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On the Moore–Penrose inverse of a closed linear relation

By TERESA ÁLVAREZ (Oviedo)

Abstract. For a closed multivalued linear operator T between complex Hilbert spaces the concept of Moore–Penrose inverse of T, denoted $T^\dagger$, is introduced and studied. We prove that if $y \in D(T^\dagger)$, then $T^\dagger y$ is the least square solution of minimal norm of the relation equation $y \in Tx$. We also approximate $T^\dagger$ by a sequence of bounded finite rank operators. Such results generalize the existing results to the case of densely defined closed operators.

1. Introduction

This paper is devoted to solve the relation equation $y \in Tx$ where T is a closed multivalued linear operator between complex Hilbert spaces H_1 and H_2 and y is a given element in H_2.

In the last years, several authors (see, for instance [1], [6], [8], [10], [14], [15] and the references therein) have paid attention to study this equation when T is a densely defined closed operator. In such case, the Moore–Penrose inverse of T has proved helpful when the operator equation does not have a solution and we look for the least square (best approximate) solutions instead. It is well known the following result:

$(*)$ *Let T be a densely defined closed operator from H_1 to H_2 and $T^\dagger$ stands for the Moore–Penrose inverse of T. For every $y \in D(T^\dagger)$, let $L(y)$ the set of all least square solutions of the equation $y = Tx$. Then $T^\dagger y \in L(y)$ and $\|T^\dagger y\| \leq \|x\|$ for all $x \in L(y)$.*

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We remark that the properties of the adjoint of T play a crucial role in the proof of the previous properties of $T^{\dagger}$. On the other hand, we can observe that for a densely defined closed operator, computing its Moore–Penrose inverse may be difficult. In a recent paper [7], Kulkarni and Ramesh proved the following result:

(∗∗) *Let T be a densely defined closed operator from* H_1 to H_2. *For each* $n \in \mathbb{N}$, *there exists a bounded finite rank operator* T_n *such that* $\lim_{n\to\infty} T_n^{\dagger} y = T^{\dagger} y$ *for all* $y \in D(T^{\dagger})$.

This approximation of $T^{\dagger}$ is very interesting since reduces the infinite dimensional problem (operator equation) to a sequence of finite dimensional operator equations (matrix equations) which can be solved with the help of the known techniques of the finite dimensional case (see, for instance, [9], [11], [12], [13]).

The important point here is to note that as remarked by Kulkarni and Ramesh [7], a large number of problems which arise naturally in applications of Quantum Mechanics and Partial Differential equations can be modeled by equations governed by non densely defined operators. The adjoints of such operators are multivalued linear operators. Hence, it would be useful if the results mentioned above could be extended to the case of closed multivalued linear operators not necessarily densely defined. This is the main purpose of this paper which is organized as follows: The Section 2 contains basic notions and results concerning multivalued linear operators in Hilbert spaces which will be frequently used throughout the remaining part of the paper. In the third section we introduce the notion of Moore–Penrose inverse of a closed multivalued linear operator and discuss some of its properties. In Section 4 we apply the results proved in the previous sections to obtain an approximation of Moore–Penrose inverse of a closed multivalued linear operator by a sequence of bounded finite rank operators which generalizes the above result (**).

2. Notations and preliminaries

In this section we set up notations and state some of the definitions and results which will be needed in the sequel.

Let $H, H_1, H_2, \ldots$ denote infinite dimensional complex Hilbert spaces. The inner product and the induced norm are denoted respectively by $\langle , \rangle$ and $\|.\|$.

If M is a closed subspace of H, then P_M is the orthogonal projection onto M and $M^{\perp}$ is the orthogonal complement of M in H. For subspaces M_1 and M_2 of H, the orthogonal direct sum is denoted by $M_1 \oplus M_2$.

A multivalued linear operator or linear relation from H_1 to H_2 is a mapping T from a subspace $D(T) \subset H_1$, called the domain of T, into the collection of nonempty subsets of H_2 such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all nonzero α, β scalars and $x_1, x_2 \in D(T)$. If T maps the points of its domain to singletons, then T is said to be a single valued or simply an operator. The linear relation T is uniquely determined by its graph, $G(T)$, which is defined by $G(T) := \{(x, y) \in H_1 \times H_2 : x \in D(T),\ y \in Tx\}$, so that in the following T will be identified with its graph. The inverse of T is defined by $T^{-1} := \{(y, x) : (x, y) \in G(T)\}$. The subsets $T(0), N(T) := T^{-1}(0)$ and $R(T) := T(D(T))$ are subspaces and if M is a subspace of H_1 such that $D(T) \cap M \neq \emptyset$, then $T\mid_{D(T)\cap M}$ is defined by $T\mid_{D(T)\cap M} := \{(x, y) : x \in D(T) \cap M,\ (x, y) \in T\}$.

For linear relations T_1 and T_2 from H_1 to H_2, the sum $T_1 + T_2$ is the linear relation defined by $T_1 + T_2 := \{(x, y+z) : (x, y) \in T_1,\ (x, z) \in T_2\}$ and the notation $T_1 \subset T_2$ means that $G(T_1) \subset G(T_2)$. Let T and S be linear relations from H_1 to H_2 and from H_2 to H_3 respectively. Then the product ST is the linear relation from H_1 to H_3 defined by $G(ST) := \{(x, z) : (x, y) \in T, (y, z) \in S \text{ for some } y \in H_2\}$. Let now T be a linear relation from H to H and let $\lambda \in \mathbb{K}$, then $\lambda - T := \lambda I - T$ where I is the identity operator in H, so that $\lambda - T := \{(x, \lambda x - y) : (x, y) \in T\}$.

Let T be a linear relation from H_1 to H_2. Then $y \in Tx$ if and only if $Tx = y + T(0)$; in particular $x \in N(T)$ if and only if $Tx = T(0)$. Moreover, T is single valued if and only if $T(0) = \{0\}$, so that $P_{T(0)^\perp} T$ is an operator and thus we say that T is continuous if $P_{T(0)^\perp} T$ is continuous, bounded if T is continuous with $D(T) = H_1$, finite rank if $\dim R(T) < \infty$ and T is called closed if its graph is a closed subspace of $H_1 \oplus H_2$. We note that if T is closed so is its inverse and the subspaces $T(0)$ and $N(T)$ are closed subspaces and further we have a Closed Graph theorem, that is, T is bounded if T is closed and everywhere defined.

The adjoint of T is the closed linear relation T^* from H_2 to H_1 defined by

$$T^* := \{(u, v) \in H_2 \times H_1 : \langle u, y \rangle = \langle v, x \rangle \quad \text{for all } (x, y) \in T\}.$$

We note that $(T^*)^{-1} = (T^{-1})^*$ and that T^{**} coincides with $\overline{T}$ where $\overline{T}$ is the linear relation whose graph is the closure of $G(T)$; in particular $T = T^{**}$ if T is closed.

By direct computation we obtain the following properties concerning the behaviour of the adjoint in sums and products:

Let S and T be linear relations from H_1 to H_2 and let R be a linear relation from H_2 to H_3. Then

$S^* + T^* \subset (S + T)^*$ with equality if S is a bounded operator.

$S^* R^* \subset (RS)^*$ with equality if R is a bounded operator.

In the sequel, $CR(H_1, H_2)$ stands for the class of all closed linear relations from H_1 to H_2.

Assume now that $T \in CR(H_1, H_2)$. The following properties are straightforward:

$R(T)$ is closed if and only if $R(T^*)$ is closed.

$T^*T(0) = T^*(0) = D(T)^\perp, T^*(0)^\perp = \overline{D(T)}, N(TT^*) = N(T^*) = R(T)^\perp$ and $N(T^*)^\perp = \overline{R(T)} = \overline{R(TT^*)}$.

The orthogonal decomposition $H_2 = T(0) \oplus T(0)^\perp$ together with the equality $T(0)^\perp = \overline{D(T^*)}$ shows that $\langle u, v\rangle = 0$ for all $u \in D(T^*), v \in T(0)$ and since $y \in Tx$ if and only if $Tx = y + T(0)$ we infer that

$$\text{for any } y \in Tx, \langle u, y\rangle = \langle u, P_{T(0)^\perp} Tx\rangle.$$

This last assertion suggests the following notion.

Definition 2.1. Let $T \in CR(H_1, H_2)$. Then

$$\langle u, Tx\rangle := \langle u, P_{T(0)^\perp} Tx\rangle \quad \text{for all } u \in D(T^*),\ x \in D(T).$$

Lemma 2.2. *Let $T \in CR(H_1, H_2)$. We have:*

(i) $\langle u, Tx\rangle = \langle T^*u, x\rangle$ *for all* $u \in D(T^*)$, $x \in D(T)$.

(ii) *If A is a subspace of $D(T)$, then*

$$(TA)^\perp \cap D(T^*) = (T^*)^{-1}(A^\perp).$$

Proof. (i) Take $u \in D(T^*)$ and $x \in D(T)$. Then $u \in T(0)^\perp$, so that $u = P_{T(0)^\perp} u$ and since $(P_{T(0)^\perp} T)^* = T^* P_{T(0)^\perp}$ it follows from the above definition that

$$\langle u, Tx\rangle = \langle u, P_{T(0)^\perp} Tx\rangle = \langle T^* P_{T(0)^\perp} u, x\rangle = \langle T^*u, x\rangle.$$

Hence (i) holds.

(ii) Let $z \in (TA)^\perp \cap D(T^*)$, so that $z \in T(0)^\perp$ and for any $y \in TA$, $0 = \langle z, y\rangle$. Hence if $a \in A$ and $y \in Ta$ we infer from (i) that $0 = \langle z, Ta\rangle = \langle T^*z, a\rangle$ which implies that $T^*z \in A^\perp$. Therefore $z \in (T^*)^{-1}T^*z \subset (T^*)^{-1}(A^\perp)$ whence $z \in (TA)^\perp \cap D(T^*)$. For the reverse inclusion, suppose that $z \in (T^*)^{-1}(A^\perp)$. Then $z \in R((T^*)^{-1}) = D(T^*)$ and $z \in (T^*)^{-1}b$ for some $b \in A^\perp$. One may conclude from this and (i) that $0 = \langle T^*z, c\rangle = \langle z, Tc\rangle$ for any $c \in A^\perp$ which implies that $z \in (TA)^\perp$, as desired. □

We recall that a linear relation T from H to H is called selfadjoint if $T = T^*$. We close this section by the following useful proposition.

Proposition 2.3. *Let $T \in CR(H_1, H_2)$. Then*

(i) *The products T^*T and TT^* are selfadjoint linear relations in H_1 and H_2 respectively.*

(ii) $\Delta_T := (I + T^*T)^{-1}$ *and* $\Delta_{T^*} := (I + TT^*)^{-1}$ *are bounded operators in H_1 and H_2 respectively. Moreover,* $N(\Delta_T) = T^*(0)$ *and* $N(\Delta_{T^*}) = T(0)$.

(iii) $T\Delta_T$ *is a bounded linear relation with* $(P_{T(0)^\perp}T\Delta_T)^* = P_{T^*(0)^\perp}T^*\Delta_{T^*}$. *Moreover,* $\Delta_{T^*}T$ *is an operator satisfying* $\Delta_{T^*}T \subset T\Delta_T$ *and* $\Delta_{T^*}T = P_{T(0)^\perp}T\Delta_T\,|_{D(T)}$.

(iv) $I - \Delta_T = P_{T^*(0)} + P_{T^*(0)^\perp}T^*T\Delta_T = P_{T^*} + P_{T^*(0)^\perp}T^*P_{T(0)^\perp}T\Delta_T$ *and* $R(I - \Delta_T) = R(T^*T)$.

(v) $\Delta_{T^*}N(TT^*) = N(TT^*)$ *and* $\overline{\Delta_{T^*}\overline{R(T)}} = \overline{R(T)} \cap T(0)^\perp$.

Proof. (i) and (ii) The proofs are along the lines of the proofs of the analogous results provided in [3, Proposition 2.1 and Corollary 2.5] for the case when $H_1 = H_2$.

(iii) As in [3, Proposition 2.1 and Corollary 2.2] we can prove that $T\Delta_T$ is a bounded linear relation and $(P_{T(0)^\perp}T\Delta_T)^* = P_{T^*(0)^\perp}T^*\Delta_{T^*}$.

To see that $\Delta_{T^*}T$ is single valued, let $z \in \Delta_{T^*}T(0)$. Then there exists $x \in H_2$ such that $z = \Delta_{T^*}x$ and $(0, x) \in T$ which implies that $(0, x) \in TT^*$ and $(z, x) \in \Delta_{T^*}^{-1} := I + TT^*$. Hence $(0, x)$ and $(z, x - z)$ belong to TT^*, so that $(z, -z) \in TT^*$. This leads to $(z, y) \in T^*$ and $(y, -z) \in T$ for some $y \in H_1$ and thus according to the definition of T^* we infer that $-\|z\|^2 = \|y\|^2$ and hence $z = 0$. Consequently $\Delta_{T^*}T(0) = \{0\}$, that is, $\Delta_{T^*}T$ is single valued, as desired.

Next, we verify that $\Delta_{T^*}T \subset T\Delta_T$. Let $(x, z) \in \Delta_{T^*}T$. Then there is $y \in H_2$ for which $(x, y) \in T$ and $(y, z) \in \Delta_{T^*}$, so that $(z, y - z) \in TT^*$ which implies that $(z, u) \in T^*$ and $(u, y - z) \in T$ for some $u \in H_1$. In consequence, $(x - u, z) \in T$ and since $(z, u) \in T^*$, it follows that $(x - u, u) \in T^*T$ and hence $(x - u, x) \in I + T^*T$. This last equality together with the fact that $(x - u, z) \in T$ ensures that $(x, z) \in T\Delta_T$, as desired. Hence $\Delta_{T^*}T \subset T\Delta_T$. This last inclusion implies that $P_{T(0)^\perp}\Delta_{T^*}T \subset P_{T(0)^\perp}T\Delta_T$ and noting that if $x \in D(T)$ then $\Delta_{T^*}Tx \in D(T^*) \subset T(0)^\perp$ we conclude that $\Delta_{T^*}T = P_{T(0)^\perp}T\Delta_T\,|_{D(T)}$.

(iv) By the part (iii) $T\Delta_T$ is everywhere defined and since $T^*T(0) = T^*(0)$ we obtain that $P_{T^*(0)^\perp}T^*T\Delta_T$ is an everywhere defined operator. Furthermore, according to the definitions, $I - \Delta_T \subset T^*T\Delta_T$, so that

$$P_{T^*(0)^\perp}(I - \Delta_T) \subset P_{T^*(0)^\perp}T^*T\Delta_T$$

and since both linear relations are everywhere defined operators we have the equality. Therefore

$$P_{T^*(0)^\perp}(I-\Delta_T) = P_{T^*(0)^\perp}T^*T\Delta_T,$$

so that,

$$I-\Delta_T = P_{T^*(0)} + P_{T^*(0)^\perp}T^*T\Delta_T.$$

Finally, it is clear that $P_{T^*(0)^\perp}T^*P_{T(0)^\perp}T\Delta_T \subset P_{T^*(0)^\perp}T^*T\Delta_T$ and since both operators have the same domain we have the equality. This completes the first part of (iv) and that $R(I-\Delta_T) = R(T^*T)$ follows easily from the definitions.

(v) Using the definitions and the facts TT^* is selfadjoint and $R(TT^*) = R(I-\Delta_{T^*})$ proved in (i) and (iv) we deduce that $\Delta_{T^*}N(TT^*) = N(TT^*)$. This equality combined with the equalities $H_2 = N(TT^*) \oplus N(TT^*)^\perp = N(TT^*) \oplus \overline{R(TT^*)} = N(TT^*) \oplus \overline{R(T)}$ gives

$$R(\Delta_{T^*}) = N(TT^*) \oplus \Delta_{T^*}\overline{R(T)}$$

But

$$R(\Delta_{T^*}) = D(TT^*) = N(TT^*) \oplus (D(TT^*) \cap N(TT^*)^\perp).$$

From this we can conclude that

$$\Delta_{T^*}\overline{R(T)} = D(TT^*) \cap N(TT^*)^\perp$$

and as

$$\begin{aligned}\overline{D(TT^*) \cap N(TT^*)^\perp} &= \overline{D(TT^*)} \cap N(TT^*)^\perp = TT^*(0)^\perp \cap N(TT^*)^\perp \\ &= T(0)^\perp \cap N(T^*)^\perp = T(0)^\perp \cap \overline{R(T)}\end{aligned}$$

we infer that

$$\overline{\Delta_{T^*}\overline{R(T)}} = T(0)^\perp \cap \overline{R(T)}.$$

This proves (v). □

3. Moore–Penrose inverse of a closed linear relation

This section is devoted to establish some properties of the linear relation $T^\dagger$ which is defined below.

Let us first recall some facts about densely defined closed operators in Hilbert spaces.

Proposition 3.1. *Let T be a densely defined closed operator from H_1 to H_2. Then there exists a unique, densely defined closed operator $T^\dagger$ from H_2 to H_1 with domain $D(T^\dagger) = R(T) \oplus R(T)^\perp$ having the following properties:*

(i) $TT^\dagger y = P_{\overline{R(T)}} y$, *for all* $y \in D(T^\dagger)$.

(ii) $T^\dagger T x = P_{N(T^\perp} x$, *for all* $x \in D(T)$.

(iii) $N(T^\dagger) = R(T)^\perp$.

The above Proposition is proved in [1], where the operator $T^\dagger$ is called the Moore–Penrose inverse of T.

Definition 3.2. The Moore–Penrose inverse of $T \in CR(H_1, H_2)$ is the linear relation

$$T^\dagger := P_{N(T)^\perp} T^{-1} P_{N(T^*)^\perp}.$$

Proposition 3.3. *Let* $T \in CR(H_1, H_2)$. *Then*

(i) $T^\dagger$ *is single valued with* $D(T^\dagger) = R(T) \oplus R(T)^\perp$, $N(T^\dagger) = T(0) \oplus R(T)^\perp$ *and* $R(T^\dagger) \subset D(T) \cap N(T)^\perp$.

(ii) $TT^\dagger = TT^{-1} P_{N(T^*)^\perp}$.

(iii) $T^\dagger T$ *is single valued with*

$$G(T^\dagger T) = \{(x, P_{N(T)^\perp} x) : x \in D(T)\}.$$

Proof. (i) Let $x \in T^\dagger(0)$, so that $(0, x) = (b, P_{N(T)^\perp} a)$ for some $(a, b) \in T$. Hence $b = 0$ and $x = P_{N(T)^\perp} a$. Consequently, $a \in N(T)$ and $x = P_{N(T)^\perp} a = 0$ which ensures that $T^\dagger(0) = \{0\}$, that is, $T^\dagger$ is an operator.

To establish that $D(T^\dagger) = R(T) \oplus R(T)^\perp$ and $N(T^\dagger) = T(0) \oplus R(T)^\perp$ it is enough to observe that

$$D(T^\dagger) := \{y \in D(P_{N(T^*)^\perp}) = H_2 : P_{N(T^*)^\perp} y \in D(P_{N(T)^\perp} T^{-1}) = D(T^{-1}) = R(T)\}$$

$$N(T^\dagger) = P^{-1}_{N(T^*)^\perp} T N(P_{N(T)^\perp}) = P^{-1}_{N(T^*)^\perp} T T^{-1}(0) = P^{-1}_{N(T^*)^\perp} T(0)$$

and

$$H_2 = N(T^*) \oplus N(T^*)^\perp = R(T)^\perp \oplus \overline{R(T)}.$$

(ii) It follows from $T^\dagger \subset T^{-1} P_{N(T^*)^\perp}$ that $TT^\dagger \subset TT^{-1} P_{N(T^*)^\perp}$. To prove the converse inclusion, let $(x, y) \in TT^{-1} P_{N(T^*)^\perp}$, so that $(z, y) \in T$ and $(P_{N(T^*)^\perp} x, z) \in T^{-1}$ for some $z \in H_1$. Decompose $z = z_1 + z_2$ with $z_1 \in N(T)^\perp$ and $z_2 \in N(T)$. Then

$$(P_{N(T^*)^\perp} x, z_1) = (P_{N(T^*)^\perp} x, z) - (0, z_2) \in T^{-1}$$

and

$$(z_1, y) = (z, y) - (z_2, 0) \in T.$$

.

Since $z_1 \in N(T)^{\perp}$ it follows that $(x, z_1) \in T^{\dagger}$, so that $(x, y) \in TT^{\dagger}$. Therefore $TT^{-1}P_{N(T^*)^{\perp}} \subset TT^{\dagger}$, as required.

(iii) We note the following chain of equalities:

$$\begin{aligned} T^{\dagger}T(0) &= P_{N(T)^{\perp}}T^{-1}P_{N(T^*)^{\perp}}T(0) = P_{N(T)^{\perp}}T^{-1}P_{\overline{R(T)}}T(0) \\ &= P_{N(T)^{\perp}}T^{-1}T(0) = P_{N(T)^{\perp}}T^{-1}(0) = P_{N(T)^{\perp}}N(T) = \{0\}. \end{aligned}$$

Hence $T^{\dagger}T(0) = \{0\}$ which shows that $T^{\dagger}T$ is single valued.

Let now $(a, b) \in T^{\dagger}T$, so that $(a, c) \in T$ and $b = T^{\dagger}c$ for some $c \in H_2$. Then, according to the definition of $T^{\dagger}$ one has $b = P_{N(T)^{\perp}}d$ for some $d \in H_1$ such that $(d, P_{N(T^*)^{\perp}}c) \in T$. One may conclude from this and the fact that $c = P_{N(T^*)^{\perp}}c$ that $(a - d, 0) \in T$, that is, $a - d \in N(T)$ which proves that $P_{N(T)^{\perp}}a = b$ and thus (iii) follows. □

As a consequence of the previous result we get the following corollary which shows that our definition of $T^{\dagger}$ coincides with the standard definition for densely defined closed operators considered in Proposition 3.1.

Corollary 3.4. *If T is a densely defined closed operator from H_1 to H_2, then its Moore–Penrose inverse has the following properties:*

(i) $D(T^{\dagger}) = R(T) \oplus R(T)^{\perp}$ *and* $N(T^{\dagger}) = R(T)^{\perp}$.

(ii) $TT^{\dagger}y = P_{\overline{R(T)}}y$ *for all* $y \in D(T^{\dagger})$.

(iii) $T^{\dagger}Tx = P_{N(T)^{\perp}}x$ *for all* $x \in D(T)$.

Proof. By virtue of Proposition 3.3 it only remains to establish that $TT^{\dagger}y = P_{\overline{R(T)}}y$ for all $y \in D(T^{\dagger})$. Let $y \in D(T^{\dagger})$. Then $y = u + v$ with $u \in R(T) \subset N(T^*)^{\perp}$ and $v \in R(T)^{\perp} = N(T^*)$. Let $x \in D(T)$ such that $Tx = u$. Then $TT^{\dagger}y = TT^{-1}P_{N(T^*)^{\perp}}y$ (by the condition (ii) of Proposition 3.3) $= TT^{-1}P_{N(T^*)^{\perp}}u + TT^{-1}P_{N(T^*)^{\perp}}v = TT^{-1}Tx = Tx = u = P_{\overline{R(T)}}u + P_{\overline{R(T)}}v = P_{\overline{R(T)}}y$. This completes the proof. □

A different notion of Moore–Penrose inverse of $T \in CR(H_1, H_2)$ is introduced in [4], as follows:

Definition 3.5. Let $T \in CR(H_1, H_2)$. The linear relation $(T^{-1})_s := P_{N(T)^{\perp}}T^{-1}$ is called a generalized inverse or Moore–Penrose inverse of T.

Note that the above definition of Moore–Penrose inverse of T coincides with our definition when $R(T)$ is a dense subspace of H_2.

We recall the notion of a least square solution of the operator equation $y = Tx$.

Definition 3.6. Let T be a densely defined closed operator from H_1 to H_2. For every $y \in D(T^\dagger)$, let

$$L(y) := \{x \in D(T) : \|Tx - y\| \leq \|Tu - y\| \quad \text{for all } u \in D(T)\}.$$

Any $u \in L(y)$ is called a least square solution of the operator equation $Tx = y$.

For a deeper discussion on the notions of Moore–Penrose inverse of T and the set of least square solutions of the operator equation $Tx = y$ when T is a densely defined closed operator between Hilbert spaces we refer to [1].

The above concept of least square solution of the operator equation $Tx = y$ can be naturally generalized for the case when T is a closed multivalued linear operator not necessarily densely defined.

Definition 3.7. Let $T \in CR(H_1, H_2)$ and let $y \in H_2$. We say that $u \in H_1$ is a least square solution of the relation $y \in Tx$ if $u \in D(T)$ and $d(y, R(T)) = \|y - z\|$ for some $z \in Tu$, where $d(y, R(T))$ is the distance between y and $R(T)$.

Note that if such a z exists, then it is unique. Of course, u need not be unique.

We now state the first main result of this paper.

Theorem 3.8. *Let $T \in CR(H_1, H_2)$ and let $y \in H_2$. We have:*

(i) *An element $u \in H_1$ is a least square solution of the relation $y \in Tx$ if and only if $u \in D(T)$ and $y \in Tu + N(T^*)$.*

(ii) *If $y \in D(T^\dagger)$, then $T^\dagger y$ is a least square solution of $y \in Tx$.*

(iii) *If $u \in H_1$ is a least square solution of $y \in Tx$, then $u \in T^\dagger y + N(T)$.*

(iv) *$\|T^\dagger y\| \leq \|u\|$ for all least square solution u of $y \in Tx$. Moreover, if u is a least square solution of $y \in Tx$ with $\| u \| \leq \| T^\dagger y \|$, then $u = T^\dagger y$.*

Proof. (i) Let $u \in H_1$ be a least square solution of $y \in Tx$. Then $u \in D(T)$ and there exists $z \in Tu$ for which $\| y - z \| = d(y, R(T))$.

$$\|P_{R(T)^\perp} y\|^2 = d(y, R(T))^2 = \|y - z\|^2 = \|P_{\overline{R(T)}} y - z\|^2 + \|P_{R(T)^\perp} y\|^2$$

which implies that $P_{\overline{R(T)}} y = z$. Therefore $y = z + P_{N(T^*)} y \in Tu + N(T^*)$.

Conversely, assume that $u \in D(T)$ and that $y = a + b$ for some $a \in Tu$ and $b \in N(T^*)$. Then

$$d(y, R(T)) = \|P_{R(T)^\perp} y\| = \|P_{R(T)^\perp} a + P_{R(T)^\perp} b\| = \|b\| := \|y - a\|.$$

(ii) Let $y \in D(T^\dagger)$. According to Proposition 3.3 we have that $y = P_{N(T^*)^\perp}y + P_{N(T^*)}y \in TT^{-1}P_{N(T^*)^\perp}y + P_{N(T^*)}y = TT^\dagger y + P_{N(T^*)}y$. This together with the part (i) leads to (ii).

(iii) If $u \in H_1$ is a least square solution of $y \in Tx$, then from Proposition 3.3 and the part (i) we infer that $T^\dagger y \in T^\dagger Tu$ and $T^\dagger y = P_{N(T)^\perp}u$ which yields

$$u = P_{N(T)^\perp}u + P_{N(T)}u \in T^\dagger y + N(T).$$

This proves (iii).

(iv) By virtue of (iii) we have that if $u \in H_1$ is a least square solution of $y \in Tx$, then $u \in T^\dagger y + v$ for some $v \in N(T)$ and noting that $T^\dagger y \in R(T^\dagger) \subset N(T)^\perp$ we deduce that

$$\|u\|^2 = \|T^\dagger y\|^2 + \|v\|^2.$$

Hence (iv) holds. □

4. Approximation of Moore–Penrose inverse of a closed linear relation

Our main objective in this section is to prove that if $T \in CR(H_1, H_2)$ then there exists a sequence of finite rank linear relations (T_n) such that for each $y \in D(T^\dagger)$, $T^\dagger y = \lim_{n\to\infty} T_n^\dagger y$.

We begin with the following result which is crucial for our purpose.

Proposition 4.1. *Let $T \in CR(H_1, H_2)$ such that $T(0)$ is a proper subspace of $R(T)$. Let (Y_n) be an increasing sequence of finite dimensional subspaces of $R(T)$ satisfying $\overline{\cup_{n=1}^\infty Y_n} = \overline{R(T)}$. For each $n \in \mathbb{N}$, let $Z_n := \Delta_{T^*}Y_n$ and $X_n := T^*Z_n \cap D(T)$. Then*

(i) $\overline{\cup_{n=1}^\infty Z_n} = \overline{R(T)} \cap T(0)^\perp$.

(ii) *(X_n) is an increasing sequence of subspaces of $R(T^*) \cap D(T)$ such that $P_{T^*(0)^\perp}X_n = X_n, P_{T^*(0)}X_n = \{0\}$ and $\dim X_n < \infty$.*

(iii) $\overline{\cup_{n=1}^\infty X_n} = \overline{R(T^*)} \cap T^*(0)^\perp$.

Proof. (i) By the equality $\overline{\Delta_{T^*}\overline{R(T)}} = \overline{R(T)} \cap T(0)^\perp$ which was proved in Proposition 2.3 (v) we obtain that

$$\overline{R(T)} \cap T(0)^\perp = \overline{\Delta_{T^*}\overline{R(T)}} = \overline{\Delta_{T^*}\overline{(\cup_{n=1}^\infty Y_n)}} = \overline{\cup_{n=1}^\infty \Delta_{T^*}Y_n} = \overline{\cup_{n=1}^\infty Z_n}.$$

(ii) Since $T^*(0)^\perp = \overline{D(T)}$, Δ_{T^*} and $P_{T^*(0)^\perp}T^*$ are operators and $\dim Z_n < \infty$ we have the assertion (ii).

(iii) By (ii), it is clear that $\overline{\cup_{n=1}^{\infty} X_n} \subset \overline{D(T)} \cap \overline{R(T^*)} = T^*(0)^{\perp} \cap \overline{R(T^*)}$. Suppose that this inclusion is strict. Then there exists $0 \neq z \in N(T)^{\perp} \cap \overline{D(T)}$ such that $z \in (\overline{\cup_{n=1}^{\infty} X_n})^{\perp}$, that is, $0 = \langle z, w \rangle$ for all $w \in \overline{\cup_{n=1}^{\infty} X_n}$ and thus it follows from the Definition 2.1 that

$$0 = \langle z, T^*\Delta_{T^*} y \rangle = \langle z, P_{T^*(0)^{\perp}} T^* \Delta_{T^*} y \rangle \quad \text{for all } y \in R(T)$$

and by the continuity of $P_{T^*(0)^{\perp}} T^* \Delta_{T^*}$, this holds for all $y \in \overline{R(T)}$. We claim that is holds for all $y \in H_2$. Let $y \in H_2$. Then $y = u + v$ for some $u \in \overline{R(T)}$ and $v \in R(T)^{\perp} = N(T^*) \subset D(T^*)$. Then by Proposition 2.3 (iii) we obtain that $P_{T^*(0)^{\perp}} T^* \Delta_{T^*} v = \Delta_T T^* v = 0$. Therefore

$$\langle z, P_{T^*(0)^{\perp}} T^* \Delta_{T^*} y \rangle = \langle z, P_{T^*(0)^{\perp}} T^* \Delta_{T^*} u \rangle = 0.$$

This proves the claim.

On the other hand, since $\overline{D(T)} \cap N(T)^{\perp} \subset \overline{D(T) \cap N(T)^{\perp}}$, there exists a sequence $(z_n) \subset D(T) \cap N(T)^{\perp}$ such that $z_n \to z$. Hence for all $y \in H_2$, by Proposition 2.3 (iii)

$$\begin{aligned} 0 = \langle z, P_{T^*(0)^{\perp}} T^* \Delta_{T^*} y \rangle &= \lim_{n\to\infty} \langle z_n, P_{T^*(0)^{\perp}} T^* \Delta_{T^*} y \rangle \\ &= \lim_{n\to\infty} \langle (P_{T^*(0)^{\perp}} T^* \Delta_{T^*})^* z_n, y \rangle = \lim_{n\to\infty} \langle P_{T(0)^{\perp}} T \Delta_T z_n, y \rangle. \end{aligned}$$

This shows that $P_{T(0)^{\perp}} T \Delta_T z_n \to 0$ (weakly) and since $P_{T(0)^{\perp}} T \Delta_T$ is a bounded operator, we have $P_{T(0)^{\perp}} T \Delta_T z = 0$, so that $P_{T^*(0)^{\perp}} T^* P_{T(0)^{\perp}} T \Delta_T z = 0$. Combining this equality with Proposition 2.3 (iv) and the fact in this setting $P_{T^*(0)} z \in P_{T^*(0)} T^*(0)^{\perp} = \{0\}$ we obtain that $z = \Delta_T z$ with $P_{T(0)^{\perp}} T \Delta_T z = 0$. Hence $\Delta_T z \in N(P_{T(0)^{\perp}} T) = T^{-1} N(P_{T(0)^{\perp}}) = T^{-1} T(0) = T^{-1}(0) = N(T)$. Consequently, $z \in N(T) \cap N(T)^{\perp}$, a contradiction to our assumption. This proves (iii). $\square$

This result generalizes an analogous result for densely defined closed operators proved in [7, Lemma 3.1].

It should be noted that the class of all linear relations T such that $\overline{R(T)} = \overline{T}(0)$ coincides with the class of all singular linear relations introduced in [5], in order to study some canonical decompositions of linear relations.

Recall the following very well known property of the limit of an increasing sequence of orthogonal projections.

Lemma 4.2. *Let (M_n) be an increasing sequence of closed subspaces of a Hilbert space H. Then the strong* $\lim P$ *of (P_{M_n}) exists and P is the projection onto $\overline{\cup_{n=1}^{\infty} M_n}$.*

We also recall the following elementary property of the inverse of a linear relation.

Lemma 4.3 (2, Corollary I.2.11)**.** *We have that $TT^{-1}y = y + T(0)$ whenever $y \in R(T)$.*

Now we are able to state our second main result.

Theorem 4.4. *Let $T \in CR(H_1, H_2)$ with separable range and $T(0)$ is a proper subspace of $R(T)$. Then for each $n \in \mathbb{N}$, there exists a finite rank linear relation T_n such that*

$$\lim_{n\to\infty} T_n^{\dagger} y = T^{\dagger} y \quad \textit{for all } y \in D(T^{\dagger}).$$

Proof. Since $R(T)$ is separable, we can find an increasing sequence of finite dimensional subspaces of $R(T)$, say (Y_n), such that $\overline{\cup_{n=1}^{\infty} Y_n} = \overline{R(T)}$.

Let Z_n and X_n be as Proposition 4.1. Let $P_n : H_2 \to H_2$ and $Q_n : H_1 \to H_1$ be sequences of orthogonal projections with $R(P_n) = Z_n$ and $R(Q_n) = X_n$. For each $n \in \mathbb{N}$, we define $T_n := P_n T$.

Then the sequence (T_n) has the following properties:

(1) $D(T_n) = D(T), N(T_n) = X_n^{\perp} \cap D(T), R(T_n) = R(P_n) = Z_n, N(T_n^*) = Z_n^{\perp}$ and $R(T_n^*) = X_n$.

Indeed, we have that

$$D(T_n) := \{x \in D(T) : Tx \cap D(P_n) \neq \emptyset\} = \{x \in D(T) : Tx \neq \emptyset\} := D(T)$$

and

$$N(T_n) := (P_n T)^{-1}(0) = T^{-1} N(P_n) = T^{-1}(Z_n^{\perp}) = (T^* Z_n)^{\perp} \cap D(T)$$

(by Lemma 2.2 (ii))$:= X_n^{\perp} \cap D(T)$.

Next we claim that $R(T_n) = R(P_n) = Z_n$. It is clear that $R(T_n) \subset R(P_n) = Z_n$. In order to show the converse inclusion it is enough to state that $N(T_n^*) \subset N(P_n)$. To see this, we note that $N(T_n^*) \subset N(P_n) \Rightarrow N(P_n)^{\perp} = Z_n \subset N(T_n^*)^{\perp} = \overline{R(T_n)} = R(T_n)$. Let now $z \in N(T_n^*) = N(T^* P_n)$, so that $P_n z \in N(T^*) = R(T)^{\perp}$ and since $P_n z \in R(P_n) = Z_n \subset \overline{R(T)}$ (by the condition (i) of Proposition 4.1) we infer that $P_n z = 0$, that is, $z \in N(P_n)$, as desired. Therefore $R(T_n) = R(P_n) = Z_n$ which further implies that $N(T_n^*) = Z_n^{\perp}$ and clearly $R(T_n^*) = T^* Z_n = X_n$. Hence (1) holds.

(2) $y \in TT^{\dagger} y$ for all $y \in R(T)$.

Follows immediately from Proposition 3.3 and Lemma 4.3.

(3) $Q_n x \to x$ for all $x \in D(T) \cap N(T)^{\perp}$.

Follows immediately from Proposition 4.1 and Lemma 4.2.

(4) $D(T_n^\dagger) = H_2$.

It follows from Proposition 2.3 (i) that $D(T_n^\dagger) = R(T_n) \oplus R(T_n)^\perp$ and by (1) $R(T_n) = Z_n$ which is finite dimensional. Therefore $T_n^\dagger$ is everywhere defined.

Our next aim is to prove that $T^\dagger y = \lim_{n\to} T_n^\dagger y$ for all $y \in D(T^\dagger)$. Let $y \in D(T^\dagger)$. Then

(5) $Q_n T^\dagger y \to T^\dagger y$.

This statement follows immediately combining (3) with the fact that $T^\dagger y \in D(T) \cap N(T)^\perp$.

Next we show that (6) $Q_n T^\dagger y = T_n^\dagger y$ for all $y \in D(T^\dagger)$.

From the facts $Q_n T^\dagger y \in X_n \subset D(T) \cap N(T)^\perp, (Q_n - I)T^\dagger y \in N(T_n) = X_n^\perp \cap D(T)$ (by (1)) and Proposition 2.3 (iii) we get that

$$Q_n T^\dagger y = T_n^\dagger T_n Q_n T^\dagger y.$$

In view of the foregoing, we have that

$$\begin{aligned} T_n^\dagger T_n Q_n T^\dagger y &= T_n^\dagger T_n Q_n T^\dagger y - T_n^\dagger P_n y + T_n^\dagger P_n y = T_n^\dagger (T_n Q_n T^\dagger y - P_n y) \\ &\quad + T_n^\dagger P_n y \subset T_n^\dagger (T_n Q_n T^\dagger y - P_n T T^\dagger y) + T_n^\dagger P_n y \\ &= T_n^\dagger (T_n Q_n - P_n T) T^\dagger y + T_n^\dagger P_n y = T_n^\dagger (T_n Q_n - T_n) T^\dagger y \\ &\quad + T_n^\dagger P_n y \subset T_n^\dagger T_n (Q_n - I) T^\dagger y + T_n^\dagger P_n y \subset T_n^\dagger T_n T_n^{-1}(0) + T_n^\dagger P_n y \\ &= T_n^\dagger T_n(0) + T_n^\dagger P_n y = T_n^\dagger P_n y. \end{aligned}$$

This proves (6). The result now follows from (5) and (6). □

The above result for densely defined closed operators was proved by Kulkarni and Ramesh in a recent paper [7, Theorem 3.3].

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TERESA ÁLVAREZ
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OVIEDO
C/ CALVO SOTELO S/N 33007
OVIEDO, ASTURIAS
SPAIN

E-mail: seco@uniovi.es

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