Soc.* **137** (2009), 2085–2092.

[14] W. Thurston, Three-Dimensional Geometry and Topology, *Princeton University Press, New Jersey*, 1997.

ERNANI RIBEIRO JR.
UNIVERSIDADE FEDERAL DO CEARÁ
DEPARTAMENTO DE MATEMÁTICA
CAMPUS DO PICI, AV. HUMBERTO MONTE
BLOCO 914, 60455-760-FORTALEZA/CE
BRAZIL

E-mail: ernani@mat.ufc.br

JOÃO FILHO SILVA
INSTITUTO DE CIÊNCIAS EXATAS
E DA NATUREZA – UNILAB
CAMPUS DOS PALMARES
CE 060, KM 51
62785-000-ACARAPE-CE
BRAZIL

E-mail: joaofilho@unilab.edu.br

(Received January 14, 2013; revised December 27, 2013)