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On a graph of a p-solvable normal subgroup

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Abstract. Let N be a p-solvable normal subgroup of a group G. In this paper, we prove that N is solvable if $\alpha > \beta > 1$ are the two maximal sizes in $cs_G(N_{p'})$ such that $(\alpha, \beta) = 1$ and β is a p'-number dividing $|N/(N \cap Z(G))|$. Moreover, the structure of N is given.

1. Introduction

All groups considered in this paper are finite. Let G be a group and x an element in G, we use x^G to denote the conjugacy class of G containing x and use $|x^G|$ to denote the size of x^G. Also we write $\mathrm{Con}(G) = \{x^G \mid x \in G\}$ and $cs(G) = \{|B| \mid B \in \mathrm{Con}(G)\}$. Furthermore, if H is a subset of G, we write $\mathrm{Con}_G(H) = \{x^G \mid x \in H\}$ and $cs_G(H) = \{|B| \mid B \in \mathrm{Con}_G(H)\}$. If B is a non-empty subset of a group G, following [1], we set $K_S = \{x \in G \mid xS = S\}$. Clearly, K_S is a subgroup of G. Furthermore, K_S is a normal subgroup of G if S is a normal subset of G. Since S is the union of right cosets of K_S, we see that $|K_S|$ divides $|S|$.

In 1904, Burnside proved that a group G is not simple if some conjugacy class size of G is a prime power, see [4] for instance. Since then, many authors began to study how the set of conjugacy class sizes may determine the properties of a group, say solvability, non-simplicity and so on (see, e.g., [5], [6], [7], [9]).

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Let N be a normal subgroup of a group G. Then N is a union of several G-conjugacy classes contained in N. Therefore, the investigation into the relation between the structure of N and the G-conjugacy class sizes contained in it has attracted interests of many authors (see, e.g., [8], [11], [12]).

Recall that a group G is called a quasi-Frobenius group if $G/Z(G)$ is a Frobenius group. The inverse images in G of the kernel and a complement of $G/Z(G)$ are called the kernel and a complement of G. And a group G is said to be p-nilpotent or p-closed if it has a normal p-complement or a normal Sylow p-subgroup respectively.

If N is a p-solvable normal subgroup of a group G, we use $N_{p'}$ to denote the set of all p'-elements in N. In this paper, we are interested in the relationship between the set $cs_G(N_{p'})$ and the property of N. We have the following theorem.

Theorem. *Let N be a p-solvable normal subgroup of a group G. Suppose that $\alpha > \beta > 1$ are the two maximal sizes in $cs_G(N_{p'})$ with $(\alpha, \beta) = 1$. If β is a p'-number and β divides $|N/(N \cap Z(G))|$, then N is solvable. Furthermore, if a p-complement K of N is not nilpotent, then K is quasi-Frobenius with abelian kernel and complements and at least one of the following conditions is satisfied:*

(1) *N is p-nilpotent;*

(2) *N is p-closed;*

(3) *$K = R \rtimes T$ with R an abelian normal Sylow r-subgroup of N and T an abelian $\{p, r\}$-complement of N.*

In the proof of this theorem, we use the graphs $\Gamma(G)$ of a group G and $\Gamma_p(G)$ of a p-solvable group G, whose vertices are the non-central G-conjugacy classes of elements and p'-elements in G respectively and two vertices are joined by an edge if their cardinalities have a common primary divisor. Furthermore, we use $n(\Gamma(G))$ and $n(\Gamma_p(G))$ to denote the number of components of $\Gamma(G)$ and $\Gamma_p(G)$ respectively. Several authors have obtained interesting results about $\Gamma(G)$ and $\Gamma_p(G)$ (see, e.g., [1], [2], [3]).

2. Preliminaries

For a positive integer m, we write $\pi(m) = \{p \mid p \text{ is a primary divisor of } m\}$, and for a non-empty set K, we use $\pi(K)$ to denote the set of primary divisors of $|K|$.

Inspired by the works of [3] and [10], we have the following generalized results. For completeness, we provide the corresponding proofs.

Lemma 2.1. *Let N be a p-solvable normal subgroup of a group G. Suppose that $B = b^G$, $C = c^G \in \mathrm{Con}_G(N_{p'})$ such that $(|B|, |C|) = 1$. Then*

(a) $G = C_G(b)C_G(c)$;

(b) $BC \in \mathrm{Con}_G(N_{p'})$ *and* $|BC|$ *divides* $|B|\,|C|$.

Proof. (a) This is obvious since $(|G : C_G(b)|, |G : C_G(c)|) = (|b^G|, |c^G|) = 1$.

(b) We first claim that $BC \in \mathrm{Con}(G)$. It suffices to prove that $b^g c^h$ is conjugate to bc for any $g, h \in G$. Since $gh^{-1} \in G = C_G(b)C_G(c)$, there exists $x \in C_G(b)$ and $y \in C_G(c)$ such that $gh^{-1} = x^{-1}y$. Then $xg = yh$ and $b^g c^h = b^{xg} c^{yh} = (bc)^{xg}$. Now, it suffices to prove that there is a p'-element in BC. In fact, let H be a p-complement of N. Then there exists $g, h \in G$ such that $b^g \in H$ and $c^h \in H$. Therefore, $b^g c^h$ is a p'-element in N and $b^g c^h \in BC$.

Since $C_G(b) \cap C_G(c) \le C_G(bc)$, we have that $|BC| = |G : C_G(bc)|$ divides $|G : C_G(b) \cap C_G(c)| = |G : C_G(b)|\,|G : C_G(c)| = |B|\,|C|$. □

Lemma 2.2. *Let N be a p-solvable normal subgroup of a group G. Then the following two properties hold:*

(a) *Suppose that $B_0 \in \mathrm{Con}_G(N_{p'})$ such that $|B_0|$ is the maximal size in $cs_G(N_{p'})$. If $C \in \mathrm{Con}_G(N_{p'})$ such that $(|B_0|, |C|) = 1$, then $C^{-1}CB_0 = B_0$ and $|\langle C^{-1}C\rangle|$ divides $|B_0|$.*

(b) *Suppose that m, n are the two maximal sizes in $cs_G(N_{p'})$ such that $m > n > 1$ and $(m, n) = 1$. If D is a non-central class in $\mathrm{Con}_G(N_{p'})$ with $(|D|, n) = 1$, then $|D| = m$.*

Proof. (a) Lemma 2.1 (b) implies that $CB_0 \in \mathrm{Con}_G(N_{p'})$, and it is obvious that $|CB_0| \ge |B_0|$. By the hypothesis, we see that $|CB_0| = |B_0|$. It follows that $C^{-1}CB_0 = B_0$, and thus $\langle C^{-1}C\rangle B_0 = B_0$. So, $\langle C^{-1}C\rangle \le K_B$, which yields that $|\langle C^{-1}C\rangle|$ divides $|B_0|$.

(b) Let $A, B \in \mathrm{Con}_G(N_{p'})$ such that $|A| = n$ and $|B| = m$. Then $DA \in \mathrm{Con}_G(N_{p'})$ by Lemma 2.1 (b). Since $|DA| \ge |A|$, $|DA| = m$ or n. If $|DA| = n$, then $D^{-1}DA \in \mathrm{Con}_G(N_{p'})$ and thus $D^{-1}DA = A$, in which case we have $\langle D^{-1}D\rangle A = A$. Therefore, $|\langle D^{-1}D\rangle|$ divides $|A|$. On the other hand, we see that $\langle D^{-1}D\rangle \subseteq \langle AA^{-1}\rangle$, so $|\langle D^{-1}D\rangle|$ divides $|\langle AA^{-1}\rangle|$. By (a) of this lemma, $|\langle AA^{-1}\rangle|$ divides $|B|$, so $|\langle D^{-1}D\rangle|$ divides $|B|$. Therefore, $|\langle D^{-1}D\rangle|$ divides $(n, m) = 1$, a contradiction. So $|DA| = m$. We conclude that $|D| = m$ since $|DA|$ divides $|D|\,|A|$. □

Lemma 2.3. *Suppose that N is a p-solvable normal subgroup of a group G and $B_0 \in \mathrm{Con}_G(N_{p'})$ such that $|B_0|$ is the maximal size in $cs_G(N_{p'})$. Let*

$$M = \langle D \mid D \in \mathrm{Con}_G(N_{p'}) \text{ such that } (|D|, |B_0|) = 1\rangle.$$

Then M is an abelian p'-subgroup of N. Furthermore, if $M \nleq Z(G)$, then $\pi(M/(Z(G) \cap M)) \subseteq \pi(B_0)$ and in this case, $|B_0|$ is not a p-number.

Proof. Let

$$K = \langle D^{-1}D \mid D \in \mathrm{Con}_G(N_{p'}) \text{ such that } (|D|, |B_0|) = 1\rangle.$$

It is easy to see that M and K are normal subgroups of G and $K = [M, G]$.

On the other hand, if $C \in \mathrm{Con}_G(N_{p'})$ such that $(|C|, |B_0|) = 1$, then $C^{-1}CB_0 = B_0$ by Lemma 2.2 (a), and thus $KB_0 = B_0$. Therefore, $|K|$ divides $|B_0|$, in particular, $\pi(K) \subseteq \pi(B_0)$. So $(|K|, |C|) = 1$. Suppose that $C = c^G$. Since $|K : C_K(c)|$ divides $(|K|, |C|)$, we have $K = C_K(c)$, and thus $K \leq Z(M)$. Therefore, M is nilpotent since $M/K \leq Z(G/K)$. We conclude that M is a p'-group since all of its generators are p'-elements.

Suppose that $M \nleq Z(G)$. Let $r \in \pi(M/(Z(G) \cap M))$ and $R \in \mathrm{Syl}_r(M)$. Then $R \trianglelefteq G$ and $1 \neq [R, G] \leq [M, G] = K$. Therefore, $r \in \pi(K) \subseteq \pi(B_0)$, and thus $\pi(M/(Z(G) \cap M)) \subseteq \pi(B_0)$.

If R is a Sylow r-subgroup of M, we can assume that $r \in \pi(M/(Z(G) \cap M))$. For any generating class d^G of M, we have that $|d^R|$ divides $(|R|, |d^G|) = 1$. Therefore, $R = C_R(d)$, which implies that $R \leq Z(M)$. Consequently, we conclude that M is abelian. □

3. Main Results

In this section, we give the proof of our main result.

Theorem. *Let N be a p-solvable normal subgroup of a group G. Suppose that $\alpha > \beta > 1$ are the two maximal sizes in $cs_G(N_{p'})$ with $(\alpha, \beta) = 1$. If β is a p'-number and β divides $|N/(N \cap Z(G))|$, then N is solvable. Furthermore, if a p-complement K of N is not nilpotent, then K is quasi-Frobenius with abelian kernel and complements and at least one of the following conditions is satisfied:*

(1) *N is p-nilpotent;*

(2) *N is p-closed;*

(3) *$K = R \rtimes T$ with R an abelian normal Sylow r-subgroup of N and T an abelian $\{p, r\}$-complement of N.*

Proof. Let K and P be a p-complement and a Sylow p-subgroup of N respectively. If K is nilpotent, then N is solvable since it is a product of two nilpotent groups. We only need to consider the case that K is not nilpotent. Then $N = PK$ and we can write $K = U \times W$, where W is the maximal Hall subgroup of K contained in $Z(G)$. Then $N_1 = PU$ is a normal subgroup of G and no Sylow subgroup of U is contained in $Z(G)$. It is easy to see that $U/Z(U) \cong K/Z(K)$ and that the theorem is true for N if and only if it is true for N_1. So, without loss of generality, we will assume that no Sylow subgroup of N with order prime to p is contained in $Z(G)$.

Since K is not nilpotent by the assumption, we have $|\pi(K)| \geq 2$.

Let
$$M = \langle D \mid D \in \mathrm{Con}_G(N_{p'}) \text{ such that } (|D|, \alpha) = 1\rangle.$$

Then M is an abelian p'-subgroup of N by Lemma 2.3, and $M \trianglelefteq G$.

Let a and b be p'-elements in N such that $|a^G| = \beta$ and $|b^G| = \alpha$. We will prove this theorem by the following steps.

Step 1. $C_G(b)$ is maximal and minimal among centralizers of all non-central p'-elements in N. In particular, b can be assumed to be a q-element for some prime $q \neq p$ and $C_N(b) = P_1Q \times L$, with $P_1 \in \mathrm{Syl}_p(C_N(b))$, $Q \in \mathrm{Syl}_q(C_N(b))$ and $L \leq Z(C_G(b))$.

Suppose that x and y are non-central p'-elements in N such that $C_G(b) \leq C_G(x)$ and $C_G(y) \leq C_G(b)$. Then $|x^G|$ divides $|b^G| = \alpha$. Therefore, $|x^G| = \alpha$ by Lemma 2.2 (b), and thus $C_G(b) = C_G(x)$. On the other hand, $\alpha = |b^G|$ divides $|y^G|$, so $|y^G| = \alpha$ and $C_G(b) = C_G(y)$ by the hypothesis of this theorem.

Now, write $b = b_1b_2 \cdots b_s$ with each b_i an element of primary order and $b_ib_j = b_jb_i$ for all i and j. We may assume that $b_1 \notin Z(G)$ and b_1 can be assumed to be a q-element for a prime $q \neq p$. It is obvious that $C_G(b) \leq C_G(b_1)$, so $C_G(b) = C_G(b_1)$ by the above paragraph. By replacing b with b_1 we can assume that b is a q-element.

For every $\{p, q\}'$-element x in $C_N(b)$, we have $C_G(bx) = C_G(b) \cap C_G(x) \leq C_G(b)$. Therefore, $C_G(bx) = C_G(b)$ by the first paragraph. It follows that $C_G(b) \leq C_G(x)$, and thus $x \in Z(C_G(b))$.

Step 2. $q \nmid \alpha$.

Suppose that $q \mid \alpha$. Then $q \nmid \beta$ and thus $q \nmid |a^N| = |N : C_N(a)| = |C_N(b) : C_N(a) \cap C_N(b)|$. Therefore, a Sylow q-subgroup of $C_N(b)$ is contained in $C_N(a) \cap C_N(b)$. Without loss of generality, we may assume that $Q \leq C_N(a)$.

Since $a \in C_N(b)$ and $L \leq Z(C_G(b))$, we have $L \leq C_N(a)$, and thus $Q \times L \leq C_N(a)$. Therefore, $|a^N|_{p'}$ divides $|b^N|_{p'}$. Since β is a p'-number, we have $|a^N| = 1$, that is $a \in Z(N)$. Write $a = a_q \cdot a_{q'}$ with a_q and $a_{q'}$ the q-component and q'-component of a respectively. If $a_{q'} \notin Z(G)$, then $C_G(ba_{q'}) = C_G(b) \cap C_G(a_{q'})$. Therefore, $C_G(b) = C_G(ba_{q'}) \leq C_G(a_{q'})$. By Step 1, we have $|a_{q'}^G| = \alpha$, and thus $|a^G| = \alpha$, which is a contradiction. Therefore, we may assume a to be a q-element. For any $\{p,q\}'$-element $x \in N = C_N(a)$, we have $C_G(ax) = C_G(a) \cap C_G(x) \leq C_G(a)$, then $C_G(ax) = C_G(a) \leq C_G(x)$ by the hypothesis, and thus $x \in Z(C_G(a))$. Since $b \in C_G(a)$, we have $x \in C_N(b)$ and thus $x \in L \leq Z(C_G(b))$. If $x \notin Z(G)$, then $C_G(x) = C_G(b)$ by Step 1. The fact that $C_G(ax) = C_G(a) \cap C_G(x)$ gives the contradiction that $\alpha\beta$ divides $|(ax)^G|$. So, $x \in Z(G)$ for all $\{p,q\}'$-elements in N, which contradicts to the assumption of the beginning of the proof.

Step 3. $K_B \cap C_G(b) = \{1\}$, where $B = b^G$. In particular, a can be assumed to be a q'-element.

By Step 1 we can assume that $|b| = q^k$ for some positive integer k.

Let $x \in K_B \cap C_G(b)$. Then $xb \in B$, and so $xb = b^g$ for some $g \in G$. As $x \in C_G(b)$, we have

$$x^{q^k} = x^{q^k} b^{q^k} = (xb)^{q^k} = (b^g)^{q^k} = (b^{q^k})^g = 1.$$

On the other hand, since $|K_B|$ divides $|B| = \alpha$, we see that $q \nmid |K_B|$ by Step 2. Therefore, $x = 1$.

Now, let $a_1 = a^s$ be the q-component of a. Then $a_1 \in C_G(b)^w$ for some $w \in G$. For every $g \in G$. Since $G = C_G(a)C_G(b)^w$, we can write $g = uv$ with $u \in C_G(a)$ and $v \in C_G(b)^w$. By Lemma 2.2 (a) and Lemma 2.3, $\langle a^G \rangle$ is a normal abelian subgroup of G and $\langle A^{-1}A \rangle \leq K_B$, where $A = a^G$. Therefore,

$$[a_1, g] = [a^s, g] = [a, g]^s = (a^{-1}a^g)^s \in K_B.$$

Also,

$$[a_1, g] = [a_1, uv] = [a_1, v] \in C_G(b)^w.$$

Therefore,

$$[a_1, g] \in K_B \cap C_G(b)^w = (K_B \cap C_G(b))^w = \{1\}.$$

So, $a_1 \in Z(G)$ and by replacing a with aa_1^{-1} we can assume that a is a q'-element.

Step 4. $(C_N(a) \cap C_N(b))_{p'} = Z(N)_{p'} = Z(G)_{p'} \cap N$.

Let x be a p'-element in $C_N(a) \cap C_N(b)$ and write $x = x_q \cdot x_{q'}$ with x_q and $x_{q'}$ the q-component and q'-component of x respectively. If $x_{q'} \notin Z(G)$, then $C_G(b) \leq C_G(x_{q'})$ by Step 1, and thus $C_G(b) = C_G(x_{q'})$ again by Step 1. It follows that $a \in C_G(b)$, and so $C_G(b) \leq C_G(a)$, which is a contradiction. Since $x_q \in C_G(a)$, we have $C_G(ax_q) = C_G(a) \cap C_G(x_q) \leq C_G(a)$. Then the hypothesis implies that $C_G(ax_q) = C_G(a) \leq C_G(x_q)$. If $x_q \notin Z(G)$, then $x_q \in M$, and thus

$q \in \pi(M/M \cap Z(G)) \subseteq \pi(\alpha)$, which contradicts to Step 2. So $(C_N(a) \cap C_N(b))_{p'} \leq Z(G)$.

Now the equalities are obvious since $Z(N)_{p'} \leq (C_N(a) \cap C_N(b))_{p'} \leq Z(G)_{p'} \cap N \leq Z(N)_{p'}$.

Step 5. $\pi(a^N) = \pi(\beta)$ and $\pi(b^N) = \pi(\alpha)$.

It is well known that $\pi(a^N) \subseteq \pi(\beta)$. On the other hand, if there exists a prime $r \in \pi(\beta) \backslash \pi(a^N)$, then $r \notin \pi(\alpha)$, and thus $r \notin \pi(b^N)$. By replacing b with a suitable conjugation, we can assume that a Sylow r-subgroup of N, say R, is contained in both $C_N(a)$ and $C_N(b)$, and thus in $C_N(a) \cap C_N(b)$. Therefore, $R \leq (C_N(a) \cap C_N(b))_{p'} \leq Z(G)$ by Step 4, which contradicts to the beginning of the proof. Hence, $\pi(a^N) = \pi(\beta)$. Similarly we have $\pi(b^N) = \pi(\alpha)$.

Step 6. $n(\Gamma_p(N)) = 2$.

For any non-central p'-element x in N, we can write $x = x_q \cdot x_{q'}$. Since $q \nmid |b^N|$ by Step 2, we can assume that $x \in Q$. We will prove that $|x^G| = \alpha$ when $x_q \notin Z(G)$ and $|x^G| = \beta$ when $x_q \in Z(G)$.

First suppose that $x_q \notin Z(G)$. If $|x_q^G| = \alpha$, then clearly $|x^G| = \alpha$. If $|x_q^G| = \beta$, then $x_q \in M$ by the definition of M and thus $q \in \pi(\alpha)$, which is a contradiction. We next show that $|x_q^G| < \beta$ may not happen. If $L \nleq Z(G)$, then we choose z a non-central $\{p, q\}'$-element in $C_N(b)$. Then $C_G(b) = C_G(z)$ by Step 1. Now $x_q \in C_N(b) = C_N(z)$, it follows that $C_G(zx_q) = C_G(z) \cap C_G(x_q) \leq C_G(z)$. Therefore, $\alpha = |z^G|$ divides $|(zx_q)^G|$, which gives that $C_G(zx_q) = C_G(z) = C_G(b) \leq C_G(x_q)$. Again by Step 1, we have $|x_q^G| = \alpha$, which contradicts to our assumption. Therefore, in this case $L \leq Z(G)$. It follows that $|a^N| = |N : C_N(a)| = |C_N(b) : C_N(a) \cap C_N(b)| = |PQL : (C_N(a) \cap C_N(b))_p Z(N)_{p'}|$ is a q-number. And so β is a q-number by Step 5. Since $\langle x_q \rangle$ acts coprimely on the abelian group $M_{q'}$, $M_{q'} = [M_{q'}, \langle x_q \rangle] \times C_{M_{q'}}(x_q)$. As $a \in M_{q'}$, we may write $a = uw$ with $u \in [M_{q'}, \langle x_q \rangle]$ and $w \in C_{M_{q'}}(x_q)$. Let $g = wx_q$. Then $C_G(g) = C_G(w) \cap C_G(x_q) \leq C_G(x_q)$ and so $|x_q^G|$ divides $|g^G|$. It is easy to see that $|g^G| \neq \alpha, \beta$. Hence $|g^G| < \beta$. Choose Q_0 to be a Sylow q-subgroup of G such that $Q \leq Q_0$. It follows that $M_{q'}Q_0 \leq G$ and

$$|M_{q'}Q_0 : C_{M_{q'}Q_0}(g)| \leq |g^G| < \beta = |Q_0| : |C_G(a)|_q.$$

Moreover, we have

$$\begin{aligned} C_{M_{q'}Q_0}(g) &= C_{M_{q'}Q_0}(w) \cap C_{M_{q'}Q_0}(x_q) = M_{q'}C_{Q_0}(w) \cap C_{M_{q'}}(x_q)C_{Q_0}(x_q) \\ &= C_{M_{q'}}(x_q)(C_{Q_0}(w) \cap C_{Q_0}(x_q)). \end{aligned}$$

Set $D = C_{Q_0}(w) \cap C_{Q_0}(x_q)$. Then

$$\frac{|Q_0|}{|C_G(a)|_q} > \frac{|M_{q'}|\,|Q_0|}{|C_{M_{q'}}(x_q)|\,|D|},$$

which implies that $|D| : |C_G(a)|_q > |M_{q'}| : |C_{M_{q'}}(x_q)| = |[M_{q'}, \langle x_q\rangle]|$. On the other hand, since $D \leq C_G(x_q)$ and $M_{q'} \trianglelefteq G$, D acts on $[M_{q'}, \langle x_q\rangle]$ by conjugation. Noticing that

$$C_D(u) = C_G(u) \cap D = C_{Q_0}(u) \cap C_{Q_0}(w) \cap C_{Q_0}(x_q) = C_{Q_0}(a) \cap C_{Q_0}(x_q),$$

we have

$$|C_D(u)| = |C_{Q_o}(a) \cap C_{Q_0}(x_q)| \leq |C_G(a)|_q.$$

Therefore,

$$|u^D| = |D| : |C_D(u)| \geq |D| : |C_G(a)|_q > |[M_{q'}, \langle x_q\rangle]|,$$

which is a contradiction. Therefore, if $x_q \notin Z(G)$, then $|x^G| = \alpha$.

Now, suppose that $x_q \in Z(G)$. Then $x_{q'} \notin Z(G)$. If $|x_{q'}^G| = \alpha$, then $|x^G| = \alpha$. Suppose that $|x_{q'}^G| \neq \alpha$. For any p'-element $y \in C_N(x_{q'}) \cap C_N(b)$, we will prove that $y \in Z(G)$. For otherwise, write $y = y_q \cdot y_{q'}$ with y_q and $y_{q'}$ the q-component and q'-component of y respectively. If $y_{q'} \notin Z(G)$, then $C_G(b) \leq C_G(y_{q'})$ and thus $C_G(y_{q'}) = C_G(b)$ by Step 1 Therefore, $x_{q'} \in C_G(b)$ and thus we see that $C_G(b) \leq C_G(x_{q'})$. Again by Step 1 we have $|x_{q'}^G| = \alpha$, which is a contradiction. Now, y can be assumed to be a q-element. Then $|y^G| = \alpha$ by the above paragraphs. Arguing similarly as above, we have the contradiction that $|x_{q'}^G| = \alpha$. Therefore, we have $(C_N(x_{q'}) \cap C_N(b))_{p'} \leq Z(G)$. Noticing that $N = C_N(a)C_N(b)$, we have $|N| = |a^N|\,|b^N|\,|C_N(a) \cap C_N(b)|$. Since β is a p'-number dividing $|N/(N \cap Z(G))|$, we see that β divides $|N|_{p'} : |N \cap Z(G)|_{p'} = |N|_{p'} : |Z(N)|_{p'}$, and thus β divides $|a^N|$. Therefore, $|a^G| = \beta = |a^N| = |C_N(b) : C_N(a) \cap C_N(b)| = |C_N(b)|_{p'} : |C_N(a) \cap C_N(b)|_{p'} = |C_N(b)|_{p'} : |Z(N)|_{p'}$. Let T be a p-complement of $C_N(b)$. Now, consider the factor group $T/Z(N)_{p'}$ and the set $x_{q'}^N$. If $zZ(N)_{p'}$ is an element in $T/Z(N)_{p'}$, we may assume that $z \in T$. For any $y \in t^N$, we define $y^{zZ(N)_{p'}} = y^z$. Then $T/Z(N)_{p'}$ acts on the set $x_{q'}^N$. By the above paragraph we have $x_{q'}^N \cap T = \emptyset$. Therefore, $T/Z(N)_{p'}$ acts on $x_{q'}^N$ without fixed point, which implies that $|T/Z(N)_{p'}|$ divides $|x_{q'}^N|$. Consequently, $|x_{q'}^N| = |x^G| = \beta$. Hence $|x^G| = |x_{q'}^G| = \beta$.

Consequently, if we denote

$$I = \{x \mid x \text{ is a } p'\text{-element in } N \text{ such that } |x^G| = \alpha\}$$

and

$$J = \{x \mid x \text{ is a } p'\text{-element in } N \text{ such that } |x^G| = \beta\}.$$

Then, from the above paragraphs and Step 4 we see that $N_{p'} = Z(N)_{p'} \cup I \cup J$ and that $|x^N| \neq 1$ and $|y^N| \neq 1$ for every $x \in I$ and $y \in J$. Therefore, $n(\Gamma_p(N)) = 2$ by [2, Theorem 1].

Step 7. Final conclusion.

Since $\Gamma_p(N)$ has two components by Step 6, we use X_1 and X_2 to denote the two components and assume that $x^N \in X_2$ where $|x^N|$ is maximal in $cs_N(N_{p'})$. Furthermore, for $i = 1$ and 2, we write

$$\pi_i = \{r \mid r \text{ is a primary divisor of } |A|, \text{ where } A = x^N \in X_i\}.$$

If p does not divide α, then all p'-elements in N have conjugacy class size in N coprime to p. Therefore, $N = P \times K$, where P is a Sylow p-subgroup in N and K is a p-complement of N by [2, Proposition 2]. In this case, $cs(K) = cs_N(N_{p'})$ and thus $n(\Gamma(K)) = 2$. So K is quasi-Frobenius with abelian kernel and complements by [1, Theorem 2].

Now suppose that p divides α. If the maximal size of conjugacy classes in $\mathrm{Con}_N(N_{p'})$ divides β, then $p \in \pi(\alpha) = \pi_1$. Therefore, N is p-nilpotent by [3, Theorem 8] and K is quasi-Frobenius with abelian kernel and complements. Otherwise, the maximal size of conjugacy classes in $\mathrm{Con}_N(N_{p'})$ divides α, it follows that $p \in \pi(\alpha) = \pi_2$. If $|\pi_2| \geq 3$, then N has a normal Sylow p-subgroup and K is quasi-Frobenius with abelian kernel and complements by [3, Theorem 12]. It is easy to see that N is solvable in the above cases.

If $|\pi_2| < 3$, then $|\pi_2| = 2$ by Lemma 2.3 and we may assume that $\pi_2 = \{p, r\}$ for some prime $r \neq p$. By [3, Theorem 9], we see that N is π_2-separable and has abelian π_2-complements. Therefore, there exists a π_2-complement T and a Sylow r-subgroup R of K such that $K = RT$. Since r does not divide β, a Sylow r-subgroup of N is contained in $C_N(a)$. it is easy to see that M is the p-complement of $C_N(a)$, so $R \leq M$. It follows that $R \trianglelefteq N$, by Schur–Zassenhaus theorem, there is a complement V of R in N. Therefore, V is a product of a Sylow p-subgroup and an abelian p-complement, hence it is solvable. Consequently, we deduce that N is solvable since $N/R \cong V$. We will finally show that K is quasi-Frobenius with abelian kernel and complements.

If $x \in Z(K)$, then $|x^N|$ is a p-number. Since β is a p'-number and $|x^N|$ divides $|x^G|$, $|x^G| \neq \beta$. If $|x^G| = \alpha$, then α is a p-number since $\pi(x^N) = \pi(\alpha)$ by Step 5, which contradicts to Lemma 2.3. Therefore, $|x^G| = 1$. It follows that x is a p'-element in $Z(G) \cap N$ and thus $x \in Z(N)_{p'}$. Therefore, $Z(K) \subseteq Z(N)_{p'}$. The fact that $Z(N)_{p'} \subseteq Z(K)$ is easy to see. So, $Z(N)_{p'} = Z(K)$.

Write $\overline{K} = K/Z(K), \overline{R} = RZ(K)/Z(K)$ and $\overline{T} = TZ(K)/Z(K)$. Then $\overline{K} = \overline{R} \rtimes \overline{T}$. It suffices to show that $C_{\overline{T}}(\overline{g}) = 1$ for any $1 \neq \overline{g} \in \overline{R}$. Suppose on contrary that there exists $1 \neq \overline{t} \in C_{\overline{T}}(\overline{g})$ for some $1 \neq \overline{g} \in \overline{R}$. Since $\overline{R} \cap \overline{T} = 1$

and $t \notin Z(N)$, we see $t \notin M$ and thus $|t^G| = \alpha$ by Step 6. On the other hand, since $M \leq C_N(g)$, $|g^N|$ divides $|N : M| = |a^N| \cdot p^s$ for some integer $s \geq 0$. If $|g^G| = \alpha$, then $|g^N|$ divides α. Therefore, $|g^N|$ is a p-number and so α is a p-number too, which contradicts to Lemma 2.3. Therefore, $|g^G| = \beta$ by Step 6 and so $(C_N(t) \cap C_N(g))_{p'} \leq Z(G)_{p'}$.

But on the other hand, since $(|\overline{t}|, |\overline{g}|) = 1$, we see that $\overline{gt}^{|\overline{g}|} = \overline{t}^{|\overline{g}|} \neq 1$. Hence,

$$\overline{gt}^{|\overline{g}|} = \overline{t}^{|\overline{g}|} \in \overline{C_N(t)} \cap \overline{C_N(gt)} = \overline{C_N(t) \cap C_N(gt)} = \overline{C_N(t) \cap C_N(g)},$$

which means that $C_N(t) \cap C_N(g)$ contains a non-central p'-element, contradicting to the above paragraph. □

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