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On the Diophantine equation $(x-1)^k + x^k + (x+1)^k = y^n$

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Abstract. In this paper, we study the Diophantine equation $(x-1)^k + x^k + (x+1)^k = y^n$, $n > 1$, and completely solve it for $k = 2, 3, 4$.

1. Introduction

The Diophantine equation

$$1^k + 2^k + \cdots + x^k = y^n$$

was studied by LUCAS [9] for $(k,n) = (2,2)$ and SCHÄFFER [12] for the general situation. There are many results on this equation, for example, see [1], [8], [11]. A generalization is to consider the equation

$$(x+1)^k + (x+2)^k + \cdots + (x+m)^k = y^n.$$

When $m = 3$, redefining variables, we will consider the equation $(x-1)^k + x^k + (x+1)^k = y^n$. CASSELS [6] proved that $x = 0, 1, 2, 24$ are the only integer solutions to this equation for $k = 3$, $n = 2$.

Our result in this paper is the following.

Theorem 1.1. *Let $k = 2, 3, 4$, then the equation*

$$(x-1)^k + x^k + (x+1)^k = y^n \tag{1}$$

has no integer solutions (x, y) with $n > 1$, unless $(x, y, k, n) = (1, \pm 3, 3, 2)$, $(2, \pm 6, 3, 2)$, $(24, \pm 204, 3, 2)$, $(\pm 4, \pm 6, 3, 3)$ or $(x, y, k) = (0, 0, 3)$.

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2. Some preliminary results

In this section, we present some lemmas which will help us to prove Theorem 1.1. The first lemma is due to NAGELL [10] and the second one is just Theorem 0.1 of [6].

Lemma 2.1. *If $2 \nmid D$ and $D \geq 3$, then the equation*

$$2 + Dx^2 = y^n, n > 2$$

has no integer solutions (x, y, n) with $n \nmid h(-2D)$, where $h(-2D)$ is the class number of $\mathbb{Q}(\sqrt{-2D}\,)$.

Lemma 2.2. *The equation*

$$3x(x^2 + 2) = y^2$$

has only the integer solutions $(x, y) = (0, 0)$, $(1, \pm 3)$, $(2, \pm 6)$, $(24, \pm 204)$.

A special case of Theorem 1.5 in [4] which we need in this paper is the following result.

Lemma 2.3. *Let $p \geq 5$ be a prime, $\alpha \geq 2$ be an integer, then the equation*

$$x^p + 3^\alpha y^p = 2z^3$$

has no solutions in coprime integers with $|xy| > 1$.

In order to discuss the small exponents for $k = 4$ we also need the following result.

Lemma 2.4. *Let $p \geq 3$ be a prime and (x, y) be an integer solution to equation*

$$3x^2 - 10 = y^p, \tag{2}$$

then

$$(\sqrt{3}x + \sqrt{10}\,)^2 = (11 + 2\sqrt{30}\,)^i (a + b\sqrt{30}\,)^p$$

for some integers a, b, i with $-\frac{p-1}{2} \leq i \leq \frac{p-1}{2}$.

PROOF. From equation (2) one has $(3x^2 - 10)^2 = y^{2p}$, that is

$$(3x^2 + 10)^2 - 30(2x)^2 = (y^2)^p.$$

It is easy to see $2 \nmid x, 5 \nmid x$, then $\gcd(3x^2+10, 2x) = 1$, together with the fact that the class number of $\mathbb{Q}(\sqrt{30})$ is 2 and $11+2\sqrt{30}$ be the fundamental unit of this field, we have

$$3x^2 + 10 + 2x\sqrt{30} = (11+2\sqrt{30})^i (a+b\sqrt{30})^p$$

with $-\frac{p-1}{2} \le i \le \frac{p-1}{2}$, that is

$$\begin{aligned}(11+2\sqrt{30})^i (a+b\sqrt{30})^p &= 3x^2 + 10 + 2x\sqrt{30} \\ &= (\sqrt{3}x + \sqrt{10})^2, -\frac{p-1}{2} \le i \le \frac{p-1}{2}.\end{aligned}$$ □

3. The modular approach

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N. For a prime of good reduction l we write $\#E(\mathbb{F}_l)$ for the number of points on E over the finite field $\mathbb{F}_l$, and let $a_l(E) = l+1-\#E(\mathbb{F}_l)$. By a *newform* f, we will always mean a cuspidal newform of weight 2 with respect to $\Gamma_0(N_0)$ for some positive integer N_0, and N_0 will be called the *level* of f. Write $f = q + \sum_{i\ge 2} c_i q^i$ the q-expansion of f, then c_n will be called the *Fourier coefficients* of f. Let $\mathbb{K} = \mathbb{Q}(c_2, c_3, \dots)$ be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of f, then $\mathbb{K}$ is a finite and totally real extension of $\mathbb{Q}$ (see e.g. [7], Chapter 15).

We shall say that the curve E arises modulo p from the newform f (and write $E \sim_p f$) if there is a prime ideal $\mathfrak{p}$ of $\mathbb{K}$ above p such that for all but finitely many primes l we have $a_l(E) \equiv c_l \pmod{\mathfrak{p}}$ (see [7], Definition 15.2.1).

We have the following result, which is just Lemma 2.1 of [5].

Proposition 3.1. *Assume that $E \sim_p f$. There exists a prime ideal $\mathfrak{p}$ of $\mathbb{K}$ above p such that for all primes l,*

(i) *if $l \nmid pNN_0$ then $a_l(E) \equiv c_l \pmod{\mathfrak{p}}$,*

(ii) *if $l \| N$ but $l \nmid pN_0$ then $\pm(l+1) \equiv c_l \pmod{\mathfrak{p}}$.*

Moreover, if f is rational, then the above can be relaxed slightly as follows: for all primes l,

(i) *if $l \nmid NN_0$ then $a_l(E) \equiv c_l \pmod{p}$,*

(ii) *if $l \| N$ but $l \nmid N_0$ then $\pm(l+1) \equiv c_l \pmod{p}$.*

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Without loss of generality, we assume $n = p$ and p is a prime. Expanding the left hand side of equation (1), one has

(i) $3x^2 + 2 = y^p$ when $k = 2$;

(ii) $3x(x^2 + 2) = y^p$ when $k = 3$;

(iii) $3x^4 + 12x^2 + 2 = y^p$ when $k = 4$.

We will discuss them separately.

(1) $k = 2$

Applying Lemma 2.1, we conclude that there are no integer solutions for $p \geq 3$ since $h(-6) = 2$. When $p = 2$, the equation $3x^2 + 2 = y^2$ modulo 3 yields a contradiction.

(2) $k = 3$

From the result of Cassels, that is Lemma 2.2, one has $(x, y) = (0, 0)$, $(1, \pm 3)$, $(2, \pm 6)$, $(24, \pm 204)$ for $p = 2$.

When $p \geq 3$ we obtain $(x, y) = (0, 0)$ if $xy = 0$. Now we assume $xy \neq 0$. Since $\gcd(x, x^2 + 2) = \gcd(x, 2) \in \{1, 2\}$, then equation

$$3x(x^2 + 2) = y^p \tag{3}$$

implies one of following cases:

(a) $x = 3^{p-1}u^p, x^2 + 2 = v^p, 2 \nmid v$; (b) $x = u^p, x^2 + 2 = 3^{p-1}v^p, 2 \nmid v$;

(c) $x = 2^{p-1} \times 3^{p-1}u^p, x^2 + 2 = 2v^p$; (d) $x = 2^{p-1}u^p, x^2 + 2 = 2 \times 3^{p-1}v^p$.

Firstly, in case (a), we can write equation (3) as $3^{p-2}(-3u^2)^p + v^p = 2$ and find it has no integer solutions when $p \geq 5$ by Lemma 2.3. When $p = 3$, one has $(3^2u^3)^2 = v^3 - 2$, modulo 9 yields a contradiction.

In case (b), equation (3) turns into $(-u^2)^p + 3^{p-1}v^p = 2$ and applying Lemma 2.3 we know it has no integer solutions when $p \geq 5$. The left equation for $p = 3$ can be written as $u^6 + 2 = 9v^3$, and no integer solutions exists since $u^6 + 2 \equiv 2, 3 \pmod 9$.

In case (c), equation (3) becomes $v^p - 2^{2p-3} \times 3^{2p-2}u^{2p} = 1$, applying Theorem 1.1 of [2], we find that the equation has no nonzero integer solutions (u, v) for $p \geq 3$.

Finally, in case (d), one has $3^{p-1}v^p - 2^{2p-3}u^{2p} = 1$, also from Theorem 1.1 of [2], we know $(u, v, p) = (\pm 4, 1, 3)$, which yields $(x, y) = (\pm 4, \pm 6)$.

(3) $k = 4$

From the equation $3(x^2+2)^2 - 10 = y^2$ we know $2 \nmid x$, then $3(x^2+2)^2 - 10 \equiv \pm 3 \not\equiv y^2 \pmod{10}$, that is there are no integer solutions for $p = 2$. When $p = 3$, one has $3(x^2+2)^2 - 10 = y^3$, that is $(9x^2+18)^2 - 270 = (3y)^3$. Applying Magma to calculate the integer points on the elliptic curve $y^2 = x^3 - 270$, we conclude that it has no integer solutions in this case.

We proceed to prove the equation

$$3x^4 + 12x^2 + 2 = y^p \tag{4}$$

has no integer solutions for prime $p \geq 11$. The remaining cases $p = 5, 7$ will be treated at the end of the paper.

Let $u = x^2 + 2, v = y$, and write equation (4) as

$$3u^2 - 10 = v^p.$$

It is easy to see $\gcd(u, v) = 1$ and $uv \neq 0$. Suppose $p \geq 7$. To a possible solution (u, v), we associate the Frey curve (see [3])

$$E_u : Y^2 = X^3 + 6uX^2 + 30X,$$

with conductor $N = 2^6 \times 3^2 \operatorname{rad}(10v) = 2^7 \times 3^2 \times 5 \operatorname{rad}_{\{2,5\}}(v)$ where

$$\operatorname{rad}_{\{2,5\}}(v) = \prod_{p|v,\, p\neq 2,5} p.$$

Then, by the result of Bennett and Skinner [3], there is a newform of level $N(E_u)_p = 2^7 \times 3^2 \times 5 = 5760$ such that $E_u \sim_p f$.

Let l be a prime and $u \equiv r \pmod{l}$. Since $u = x^2 + 2$, one has the following table:

l	r
7	2, 3, 4, 6
11	0, 2, 3, 5, 6, 7
13	1, 2, 3, 5, 6, 11, 12
17	0, 1, 2, 3, 4, 6, 10, 11, 15
19	0, 2, 3, 6, 7, 8, 9, 11, 13, 18

Recall the definition of a_l and c_l in Section 3, that is $a_l = a_l(E) = l + 1 - \#E(\mathbb{F}_l)$, and $c_l = c_l(f)$ the Fourier coefficient of f. Therefore, calculating by Pari we obtain

(i) $7|N$ or $a_7(E_u) \in \{0, -4\}$; (ii) $a_{11}(E_u) \in \{0, \pm 2, -4, \pm 6\}$; (iii) $13|N$ or $a_{13}(E_u) \in \{\pm 2, -6\}$; (iv) $17|N$ or $a_{17}(E_u) \in \{2, \pm 6\}$; (v) $a_{19}(E_u) \in \{\pm 6\}$.

For rational newforms at level 5760 numbered in Stein's Table [13] , we get a bound for p by Proposition 3.1, that is from $p|a_l(E_u) - c_l(f)$ when $l \nmid N$ or $p| \pm (l+1) - c_l(f)$ when $l|N$. We list these bounds in the following table.

l	f	p
7	$f_{2+i}, f_{16+j}, f_{26+k}, f_{40+m}, 1 \le i, k \le 6, 1 \le j, m \le 8$	≤ 5
11	$f_{15}, f_{16}, f_{39}, f_{40},$	≤ 5
13	$f_1, f_2, f_9, f_{10}, f_{12}, f_{14}, f_{26}, f_{33}, f_{34}, f_{35}, f_{37}$	≤ 7
17	f_{25}	≤ 3
19	$f_{11}, f_{13}, f_{36}, f_{38}$	≤ 7

For the nonrational newforms $f_{49}, f_{50}, \ldots, f_{64}$, we using $p = l$ or $p | N_{\mathbb{K}/\mathbb{Q}}(a_l(E_u) - c_l(f))$ or $p | N_{\mathbb{K}/\mathbb{Q}}(\pm(l+1) - c_l(f))$ to bound p.

For $f = f_{49}$, one has $c_{13}^2(f) - 20 = 0$, $c_{17}^2(f) - 20 = 0$. Take $l = 13$, then $N_{\mathbb{K}/\mathbb{Q}}(a_l(E_u) - c_l(f)) = \pm 16$, $N_{\mathbb{K}/\mathbb{Q}}(\pm(l+1) - c_l(f)) = 2^4 \times 11$, which implies $p \le 5$ or $p = 11, 13$. Take $l = 17$, then $N_{\mathbb{K}/\mathbb{Q}}(a_l(E_u) - c_l(f)) = \pm 16$, $N_{\mathbb{K}/\mathbb{Q}}(\pm(l+1) - c_l(f)) = 2^4 \times 19$, which implies $p \le 5$ or $p = 17, 19$. Combining these two bounds yields $p \le 5$.

For $f = f_{56}, f_{60}, f_{61}$, take $l = 13, 17$, and for the left 12 nonrational newforms take $l = 7, 13$, then the same argument as f_{49}, we get $p \le 5$.

From the discussion above, we know there is no newform of level 5760 corresponding to E_u when $p \ge 11$. It remains to deal with the prime $p = 5, 7$. We prove that there are no integer solutions to equation

$$3x^2 - 10 = y^p$$

for $p = 5, 7$.

We discuss the case $p = 5$ in detail. By Lemma 2.4 we get

$$(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})^i (a + b\sqrt{30})^5 \tag{5}$$

for some integers a, b, i with $-2 \le i \le 2$. Replacing x by $-x$, we only need to consider the cases $0 \le i \le 2$.

If $i = 0$, expanding both sides of equation (5) we obtain

$$2x = 5a^4 b + 300a^2 b^3 + 900b^5,$$

so that $5|x$, an impossibility.

If $i = 1$, equation (5) can be written as

$$(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})(a + b\sqrt{30})(a + b\sqrt{30})^4,$$

thus

$$(11 + 2\sqrt{30})(a + b\sqrt{30}) = (\sqrt{3}u + \sqrt{10}v)^2$$

for some integers u, v. Expanding this equality we get

$$\begin{cases} 11a + 60b = 3u^2 + 10v^2 \\ 2a + 11b = 2uv, \end{cases}$$

that is

$$\begin{cases} a = 33u^2 + 110v^2 - 120uv \\ b = -6u^2 - 20v^2 + 22uv. \end{cases}$$

Substitution into

$$\sqrt{3}x + \sqrt{10} = (\sqrt{3}u + \sqrt{10}v)(a + b\sqrt{30}\,)^2$$

yields the Thue equation

$$-1188u^5 + 10845u^4v - 39600u^3v^2 + 72300u^2v^3 - 66000uv^4 + 24100v^5 = 1.$$

According to Magma one obtains no integer solutions.

If $i = 2$, we write equation (5) as

$$(\sqrt{3}x + \sqrt{10}\,)^2 = (11 + 2\sqrt{30}\,)^2(a + b\sqrt{30}\,)(a + b\sqrt{30}\,)^4,$$

and then

$$a + b\sqrt{30} = (\sqrt{3}u + \sqrt{10}v)^2$$

for some integers u, v, therefore

$$\sqrt{3}x + \sqrt{10} = (11 + 2\sqrt{30}\,)(\sqrt{3}u + \sqrt{10}v)^5.$$

Expanding the right hand side of the equation yields the Thue equation

$$54u^5 + 495u^4v + 1800u^3v^2 + 3300u^2v^3 + 3000uv^4 + 1100v^5 = 1$$

and again we find no integer solutions after appealing to Magma.

For the case $p = 7$, the same argument as in case $p = 5$, solving the corresponding Thue equations, we know the equation $3x^2 - 10 = y^7$ has no integer solutions. From the discussion above, this completes the proof of Theorem 1.1. □

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