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On some applications of Eisenstein series

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Abstract. We derive the uniqueness of the theta functions associated with certain quadratic forms. Furthermore, we show some partially multiplicative relations between the representation numbers of such quadratic forms. To this end we apply Fricke involutions and Hecke operators to Eisenstein series.

1. Introduction

For positive integers k and N let $\mathrm{Mat}(k,N)$ be the set of $2k\times 2k$ integral positive definite symmetric matrices A with $\det(A)=N$ for which both A and NA^{-1} have even diagonal entries. For such a matrix A in $\mathrm{Mat}(k,N)$ let Q be its associated quadratic form and $r_Q(n)$ be the representation number by Q for a nonnegative integer n, namely

$$Q(\mathbf{x})=\frac{1}{2}\mathbf{x}^TA\mathbf{x} \text{ for } \mathbf{x}=(x_1,x_2,\dots,x_{2k})\in\mathbb{Z}^{2k} \quad\text{and}\quad r_Q(n)=\#\{\mathbf{x}\in\mathbb{Z}^{2k};\ Q(\mathbf{x})=n\}.$$

We consider the theta function

$$\Theta_Q(\tau)=\sum_{n=0}^{\infty}r_Q(n)e^{2\pi i\tau n},$$

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which belongs to the space $\mathcal{M}_k(N, \chi_{(-1)^k N})$ of weight k modular forms for $\Gamma_0(N)$ associated with the character $\chi_{(-1)^k N}(\cdot) = \left(\frac{(-1)^k N}{\cdot}\right)$ [7, Corollary 4.9.5(3)].

In particular, when $(k, N) = (2, 13)$ and $A = \left[\begin{smallmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{smallmatrix}\right]$, we have $Q = x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_1x_3 + x_1x_4 + x_2x_4$, and $\Theta_Q(\tau)$ lies in $\mathcal{M}_2(13, \chi_{13})$. Eum *et al.* [2, Example 3.4] provided a basis of the space $\mathcal{M}_2(\Gamma_1(13))$ of modular forms of weight 2 for $\Gamma_1(13)$ in terms of Klein forms, and expressed $\Theta_Q(\tau)$ as a linear combination of such basis elements. In the process they found the interesting identity

$$r_Q(1)r_Q(p^2 n) = r_Q(p^2)r_Q(n) \quad \text{for any prime } p \nmid 13 \text{ and any positive integer } n \text{ such that } p \nmid n. \quad (1)$$

Recently they proved (1) in [3], by combining $\dim_{\mathbb{C}} \mathcal{M}_2(13, \chi_{13}) = 2$ with Hecke's two Eisenstein series [4].

In this paper we shall further develop the above result as follows. Let $k \geq 2$ and N be any positive integers such that $(-1)^k N$ is the discriminant of a quadratic field. Let A be a matrix in $\mathrm{Mat}(k, N)$ and let Q be its associated quadratic form. Suppose that

$$\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2. \quad (2)$$

Then we shall first show that there are only finitely many pairs (k, N) that satisfy (2) (Corollary 2.4(iii)). We shall also prove that the theta series $\Theta_Q(\tau)$ depends only on (k, N) (Theorem 6.2 and Corollary 6.3); hence, the representation numbers $r_Q(n)$ can be written in terms of a generalized Bernoulli number (Remark 6.5). For this, we shall investigate the action of the Fricke involution $\cdot|_k \omega_N$ on Hecke's two Eisenstein series (Corollary 4.4). On the other hand, let $\mathcal{S}_k(N, \chi_{(-1)^k N})$ be the subspace of $\mathcal{M}_k(N, \chi_{(-1)^k N})$ consisting of cusp forms. If we further assume that $\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 0$, which is weaker than (2) (Corollary 2.4(ii),(iii)), then we are able to claim the relation (1) for these finitely many pairs (k, N) (Theorem 6.4 and Example 7.2) by applying Hecke operators to these Eisenstein series ($5).

2. Modular forms

We denote by $\mathbb{H}$ the complex upper half-plane. Let k be an integer and $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be an element of $\mathrm{GL}_2^+(\mathbb{R})$. We define the *slash operator* $\cdot|_k \alpha$ on the functions $f(\tau)$ on $\mathbb{H}$ by

$$f(\tau)|_k\alpha := \det(\alpha)^{k/2}(c\tau + d)^{-k}(f(\tau)\circ\alpha),$$

where α acts on $\mathbb{H}$ as the fractional linear transformation $\tau \mapsto (a\tau + b)/(c\tau + d)$. Note that

$$(f(\tau)|_k\alpha)|_k\beta = f(\tau)|_k\alpha\beta \quad (\alpha, \beta \in \mathrm{GL}_2^+(\mathbb{R})). \tag{3}$$

Let N be a positive integer and let Γ be one of the following congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma_1(N) := \left\{\alpha \in \mathrm{SL}_2(\mathbb{Z});\ \alpha \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod N\right\},$$

$$\Gamma_0(N) := \left\{\alpha \in \mathrm{SL}_2(\mathbb{Z});\ \alpha \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \pmod N\right\}.$$

A holomorphic function $f(\tau)$ on $\mathbb{H}$ is called a *modular form* of weight k for Γ if

(i) $f(\tau)|_k\alpha = f(\tau)$ for all $\alpha \in \Gamma$,

(ii) $f(\tau)$ is holomorphic at every cusp ($\in \mathbb{Q} \cup \{\infty\}$). In particular, since $f(\tau)|_k \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] = f(\tau+1) = f(\tau)$ by (i), $f(\tau)$ has a Laurent series expansion with respect to

$$q := e^{2\pi i\tau}$$

of the form

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \quad (a(n) \in \mathbb{C}),$$

which is called the *Fourier expansion* of $f(\tau)$ (at the cusp ∞).

Moreover, if a modular form vanishes at every cusp, then it is called a *cusp form*. We denote the space of all modular forms (respectively, cusp forms) of weight k for Γ by $\mathcal{M}_k(\Gamma)$ (respectively, $\mathcal{S}_k(\Gamma)$).

For a Dirichlet character χ modulo N we define a character χ of $\Gamma_0(N)$ by

$$\chi(\alpha) := \chi(d) \quad \text{for } \alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N).$$

We let

$$\mathcal{M}_k(N,\chi) := \{f(\tau) \in \mathcal{M}_k(\Gamma_1(N));\ f(\tau)|_k\alpha = \chi(\alpha)f(\tau) \text{ for all } \alpha \in \Gamma_0(N)\},$$

$$\mathcal{S}_k(N,\chi) := \mathcal{M}_k(N,\chi) \cap \mathcal{S}_k(\Gamma_1(N)),$$

which are subspaces of $\mathcal{M}_k(\Gamma_1(N))$ and $\mathcal{S}_k(\Gamma_1(N))$, respectively. Then we have the decomposition

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_k(N,\chi),$$

where χ runs over all Dirichlet characters modulo N [7, Lemma 4.3.1]. Here we observe that if $\chi(-1) \neq (-1)^k$, then the space $\mathcal{M}_k(N,\chi)$ is known to be $\{0\}$ [7, Lemma 4.3.2(1)].

Proposition 2.1. *Let N be a positive integer.*

(i) $\dim_{\mathbb{C}} \mathcal{M}_k(\Gamma_1(N)) = 0$ *for any negative integer k.*

(ii) $\dim_{\mathbb{C}} \mathcal{M}_0(\Gamma_1(N)) = 1$, *and hence* $\dim_{\mathbb{C}} \mathcal{M}_0(N,\chi) = 0$ *if χ is nontrivial.*

PROOF. See [7, Theorems 2.5.2 and 2.5.3]. □

Proposition 2.2. *For an integer m, let*

$$\nu_m := \begin{cases} 0 & \text{if } m \text{ is odd}, \\ -1/4 & \text{if } m \equiv 2 \pmod 4, \\ 1/4 & \text{if } m \equiv 0 \pmod 4, \end{cases} \quad \text{and} \quad \mu_m := \begin{cases} 0 & \text{if } m \equiv 1 \pmod 3, \\ -1/3 & \text{if } m \equiv 2 \pmod 3, \\ 1/3 & \text{if } m \equiv 0 \pmod 3. \end{cases}$$

Let k be an integer and χ be a primitive Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. Then we have the dimension formula

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M}_k(N,\chi) - \dim_{\mathbb{C}} \mathcal{S}_{2-k}(N,\chi) &= \frac{(k-1)N}{12} \prod_{p|N}(1+p^{-1}) \\ &\quad + \frac{1}{2} \prod_{p|N} 2 - \nu_{2-k}\alpha(\chi) - \mu_{2-k}\beta(\chi), \end{aligned} \tag{4}$$

where

$$\alpha(\chi) := \sum_{\substack{x \pmod N \\ x^2+1\equiv 0 \pmod N}} \chi(x) \quad \text{and} \quad \beta(\chi) := \sum_{\substack{x \pmod N \\ x^2+x+1\equiv 0 \pmod N}} \chi(x).$$

PROOF. See [1, Théorèm 1] or [8, Theorem 1.34]. □

Remark 2.3. (i) If we replace k by $2-k$ in the formula (4), then we obtain

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M}_{2-k}(N,\chi) - \dim_{\mathbb{C}} \mathcal{S}_k(N,\chi) &= \frac{(1-k)N}{12} \prod_{p|N}(1+p^{-1}) \\ &\quad + \frac{1}{2} \prod_{p|N} 2 - \nu_k\alpha(\chi) - \mu_k\beta(\chi). \end{aligned} \tag{5}$$

Suppose $k \geq 2$ and χ is nontrivial. Then
$\dim_{\mathbb{C}} \mathcal{M}_{2-k}(N,\chi) = \dim_{\mathbb{C}} \mathcal{S}_{2-k}(N,\chi) = 0$ by Proposition 2.1. And, we see that $\nu_{2-k} + \nu_k = \mu_{2-k} + \mu_k = 0$. Thus we derive by adding (4) and (5)

$$\dim_{\mathbb{C}} \mathcal{M}_k(N,\chi) - \dim_{\mathbb{C}} \mathcal{S}_k(N,\chi) = \prod_{p|N} 2. \tag{6}$$

(ii) If N is the product of ℓ distinct prime numbers, then the equation $x^2 + 1 \equiv 0 \pmod{N}$ has at most 2^ℓ solutions in $\mathbb{Z}/N\mathbb{Z}$ by the Chinese remainder theorem. Hence we get $|\alpha(\chi)| \leq 2^\ell$. Similarly, $|\beta(\chi)| \leq 2^\ell$.

(iii) The equations $x^2 + 1 \equiv 0 \pmod{4}$ and $x^2 + x + 1 \equiv 0 \pmod{4}$ are not solvable. So, if $4|N$, then $\alpha(\chi) = \beta(\chi) = 0$.

For a nonzero integer N with $N \equiv 0$ or $1 \pmod{4}$, by χ_N we mean the Dirichlet character modulo $|N|$ which is defined by

$$\chi_N(d) := \text{the Kronecker symbol } \left(\frac{N}{d}\right) \text{ for } d \in (\mathbb{Z}/|N|\mathbb{Z})^\times.$$

Observe that

$$\left(\frac{N}{-1}\right) := \begin{cases} 1 & \text{if } N > 0, \\ -1 & \text{if } N < 0. \end{cases} \tag{7}$$

In particular, if N is the discriminant of a quadratic field, namely

$$N = \begin{cases} m & \text{if } m \equiv 1 \pmod{4} \\ 4m & \text{if } m \not\equiv 1 \pmod{4} \end{cases} \quad \text{for a square-free integer } m\ (\neq 1),$$

then χ_N becomes a primitive Dirichlet character modulo $|N|$ [7, pp. 82–84].

Corollary 2.4. *Let k (≥ 2) and N be positive integers such that $(-1)^k N$ is the discriminant of a quadratic field.*

(i) $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) - \dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 2$ *if and only if*

$$\begin{cases} k (\geq 3) \text{ is odd and } N = 4, \\ k \geq 2 \text{ and } N = 8, \\ k \geq 2 \text{ and } N \text{ is a prime such that } N \equiv (-1)^k \pmod{4}. \end{cases}$$

(ii) $\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 0$ *if and only if*

$$(k,N) = (2,5), (2,8), (2,12), (2,13), (2,17), (2,21), (3,3), (3,4), (4,5), (5,3).$$

(iii) $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$ *if and only if*

$$(k,N) = (2,5),(2,8),(2,13),(2,17),(3,3),(3,4),(4,5),(5,3).$$

PROOF. (i) We deduce from (6) that

$$\dim_{\mathbb{C}} \mathcal{M}_k(N,\chi_{(-1)^k N}) - \dim_{\mathbb{C}} \mathcal{S}_k(N,\chi_{(-1)^k N}) = 2 \iff N \text{ has only one prime factor.}$$

Since $(-1)^k N$ is the discriminant of a quadratic field, we get the assertion (i).

(ii) Since $k \geq 2$ and $\chi_{(-1)^k N}$ is nontrivial, $\dim_{\mathbb{C}} \mathcal{M}_{2-k}(N, \chi_{(-1)^k N}) = 0$ by Proposition 2.1. It then follows from (5) that

$$\dim_{\mathbb{C}} \mathcal{S}_k(N,\chi_{(-1)^k N}) = \frac{(k-1)N}{12}\prod_{p|N}(1+p^{-1}) - \frac{1}{2}\prod_{p|N} 2 + \nu_k \alpha(\chi_{(-1)^k N}) + \mu_k \beta(\chi_{(-1)^k N}). \tag{8}$$

First, we consider the case where N has the prime factorization $N = \prod_{j=1}^{\ell} p_j$ with p_j odd and $p_j < p_{j+1}$. Now we see that

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{S}_k(N,\chi_{(-1)^k N}) &\geq \frac{k-1}{12}\prod_{j=1}^{\ell}(p_j+1) - \frac{1}{2}2^{\ell} - \frac{1}{4}2^{\ell} - \frac{1}{3}2^{\ell} \\
&\qquad\qquad \text{by (8) and Remark 2.3(ii)} \\
&\geq \begin{cases} \dfrac{(k-1)(p_1+1)}{12} - \dfrac{13}{6} & \text{if } \ell = 1, \\[2ex] \dfrac{(k-1)(p_1+1)(p_2+1)^{\ell-1}}{12} - \dfrac{13\cdot 2^{\ell-1}}{6} & \text{if } \ell \geq 2, \end{cases} \\
&= \begin{cases} \dfrac{(k-1)(p_1+1)-26}{12} & \text{if } \ell = 1, \\[2ex] \dfrac{13\cdot 2^{\ell-1}}{6}\left(\dfrac{(k-1)(p_1+1)}{26}\left(\dfrac{p_2+1}{2}\right)^{\ell-1} - 1\right) & \text{if } \ell \geq 2. \end{cases}
\end{aligned}$$

If $\ell \geq 3$, then the above inequality yields

$$\dim_{\mathbb{C}} \mathcal{S}_k(N,\chi_{(-1)^k N}) \geq \frac{13\cdot 2^2}{6}\left(\frac{(k-1)(3+1)}{26}\left(\frac{5+1}{2}\right)^2 - 1\right) > 0,$$

because $p_1 \geq 3$ and $p_2 \geq 5$. Thus we again achieve from the above inequality that $\dim_{\mathbb{C}} \mathcal{S}_k(N,\chi_{(-1)^k N}) = 0$ implies

$$(k,N) \in \begin{cases} \{(2,5),(2,13),(2,17),(3,3),(3,7),(3,11),(4,5),(5,3),(7,3)\} & \text{if } \ell = 1, \\ \{(2,21),(2,33),(3,15)\} & \text{if } \ell = 2. \end{cases}$$

And, the formula (8) leads us to the fact

$$\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 0 \Longleftrightarrow (k, N) = (2,5), (2,13), (2,17), (3,3), (4,5), (5,3), (2,21).$$

Next, suppose that $N = 4 \cdot 2^m \prod_{j=1}^{\ell} p_j$ with $m \in \{0,1\}$, $\ell \geq 1$, p_j odd and $p_j < p_{j+1}$. Then we have that

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) &= \frac{2^m(k-1)}{2} \prod_{j=1}^{\ell} (p_j + 1) - \frac{1}{2} 2^{\ell+1} \\
&\qquad\qquad\qquad\qquad \text{by (8) and Remark 2.3(iii)} \\
&\geq \begin{cases} \dfrac{2^m(k-1)(p_1+1)}{2} - 2 & \text{if } \ell = 1, \\[2ex] \dfrac{2^m(k-1)(p_1+1)(p_2+1)^{\ell-1}}{2} - 2^{\ell} & \text{if } \ell \geq 2, \end{cases} \\
&= \begin{cases} \dfrac{2^m(k-1)(p_1+1) - 4}{2} & \text{if } \ell = 1, \\[2ex] 2^{\ell}\left(\dfrac{2^m(k-1)(p_1+1)}{4}\left(\dfrac{p_2+1}{2}\right)^{\ell-1} - 1\right) & \text{if } \ell \geq 2. \end{cases} \qquad (9)
\end{aligned}$$

One can then readily show that if $m = 1$ or $\ell \geq 2$, $\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) > 0$. And, (9) implies

$$\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 0 \Longleftrightarrow (k, N) = (2, 12).$$

Lastly, let $N = 4$ or 8. We get by (8) and Remark 2.3(iii)

$$\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = \begin{cases} \dfrac{k-1}{2} - 1 & \text{if } N = 4, \\[2ex] (k-1) - 1 & \text{if } N = 8. \end{cases}$$

Hence it follows that

$$\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 0 \Longleftrightarrow (k, N) = (3,4), (2,8).$$

This completes the proof of (ii).

(iii) By (6) we deduce

$$\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = \dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) + \prod_{p \mid N} 2,$$

from which we conclude that

$$\begin{aligned}
&\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2 \Longleftrightarrow \dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 0, \\
&\qquad \text{and } N \text{ has only one prime factor} \\
&\Longleftrightarrow (k, N) = (2,5), (2,8), (2,13), (2,17), (3,3), (3,4), (4,5), (5,3) \quad \text{by (ii).} \quad \square
\end{aligned}$$

Remark 2.5. If $(k, N) = (2, 12)$ or $(2, 21)$, then $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 4$ by (6).

3. Eisenstein series

Let χ be a nontrivial primitive Dirichlet character modulo N. The *Dirichlet L-function* for χ is defined by

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (s \in \mathbb{C}),$$

where we set $\chi(n) = 0$ if $\gcd(n, N) \neq 1$. This series converges for $\mathrm{Re}(s) > 1$ and extends to an entire function [6, Chapter XIV, Theorem 2.2(ii)].

Lemma 3.1. *Let k be a positive integer.*

(i) *$L(1-k, \chi) \neq 0$ if and only if $\chi(-1) = (-1)^k$.*

(ii) *We have*

$$L(1-k, \chi) = -\frac{B_{k,\chi}}{k},$$

where $B_{k,\chi}$ is a generalized Bernoulli number defined by

$$\sum_{a=1}^{N-1} \chi(a) \frac{t e^{at}}{e^{Nt} - 1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}.$$

Proof. (i) See [6, Chapter XIV, Corollary of Theorem 2.2].

(ii) See [6, Chapter XIV, Theorem 2.3]. $\square$

Proposition 3.2. *Let k be a positive integer, and let ψ_1 and ψ_2 be Dirichlet characters modulo N_1 and N_2, respectively, such that $\psi_1(-1)\psi_2(-1) = (-1)^k$. Suppose that ψ_1 and ψ_2 satisfy the following condition:*

$$\begin{cases} \text{if } k = 2 \text{ and both } \psi_1 \text{ and } \psi_2 \text{ are trivial,} \\ \text{then } N_1 = 1 \text{ and } N_2 \text{ is a prime,} \\ \text{otherwise, } \psi_1 \text{ and } \psi_2 \text{ are primitive characters.} \end{cases} \tag{10}$$

Let $N = N_1 N_2$ and $\chi = \psi_1\psi_2$. Define

$$E_{k,\psi_1,\psi_2}(\tau) := a_0 + \sum_{n=1}^{\infty}\Bigg(\sum_{d>0,d|n}\psi_1(n/d)\psi_2(d)d^{k-1}\Bigg)q^n, \tag{11}$$

where

$$a_0 = \begin{cases} 0 & \text{if } k \neq 1 \text{ and } \psi_1 \text{ is nontrivial,} \\ & \text{or if both } \psi_1 \text{ and } \psi_2 \text{ are nontrivial,} \\ (N-1)/24 & \text{if } k = 2, \text{ and both } \psi_1 \text{ and } \psi_2 \text{ are trivial,} \\ -B_{k,\chi}/2k & \text{otherwise.} \end{cases}$$

Then $E_{k,\psi_1,\psi_2}(\tau)$ defines an element of $\mathcal{M}_k(N,\chi)$, which is called an Eisenstein series.

Proof. See [7, Theorem 4.7.1]. □

Proposition 3.3. *Let k be a positive integer and χ be a Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. Set*

$$\mathcal{E}_k(N,\chi) := \mathrm{Span}_{\mathbb{C}}\{E_{k,\psi_1,\psi_2}(\ell\tau) \mid \psi_1\psi_2 = \chi,\ \psi_1 \text{ and } \psi_2 \text{ satisfy the condition (10)},\ \ell > 0,\ \ell N_1 N_2 | N\}.$$

Then we have the decomposition

$$\mathcal{M}_k(N,\chi) = \mathcal{E}_k(N,\chi) \oplus \mathcal{S}_k(N,\chi).$$

Proof. See [7, Theorems 2.1.7(1) and 4.7.2]. □

Corollary 3.4. *Let k (≥ 2) and N be positive integers such that $(-1)^k N$ is the discriminant of a quadratic field. Then the Eisenstein series*

$$\begin{aligned} G_{k,N}(\tau) &:= E_{k,\chi_0,\chi_{(-1)^k N}}(\tau) = -\frac{B_{k,\chi_{(-1)^k N}}}{2k} \\ &\quad + \sum_{n=1}^{\infty}\Bigg(\sum_{d>0,d|n}\chi_{(-1)^k N}(d)d^{k-1}\Bigg)q^n, \end{aligned} \tag{12}$$

$$H_{k,N}(\tau) := E_{k,\chi_{(-1)^k N},\chi_0}(\tau) = \sum_{n=1}^{\infty}\Bigg(\sum_{d>0,d|n}\chi_{(-1)^k N}(n/d)d^{k-1}\Bigg)q^n, \tag{13}$$

with χ_0 the principal character, are linearly independent elements of $\mathcal{E}_k(N,\chi_{(-1)^k N})$.

Proof. See [3, Corollary 2.7]. □

Remark 3.5. The constant term of $G_{k,N}(\tau)$ does not vanish by Lemma 3.1.

For a positive integer k, let ψ_1 and ψ_2 be Dirichlet characters modulo N_1 and N_2, respectively, such that $\psi_1(-1)\psi_2(-1) = (-1)^k$. We define a function

$$E_k(\tau, s; \psi_1, \psi_2) := \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \psi_1(m)\psi_2(n)(m\tau+n)^{-k}|m\tau+n|^{-2s} \qquad (\tau\in\mathbb{H},\ s\in\mathbb{C}). \tag{14}$$

Then the right-hand side is uniformly and absolutely convergent for $k+2\mathrm{Re}(s) \geq 2+\varepsilon$ ($\varepsilon > 0$), and so it is holomorphic on $k+2\mathrm{Re}(s) > 2$. Furthermore, it is analytically continued to a holomorphic function

$$H_k(\tau, s; \psi_1, \psi_2) \text{ for } \begin{cases} \text{the whole } s\text{-plane} & \text{if } \psi_2 \text{ is nontrivial,} \\ \mathrm{Re}(s) > (1-k)/2 & \text{if } \psi_2 \text{ is trivial and } k \geq 2, \\ \mathrm{Re}(s) > -1/2 & \text{if } \psi_2 \text{ is trivial and } k = 1 \end{cases} \tag{15}$$

[7, Theorem 7.2.9 and Corollary 7.2.10]. Since $H_k(\tau, s; \psi_1, \psi_2)$ is holomorphic at $s = 0$, we set

$$E_k(\tau; \psi_1, \psi_2) := H_k(\tau, 0; \psi_1, \psi_2). \tag{16}$$

Then $E_k(\tau; \psi_1, \psi_2)$ becomes a holomorphic function of τ on $\mathbb{H}$ except for the case where $k = 2$ and both ψ_1 and ψ_2 are trivial [7, Corollary 7.2.14].

Proposition 3.6. *Let ψ_1 and ψ_2 be primitive Dirichlet characters modulo N_1 and N_2, respectively, such that $\psi_1(-1)\psi_2(-1) = (-1)^k$. Except for the case where $k = 2$ and $N_1 = N_2 = 1$ we get the relation*

$$E_k(N_2\tau; \psi_1, \psi_2) = A_{k,\psi_2} E_{k,\psi_1,\overline{\psi}_2}(\tau),$$

where

$$A_{k,\psi_2} := \frac{2(-2\pi i)^k W(\psi_2)}{N_2^k (k-1)!} \quad \text{and} \quad W(\psi_2) := \sum_{a=0}^{N_2-1} \psi_2(a) e^{2\pi i a/N_2}.$$

Proof. See [7, (7.1.13) and Lemma 7.2.19]. □

Remark 3.7. It is well-known that $|W(\psi_2)| = \sqrt{N_2}$ [7, Lemma 3.1.1(4)].

4. Fricke involutions

Let k and N be positive integers. The slash operator

$$\cdot|_k\omega_N \quad \text{with} \quad \omega_N := \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}$$

on the functions $f(\tau)$ ($\tau \in \mathbb{H}$) is called the *Fricke involution* of weight k and level N. We derive by the property (3) that

$$(f(\tau)|_k\omega_N)|_k\omega_N = f(\tau)|_k\omega_N^2 = f(\tau)|_k \begin{bmatrix} -N & 0 \\ 0 & -N \end{bmatrix} = (-1)^k f(\tau). \tag{17}$$

Proposition 4.1. *Let k be an integer and χ be a Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. Then the correspondence $f(\tau) \mapsto f(\tau)|_k\omega_N$ induces isomorphisms*

$$\mathcal{M}_k(N,\chi) \simeq \mathcal{M}_k(N,\overline{\chi}),\ \mathcal{S}_k(N,\chi) \simeq \mathcal{S}_k(N,\overline{\chi}) \quad \text{and} \quad \mathcal{E}_k(N,\chi) \simeq \mathcal{E}_k(N,\overline{\chi}).$$

Proof. See [7, Lemma 4.3.2(2)]. □

Proposition 4.2. *Let ψ_1 and ψ_2 be Dirichlet characters modulo N_1 and N_2, respectively, such that $\psi_1(-1)\psi_2(-1) = (-1)^k$. Then we have the property*

$$E_k(\tau;\psi_1,\psi_2)|_k\omega_N = N^{k/2}\psi_1(-1)E_k(N\tau;\psi_2,\psi_1). \tag{18}$$

Proof. By Definition of $\cdot|_k\omega_N$ we obtain

$$E_k(\tau;\psi_1,\psi_2)|_k\omega_N = N^{k/2}(N\tau)^{-k}E_k(-1/N\tau;\psi_1,\psi_2). \tag{19}$$

On the other hand, we derive that

$$\begin{aligned}
&H_k(-1/N\tau,s;\psi_1,\psi_2) = E_k(-1/N\tau,s;\psi_1,\psi_2) \quad \text{for } \mathrm{Re}(s) > 1-k/2 \\
&= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \psi_1(m)\psi_2(n)(-m/N\tau+n)^{-k}|-m/N\tau+n|^{-2s} \text{ by Definition (14)} \\
&= (N\tau)^k|N\tau|^{2s}\psi_1(-1) \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \psi_1(-m)\psi_2(n)(nN\tau-m)^{-k}|nN\tau-m|^{-2s} \\
&= (N\tau)^k|N\tau|^{2s}\psi_1(-1)E_k(N\tau,s;\psi_2,\psi_1) \quad \text{by Definition (14)} \\
&= (N\tau)^k|N\tau|^{2s}\psi_1(-1)H_k(N\tau,s;\psi_2,\psi_1).
\end{aligned}$$

It then follows that

$$H_k(-1/N\tau,s;\psi_1,\psi_2)=(N\tau)^k|N\tau|^{2s}\psi_1(-1)H_k(N\tau,s;\psi_2,\psi_1) \quad \text{on the domain in (15),}$$

because the analytic continuation is unique. Setting $s=0$ we get by Definition (16)

$$E_k(-1/N\tau;\psi_1,\psi_2)=(N\tau)^k\psi_1(-1)E_k(N\tau;\psi_2,\psi_1). \tag{20}$$

Therefore (19) and (20) give rise to the relation (18). □

Lemma 4.3. *Let k and N be positive integers such that $(-1)^kN$ is the discriminant of a quadratic field. Then $W(\chi_{(-1)^kN})=i^{k^2}\sqrt{N}$.*

Proof. See [7, Lemma 4.8.1]. □

Corollary 4.4. *Let $k\geq 2$ and N be positive integers such that $(-1)^kN$ is the discriminant of a quadratic field. Then we have*

$$G_{k,N}(\tau)|_k\omega_N=i^{-k^2}N^{(k-1)/2}H_{k,N}(\tau) \text{ and } H_{k,N}|_k\omega_N=i^{k^2}N^{(1-k)/2}(-1)^kG_{k,N}(\tau).$$

Proof. Since $\chi_{(-1)^kN}$ is a real-valued primitive character modulo N, we have

$$\begin{aligned}
H_{k,N}(\tau)|_k\omega_N &= E_{k,\chi_{(-1)^kN},\chi_0}(\tau)|_k\omega_N \quad \text{by Definition (13)}\\
&= A_{k,\chi_0}^{-1}E_k(\tau;\chi_{(-1)^kN},\chi_0)|_k\omega_N \quad \text{by Proposition 3.6}\\
&= A_{k,\chi_0}^{-1}N^{k/2}\chi_{(-1)^kN}(-1)E_k(N\tau;\chi_0,\chi_{(-1)^kN}) \quad \text{by Proposition 4.2}\\
&= A_{k,\chi_0}^{-1}N^{k/2}(-1)^kE_k(N\tau;\chi_0,\chi_{(-1)^kN}) \quad \text{because } \chi_{(-1)^kN}(-1)=(-1)^k\\
&= A_{k,\chi_0}^{-1}N^{k/2}(-1)^kA_{k,\chi_{(-1)^kN}}E_{k,\chi_0,\chi_{(-1)^kN}}(\tau) \quad \text{by Proposition 3.6}\\
&= i^{k^2}N^{(1-k)/2}(-1)^kG_{k,N}(\tau) \quad \text{by Lemma 4.3 and Definition (12).}
\end{aligned}$$

Thus

$$\begin{aligned}
(H_{k,N}(\tau)|_k\omega_N)|_k\omega_N &= (-1)^kH_{k,N}(\tau) \quad \text{by (17)}\\
&= i^{k^2}N^{(1-k)/2}(-1)^kG_{k,N}(\tau)|_k\omega_N \text{ by the first part of the proof,}
\end{aligned}$$

from which we conclude

$$G_{k,N}(\tau)|_k\omega_N=i^{-k^2}N^{(k-1)/2}H_{k,N}(\tau).$$

□

5. Hecke operators

Let k be an integer and χ be a Dirichlet character modulo N such that $\chi(-1)=(-1)^k$. For a positive integer m, the *Hecke operator* $\cdot|T_{k,\chi}(m)$ is defined on the functions $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{M}_k(N,\chi)$ by the rule

$$f(\tau)|T_{k,\chi}(m) := \sum_{n=0}^{\infty} \Bigg(\sum_{d>0, d|\gcd(m,n)} \chi(d) d^{k-1} a(mn/d^2) \Bigg) q^n. \tag{21}$$

Here we set $\chi(d) = 0$ if $\gcd(N,d) \neq 1$. Then the operator $\cdot|T_{k,\chi}(m)$ preserves the space $\mathcal{M}_k(N,\chi)$ [5, Chapter 3, Propositions 36 and 39].

Lemma 5.1. *Let m be a positive integer. Suppose that $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{M}_k(N,\chi)$ is an eigenfunction of $\cdot|T_{k,\chi}(m)$ with eigenvalue $t(m)$, that is, $f(\tau)|T_{k,\chi}(m) = t(m)f(\tau)$.*

(i) *$a(m) = t(m)a(1)$.*

(ii) *If n is a nonnegative integer such that $\gcd(m,n) = 1$, then $a(1)a(mn) = a(m)a(n)$.*

Proof. Let $f(\tau)|T_{k,\chi}(m) = \sum_{n=0}^{\infty} b(n)q^n$.

(i) We see that

$$b(1) = a(m) \text{ by Definition (21)} = t(m)a(1) \text{ by the assumption}$$
$$f(\tau)|T_{k,\chi}(m) = t(m)f(\tau) = t(m)a(0) + t(m)a(1)q + \dots.$$

(ii) We achieve that

$$\begin{aligned} a(1)b(n) &= a(1) \sum_{d>0, d|\gcd(m,n)} \chi(d)d^{k-1}a(mn/d^2) \quad \text{by Definition (21)} \\ &= a(1)a(mn) \quad \text{since } \gcd(m,n) = 1 \\ &= a(1)t(m)a(n) \quad \text{by the assumption } f(\tau)|T_{k,\chi}(m) = t(m)f(\tau) \\ &= a(m)a(n) \quad \text{by (i),} \end{aligned}$$

which proves (ii). □

Proposition 5.2. *For a positive integer k, let ψ_1 and ψ_2 be Dirichlet characters modulo N_1 and N_2, respectively, that satisfy $\psi_1(-1)\psi_2(-1) = (-1)^k$ and the condition* (10)*. Let $N = N_1N_2$ and $\chi = \psi_1\psi_2$. Then $E_{k,\psi_1,\psi_2}(\tau)$ is a common eigenfunction of $\cdot|T_{k,\chi}(m)$ for all positive integers m such that $\gcd(N,m) = 1$.*

Proof. See [7, (4.7.16)]. □

Remark 5.3. Suppose that ψ_1 and ψ_2 are real valued characters, in other words their values are 1, -1 or 0. Let p be a prime such that $p \nmid N$. Then $E_{k,\psi_1,\psi_2}(\tau)$ is an eigenfunction of $\cdot|T_{k,\chi}(p^2)$ with eigenvalue $1+\chi(p)p^{k-1}+p^{2(k-1)}$ by Lemma 5.1(i) and Definition (11).

Corollary 5.4. *Let k (≥ 2) and N be positive integers such that $(-1)^k N$ is the discriminant of a quadratic field. Let p be a prime such that $p \nmid N$. Then any linear combination of $G_{k,N}(\tau)$ and $H_{k,N}(\tau)$ is an eigenfunction of $\cdot|T_{k,\chi_{(-1)^k N}}(p^2)$.*

Proof. Both $G_{k,N}(\tau) = E_{k,\chi_0,\chi_{(-1)^k N}}(\tau)$ and $H_{k,N}(\tau) = E_{k,\chi_{(-1)^k N},\chi_0}(\tau)$ are eigenfunctions of $\cdot|T_{k,\chi_{(-1)^k N}}(p^2)$ with the same eigenvalue by Proposition 5.2 and Remark 5.3. This implies that any linear combination of $G_{k,N}(\tau)$ and $H_{k,N}(\tau)$ is again an eigenfunction of $\cdot|T_{k,\chi_{(-1)^k N}}(p^2)$. □

6. Application to quadratic forms

Let A be an $r \times r$ positive definite symmetric matrix over $\mathbb{Z}$ with even diagonal entries. Let Q be its associated quadratic form, namely

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} \quad \text{for } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} \in \mathbb{Z}^r.$$

We define on $\mathbb{H}$ the theta function $\Theta_Q(\tau)$ associated with Q by

$$\Theta_Q(\tau) := \sum_{\mathbf{x}\in\mathbb{Z}^r} e^{2\pi i Q(\mathbf{x})\tau} = \sum_{n=0}^{\infty} r_Q(n) q^n,$$

where

$$r_Q(n) := \#\{\mathbf{x} \in \mathbb{Z}^r; \quad Q(\mathbf{x}) = n\}$$

is the representation number of n by Q.

Proposition 6.1. *With the notations as above we further assume that $r = 2k$ is even. Let N be a positive integer such that NA^{-1} is an integral matrix with even diagonal entries.*

(i) *$\Theta_Q(\tau)$ belongs to $\mathcal{M}_k(N, \chi_{(-1)^k \det(A)})$.*

(ii) *$\Theta_Q(\tau)|_k\omega_N = N^{k/2}\det(A)^{-1/2} i^k (-1)^k \Theta_{Q^*}(\tau)$, where Q^* is the quadratic form associated with NA^{-1}.*

Proof. See [7, Corollary 4.9.5(3)]. □

Let k and N be positive integer. We let $\mathrm{Mat}(k, N)$ be the set of $2k \times 2k$ integral positive definite symmetric matrices with $\det(A) = N$ such that both A and NA^{-1} have even diagonal entries.

Theorem 6.2. *Let $k \geq 2$ and N be positive integers such that $(-1)^k N$ is the discriminant of a quadratic field. Assume that*

$$\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) - \dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 2. \tag{22}$$

(Such pairs (k, N) are given in Corollary 2.4(i).) Let A be a matrix in $\mathrm{Mat}(k, N)$ *and Q be its associated quadratic form. Then the Eisenstein series part of $\Theta_Q(\tau)$ depends only on k and N.*

Proof. We have the decomposition

$$\mathcal{M}_k(N, \chi_{(-1)^k N}) = \mathrm{Span}_{\mathbb{C}}\{G_{k,N}(\tau), H_{k,N}(\tau)\} \oplus \mathcal{S}_k(N, \chi_{(-1)^k N})$$

by the assumption (22), Proposition 3.3 and Corollary 3.4. Since $\Theta_Q(\tau)$ belongs to $\mathcal{M}_k(N, \chi_{(-1)^k N})$ by Proposition 6.1(i), we can write it as

$$\Theta_Q(\tau) = c_1 G_{k,N}(\tau) + c_2 H_{k,N}(\tau) + \text{a cusp form} \tag{23}$$

for some $c_1, c_2 \in \mathbb{C}$. Comparing the constant terms of both sides we obtain

$$r_Q(0) = 1 = c_1\left(- \frac{B_{k,\chi_{(-1)^k N}}}{2k}\right).$$

Now, applying the Fricke involution $\cdot|_k \omega_N$ on both sides of (23) we get by Proposition 6.1(ii), Corollary 4.4 and Proposition 4.1

$$\begin{aligned} N^{(k-1)/2} i^k (-1)^k \Theta_{Q^*}(\tau) = c_1 i^{-k^2} N^{(k-1)/2} H_{k,N}(\tau) \\ + c_2 i^{k^2} N^{(1-k)/2} (-1)^k G_{k,N}(\tau) + \text{a cusp form}, \end{aligned}$$

where Q^* is the quadratic form associated with NA^{-1}. By comparing the constant terms of both sides we derive

$$N^{(k-1)/2} i^k (-1)^k r_{Q^*}(0) = N^{(k-1)/2} i^k (-1)^k = c_2 i^{k^2} N^{(1-k)/2} (-1)^k \left(- \frac{B_{k,\chi_{(-1)^k N}}}{2k}\right).$$

Therefore we obtain

$$c_1 = -\frac{2k}{B_{k,\chi_{(-1)^k N}}} \quad \text{and} \quad c_2 = N^{k-1} i^{-k^2+k}\left(- \frac{2k}{B_{k,\chi_{(-1)^k N}}}\right), \tag{24}$$

which depend only on k and N. This prove the theorem. $\square$

As a direct consequence of Theorem 6.2 we have the following corollary.

Corollary 6.3. *Let $k \geq 2$ and N be positive integers such that $(-1)^k N$ is the discriminant of a quadratic field. Assume that $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$. (Such pairs (k, N) are given in Corollary 2.4(iii).) Let A be a matrix in $\mathrm{Mat}(k, N)$ and Q be its associated quadratic form. Then $\Theta_Q(\tau)$ depends only on k and N.*

Theorem 6.4. *With the same notations and assumptions as Corollary 6.3, let $\Theta_Q(\tau) = \sum_{n=0}^{\infty} r_Q(n) q^n$. If p is a prime such that $p \nmid N$, then $r_Q(1) \neq 0$ and*

$$\frac{r_Q(p^2 n)}{r_Q(1)} = \frac{r_Q(p^2)}{r_Q(1)} \cdot \frac{r_Q(n)}{r_Q(1)} \quad \text{for all positive integers } n \text{ such that } p \nmid n. \tag{25}$$

Proof. By the assumption $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$ one can derive in a similar way as in the proof of Theorem 6.2 that

$$\Theta_Q(\tau) = c_1 G_{k,N}(\tau) + c_2 H_{k,N}(\tau) \tag{26}$$

with c_1 and c_2 described in (24). If we compare the Fourier coefficients of the term q, we see that $r_Q(1) = c_1 + c_2 = (1 + N^{k-1} i^{-k^2+k})(-2k/B_{k,\chi_{(-1)^k N}})$, which is nonzero due to the fact $N \geq 2$.

On the other hand, if p is a prime such that $p \nmid N$, then $\Theta_Q(\tau)$ is an eigenfunction of $\cdot|T_{k,\chi_{(-1)^k N}}(p^2)$ by Corollary 5.4. Hence it follows from Lemma 5.1(ii) that

$$r_Q(1) r_Q(p^2 n) = r_Q(p^2) r_Q(n) \text{ for all positive integers } n \text{ such that } p \nmid n.$$

Now, dividing both sides by $r_Q(1)^2$ we have the relation (25). $\square$

Remark 6.5. We obtain by (26) that $r_Q(0) = 1$ and for $n \geq 1$

$$r_Q(n) = -\frac{2k}{B_{k,\chi_{(-1)^k N}}} \sum_{d>0, d|n} \left(\chi_{(-1)^k N}(d) + N^{k-1} i^{-k^2+k} \chi_{(-1)^k N}(n/d) \right) d^{k-1}.$$

7. Examples

We shall give some examples of matrices in $\mathrm{Mat}(k, N)$ for (k, N) stated in Corollary 2.4(ii).

Example 7.1. Let $(k,N)=(2,8)$ or $(3,4)$. Take

$$A=\begin{cases}\begin{bmatrix}2&1&1&0\\1&2&1&0\\1&1&2&0\\0&0&0&2\end{bmatrix} & \text{if } (k,N)=(2,8),\\ \begin{bmatrix}4&2&0&0&0&0\\2&2&1&0&0&0\\0&1&2&1&0&0\\0&0&1&2&1&0\\0&0&0&1&2&1\\0&0&0&0&1&2\end{bmatrix} & \text{if } (k,N)=(3,4),\end{cases}$$

whose associated quadratic form is

$$Q=\begin{cases}x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_1x_3+x_2x_3 & \text{if } (k,N)=(2,8),\\ 2x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+2x_1x_2+x_2x_3 & \\ \qquad\qquad +x_3x_4+x_4x_5+x_5x_6 & \text{if } (k,N)=(3,4).\end{cases}$$

Since $\dim_{\mathbb{C}}\mathcal{M}_k(N,\chi_{(-1)^kN})=2$ and $\dim_{\mathbb{C}}\mathcal{S}_k(N,\chi_{(-1)^kN})=0$, we obtain by the proof of Theorem 6.2 that

$$\Theta_Q(\tau)=\begin{cases}-2G_{2,8}(\tau)+16H_{2,8}(\tau) & \text{if } (k,N)=(2,8),\\ -4G_{3,4}(\tau)+64H_{3,4}(\tau) & \text{if } (k,N)=(3,4).\end{cases}$$

For the cases where $(k,N)=(2,5),(2,13),(2,17),(3,3),(4,5),(5,3)$ one can refer to [3, Table 1].

Example 7.2. Let $(k,N)=(2,12)$ or $(2,21)$. Since

$$\dim_{\mathbb{C}}\mathcal{M}_k(N,\chi_{(-1)^kN})=4 \quad\text{and}\quad \dim_{\mathbb{C}}\mathcal{S}_k(N,\chi_{(-1)^kN})=0,$$

we get by Proposition 3.3 that

$$\mathcal{M}_k(N,\chi_{(-1)^kN})=\mathcal{E}_k(N,\chi_{(-1)^kN})$$
$$=\begin{cases}\mathrm{Span}_{\mathbb{C}}\{G_{2,12}(\tau),H_{2,12}(\tau),E_{2,\chi_{-3},\chi_{-4}}(\tau),E_{2,\chi_{-4},\chi_{-3}}(\tau)\} & \text{if } (k,N)=(2,12),\\ \mathrm{Span}_{\mathbb{C}}\{G_{2,21}(\tau),H_{2,21}(\tau),E_{2,\chi_{-3},\chi_{-7}}(\tau),E_{2,\chi_{-7},\chi_{-3}}(\tau)\} & \text{if } (k,N)=(2,21).\end{cases}$$

Now, consider a matrix

$$A=\begin{cases}\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&2&1\\0&0&1&2\end{bmatrix} & \text{if } (k,N)=(2,12),\\ \begin{bmatrix}6&5&5&5\\5&6&5&5\\5&5&6&5\\5&5&5&6\end{bmatrix} & \text{if } (k,N)=(2,21),\end{cases}$$

whose associated quadratic form is

$$Q = \begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_3x_4 & \text{if } (k,N) = (2,12), \\ 3x_1^2 + 3x_2^2 + 3x_3^2 + 3x_4^2 + 5x_1x_2 + 5x_1x_3 + 5x_1x_4 & \\ \qquad +5x_2x_3 + 5x_2x_4 + 5x_3x_4 & \text{if } (k,N) = (2,21). \end{cases}$$

Then we deduce that

$$\begin{aligned} \Theta_Q(\tau) &= \sum_{n=0}^{\infty} r_Q(n)q^n \\ &= \begin{cases} 1 + 10q + 28q^2 + 30q^3 + \ldots & \text{if } (k,N) = (2,12), \\ = -G_{2,12}(\tau) + 12H_{2,12}(\tau) & \\ -4E_{2,\chi_{-3},\chi_{-4}}(\tau) + 3E_{2,\chi_{-4},\chi_{-3}}(\tau) & \\ 1 + 12q + 6q^2 + 32q^3 + \ldots & \text{if } (k,N) = (2,21). \\ = -\frac{1}{2}G_{2,21}(\tau) + \frac{21}{2}H_{2,21}(\tau) & \\ -\frac{3}{2}E_{2,\chi_{-3},\chi_{-7}}(\tau) + \frac{7}{2}E_{2,\chi_{-7},\chi_{-3}}(\tau) & \end{cases} \end{aligned}$$

Let p be a prime such that $p \nmid N$. Note that all the generators of $\mathcal{M}_k(N, \chi_{(-1)^k N})$ belong to the same eigenspace of $\cdot|T_{k,\chi_{(-1)^k N}}(p^2)$ by Proposition 5.2 and Remark 5.3. Hence Lemma 5.1(ii) leads to the relation

$$r_Q(1)r_Q(p^2n) = r_Q(p^2)r_Q(n) \quad \text{for all positive integers } n \text{ such that } p \nmid n.$$

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