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On the restricted summability of two-dimensional Walsh–Fejér means

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Abstract. The main aim of this paper is to discuss the properties of the maximal operator of two-dimensional Walsh–Fejér means, where the set of indices is inside a cone-like set L. We show that the maximal operator σ_L^* is not bounded from the Hardy space $H_{1/2}^\alpha$ to the space $L_{1/2}$. For $p > 1/2$ Weisz [14] showed that the maximal operator σ_L^* is bounded from the Hardy space H_p^α to the space L_p. That is, we show that the assumption $p > 1/2$ is essential in the theorem of Weisz.

1. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis [1], [10].

Denote G the Walsh group, and μ the normalized Haar measure on G. Dyadic intervalls are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, x_1, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $x \in G, n \in \mathbf{P}$. Let $0 := (x_0 = 0, x_1 = 0, \dots)$ be the nullelement of G and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

The Rademacher functions are defined by

$$r_k(x) := (-1)^{x_k} \quad (x \in G,\ k \in \mathbf{N}).$$

Each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ $(i \in \mathbf{N})$. Define the order $|n|$ of n by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that

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is $2^{|n|} \le n < 2^{|n|+1}$.

Define the Walsh–Paley functions by

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k}.$$

Let us consider the Dirichlet and Fejér kernel functions

$$D_n := \sum_{k=0}^{n-1} w_k, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, \quad D_0 = K_0 := 0.$$

The Fourier coefficients, the nth partial sum of Fourier series and the Fejér means are defined in the usual way for $f \in L^1(G)$.

The norm (or quasinorm) of the space L_p is defined by

$$\|f\|_p := \left(\int_{K^2} \left| f\left(x^1, x^2\right) \right|^p d\mu\left(x^1, x^2\right) \right)^{1/p} \quad (0 < p < +\infty).$$

For $x = (x^1, x^2) \in G^2$ and $n = (n_1, n_2) \in \mathbf{N}^2$ the 2-dimensional rectangles $I_n(x) := I_{n_1}(x^1) \times I_{n_2}(x^2)$ have got measure $2^{-(n_1+n_2)}$. For $n = (n_1, n_2)$ the σ-algebra generated by the rectangles $\{I_n(x), x \in G^2\}$ is denoted by $\mathcal{F}_n$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n.

Suppose that $\alpha : [1, +\infty) \to [1, +\infty)$ be a strictly monotone increasing continuous function with property $\lim_{+\infty} \alpha = +\infty$, $\alpha(1) = 1$. Moreover, suppose that there exist $\zeta, \gamma_1, \gamma_2 > 1$ such that the inequality

$$\gamma_1 \alpha(x) \le \alpha(\zeta x) \le \gamma_2 \alpha(x) \tag{1}$$

holds for each $x \ge 1$. In this case the function α is called CRF (cone-like restriction function). Let $\beta \ge 1$ be fixed. We will investigate the maximal operator of the two-dimensional Fejér means and the convergence over a cone-like set L (with respect to the first dimension), where

$$L := \{n \in \mathbf{N}^2 : \beta^{-1}\alpha(n_1) \le n_2 \le \beta\alpha(n_1)\}.$$

The cone-like sets was introduced by Gát in [4]. The condition (1) on the function α is natural, because Gát [4] proved that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if the inequality (1) holds.

Weisz defined a new type martingale Hardy space depending on the function α (see [14]). For a given $n_1 \in \mathbf{N}$ set $n_2 := |\alpha(2^{n_1})|$, that is the order of $\alpha(2^{n_1})$

(this means that $2^{n_2} \le \alpha(2^{n_1}) < 2^{n_2+1}$). Let $\overline{n}_1 := (n_1, n_2)$. Since, the function α is increasing, the sequence $(\overline{n}_1,\ n_1 \in \mathbf{N})$ is increasing, too. We investigate the class of one-parameter martingales $f = (f_{\overline{n}_1},\ n_1 \in \mathbf{N})$ with respect to the σ-algebras $(\mathcal{F}_{\overline{n_1}},\ n_1 \in \mathbf{N})$. The maximal function of a martingale f is defined by $f^* = \sup_{n_1 \in \mathbf{N}} |f_{\overline{n}_1}|$. For $0 < p < \infty$ the martingale Hardy space $H_p^\alpha(G^2)$ consists of all martingales for which

$$\|f\|_{H_p^\alpha} := \|f^*\|_p < \infty.$$

It is known (see [13]) that $H_p^\alpha \sim L_p$ for $1 < p \le \infty$, where $\sim$ denotes the equivalence of the norms and spaces.

The Kronecker product $(w_{n,m} : n,\ m \in \mathbf{N})$ of two Walsh system is said to be the two-dimensional Walsh system. That is, $w_{n,m}(x^1, x^2) = w_n(x^1) w_m(x^2)$.

If $f \in L(G^2)$, then the number $\widehat{f}(n,m) := \int_{G^2} f w_{n,m} \quad (n, m \in \mathbf{N})$ is said to be the (n,m)-th Walsh–Fourier coefficient of f. We can extend this definition to martingales in the usual way (see Weisz [12], [13]). Denote by $S_{n,m}$ the (n,m)th rectangular partial sum of the Walsh–Fourier series of a martingale f. Namely,

$$S_{n,m}(f; x^1, x^2) := \sum_{k=0}^{n-1} \sum_{i=0}^{m-1} \widehat{f}(k,i) w_{k,i}(x^1, x^2).$$

The $n = (n_1, n_2)$-th Fejér mean is defined by

$$\sigma_n f(x^1, x^2) := \frac{1}{n_1 n_2} \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} S_{k,l}(f; x^1, x^2).$$

It is known that

$$K_{n_1,n_2}(x^1, x^2) = K_{n_1}(x^1) K_{n_2}(x^2).$$

Define the maximal operator σ_L^* by

$$\sigma_L^* f(x^1, x^2) := \sup_{n \in L} |\sigma_n f(x^1, x^2)|.$$

For double Walsh–Fourier series, Móricz, Schipp and Wade [7] proved that $\sigma_n f$ converge to f a.e. in the Pringsheim sense (that is, no restriction on the indices other than $\min(n_1, n_2) \to \infty$) for all functions $f \in L \log^+ L$. In the paper [3] Gát proved that the theorem of Móricz, Schipp and Wade can not be sharpened. Namely, the following was proved. Let $\delta : [0, +\infty) \to [0, +\infty)$ be a measurable function with property $\lim_{+\infty} \delta = 0$, then there exists a function $f \in L \log^+ L \delta(L)$ such that $\sigma_n f$ does not converge to f a.e. as $\min(n_1, n_2) \to \infty$.

The a.e. convergence of Fejér means $\sigma_n f$ of integrable functions, where the set of indices is inside a positive cone around the identical function, that is $\beta^{-1} \leq n_1/n_2 \leq \beta$ is provided with some fixed parameter $\beta \geq 1$, was proved by Gát [5] and Weisz [15]. Moreover, in the paper of Weisz [14] the properties of the maximal operator was treated. That is, he proved that the maximal operator of Walsh–Fejér means is of type (H_p, L_p) for all $p > 1/2$ and of weak type $(1,1)$, provided that the supremum in the maximal operator is taken over a positive cone.

A common generalization of results of Móricz, Schipp, Wade [7] and Gát [5], Weisz [15] for cone-like set was given by the author and Gát in [6]. That is, a necessary and sufficient condition for cone-like sets in order to preserve the convergence property, was given. Recently, the properties of the maximal operator of the Walsh–Fejér means provided that the supremum in the maximal operator is taken over a cone-like set, was discussed by Weisz [14]. Namely, it was proved that the maximal operator is bounded from H_p^α to L_p for $1/2 < p < \infty$ and is of weak type $(1,1)$. Moreover, the theorem of Gát and the author was reached as a corollary, that is, the a.e. convergence of the Walsh–Fejér means of integrable functions provided that the set of the indices is inside a cone-like set, was proved. The endpoint case $p = 1/2$ was not treated. The main aim of this paper is to discuss what does happen at the endpoint $p = 1/2$. We show that the maximal operator σ_L^* is not bounded from the Hardy space $H_{1/2}^\alpha$ to the space $L_{1/2}$.

It is important to note that, the a.e. convergence two-dimensional Fejér means of integrable functions, provided that the set of the indices is inside a cone-like set, was generalized by the author for representative product systems [9] and for Vilenkin-like systems [8].

2. Auxiliary propositions and main result

To prove our Theorem we need the following definition of Weisz [14] and Lemma of Gát [2]:

A bounded measurable function a is a p-atom, if there exists a dyadic two-dimensional rectangle $I \in \mathcal{F}_{\overline{n}_1}$, such that

a) $\int_I a d\mu = 0$,

b) $\|a\|_\infty \leq \mu(I)^{-1/p}$,

c) $\operatorname{supp} a \subseteq I$.

Lemma 1 (Gát [2]). *Let $A, t \in \mathbf{N}$, $A > t$. Suppose that $x \in I_t \backslash I_{t+1}$. Then*

$$K_{2^A}(x) = \begin{cases} 0, & \text{if } x - e_t \notin I_A, \\ 2^{t-1}, & \text{if } x - e_t \in I_A. \end{cases}$$

If $x \in I_A$ then $K_{2^A}(x) = 2^{A-1} + 1/2$.

Now, we formulate our main theorem.

Theorem 1. *Let α be CRF. The maximal operator σ_L^* is not bounded from the Hardy space $H_{1/2}^{\alpha}$ to the space $L_{1/2}$.*

Proof. Let

$$f_{\overline{n_1}}(x^1, x^2) := (D_{2^{n_1+1}}(x^1) - D_{2^{n_1}}(x^1))(D_{2^{n_2+1}}(x^2) - D_{2^{n_2}}(x^2)).$$

It is a one-parameter martingale, where n_2 is defined to n_1, earlier by Weisz. $\operatorname{supp} f_{\overline{n_1}} = I_{n_1} \times I_{n_2} \in \mathcal{F}_{\overline{n}_1}$, $\int_{I_{n_1} \times I_{n_2}} f_{\overline{n_1}} d\mu = 0$ and $\|f_{\overline{n_1}}\|_\infty \leq \mu(I_{n_1} \times I_{n_2})^{-2}$, That is, $f_{\overline{n_1}}$ is an 1/2-atom in $H_{1/2}^{\alpha}$ and

$$\|f_{\overline{n_1}}\|_{H_{1/2}^{\alpha}} = \|D_{2^{n_1}}\|_{1/2} \|D_{2^{n_2}}\|_{1/2} = 2^{-(n_1+n_2)}. \tag{2}$$

It is simple to calculate the Fourier coefficients and partial sums. That is,

$$\hat{f}_{\overline{n_1}}(i,k) = \begin{cases} 1, & \text{if } i = 2^{n_1}, \dots, 2^{n_1+1} - 1,\ k = 2^{n_2}, \dots, 2^{n_2+1} - 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$S_{i,j}(f_{\overline{n_1}}; x^1, x^2)$$

$$= \begin{cases} (D_i(x^1) - D_{2^{n_1}}(x^1))(D_j(x^2) - D_{2^{n_2}}(x^2)), & \text{if } i = 2^{n_1}+1, \dots, 2^{n_1+1} - 1, \\ & \text{and } j = 2^{n_2}+1, \dots, 2^{n_2+1} - 1, \\ (D_i(x^1) - D_{2^{n_1}}(x^1))(D_{2^{n_2+1}}(x^2) - D_{2^{n_2}}(x^2)), & \text{if } i = 2^{n_1}+1, \dots, 2^{n_1+1} - 1, \\ & \text{and } j \geq 2^{n_2+1}, \\ (D_{2^{n_1+1}}(x^1) - D_{2^{n_1}}(x^1))(D_j(x^2) - D_{2^{n_2}}(x^2)), & \text{if } j = 2^{n_2}+1, \dots, 2^{n_2+1} - 1, \\ & \text{and } i \geq 2^{n_1+1}, \\ f_{\overline{n_1}}(x^1, x^2), & \text{if } i \geq 2^{n_1+1}, \text{and } j \geq 2^{n_2+1} \\ 0, & \text{otherwise.} \end{cases}$$

We can write the nth Dirichlet kernel with respect to the Walsh system in the following form:

$$D_n(x) = D_{2^{|n|}}(x) + r_{|n|}(x) D_{n-2^{|n|}}(x). \tag{3}$$

First, we calculate $\sigma_{N_1,N_2} f_{\overline{n_1}}$ for special indices. Set $N_1 := 2^{n_1} + 2^s$ $(0 < s < n_1)$

and $N_2 := [\alpha(2^{n_1} + 2^s)] + 1$, (where $[x]$ denotes the integer part of x). Taking account that $\beta > 1$ we could see easily that $(N_1, N_2) \in L$. We have two cases, $N_2 < 2^{n_2+1}$ and $N_2 \geq 2^{n_2+1}$.

First, we set $N_2 < 2^{n_2+1}$. By the equality (3) we immediately have that

$$\begin{aligned}
|\sigma_{N_1,N_2} f_{\overline{n_1}}(x^1, x^2)| &= \frac{1}{N_1 N_2} \left| \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} S_{k,l}(f_{\overline{n_1}}; x^1, x^2) \right| \\
&= \frac{1}{N_1 N_2} \left| \sum_{k=2^{n_1}+1}^{N_1} \sum_{l=2^{n_2}+1}^{N_2} (D_k(x^1) - D_{2^{n_1}}(x^1))(D_l(x^2) - D_{2^{n_2}}(x^2)) \right| \\
&= \frac{1}{N_1 N_2} \left| \sum_{k=2^{n_1}+1}^{N_1} \sum_{l=2^{n_2}+1}^{N_2} r_{n_1}(x^1) D_{k-2^{n_1}}(x^1) r_{n_2}(x^2) D_{l-2^{n_2}}(x^2) \right| \\
&= \frac{1}{N_1 N_2} \left| \sum_{k=1}^{2^s} \sum_{l=1}^{N_2-2^{n_2}} D_k(x^1) D_l(x^2) \right| \\
&\geq \frac{1}{2^{n_1+1} 2^{n_2+1}} |2^s K_{2^s}(x^1)(N_2 - 2^{n_2}) K_{N_2-2^{n_2}}(x^2)|.
\end{aligned}$$

Now, we set $N_2 \geq 2^{n_2+1}$. By equality (3) we have again that

$$\begin{aligned}
&|\sigma_{N_1,N_2} f_{\overline{n_1}}(x^1, x^2)| = \frac{1}{N_1 N_2} \left| \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} S_{k,l}(f_{\overline{n_1}}; x^1, x^2) \right| \\
&= \frac{1}{N_1 N_2} |2^s K_{2^s}(x^1)| \left| r_{n_2}(x^2) \sum_{l=2^{n_2}+1}^{2^{n_2+1}-1} D_{l-2^{n_2}}(x^2) + \sum_{l=2^{n_2+1}}^{N_2} (D_{2^{n_2+1}}(x^2) - D_{2^{n_2}}(x^2)) \right| \\
&\geq \frac{1}{2^{n_1+1} N_2} |2^s K_{2^s}(x^1)| \\
&\quad \times \left| r_{n_2}(x^2)(2^{n_2} - 1) K_{2^{n_2}-1}(x^2) + (N_2 - 2^{n_2+1} + 1)(D_{2^{n_2+1}}(x^2) - D_{2^{n_2}}(x^2)) \right| \\
&= \frac{1}{2^{n_1+1} N_2} |2^s K_{2^s}(x^1)| \left| (2^{n_2} - 1) K_{2^{n_2}-1}(x^2) + (N_2 - 2^{n_2+1} + 1) D_{2^{n_2}}(x^2) \right|.
\end{aligned}$$

By inequality (2) we write that

$$\sigma_L^* f_{\overline{n_1}}(x^1, x^2) = \sup_{n \in L} |\sigma_n f_{\overline{n_1}}(x^1, x^2)| \geq \max_{1 \leq 2^s < 2^{n_1}} |\sigma_{(2^{n_1}+2^s, [\alpha(2^{n_1}+2^s)]+1)} f_{\overline{n_1}}(x^1, x^2)|$$

and

$$\begin{aligned}
&\frac{\|\sigma_L^* f_{\overline{n_1}}\|_{1/2}}{\|f_{\overline{n_1}}\|_{H_{1/2}^{\alpha}}} \\
&\geq \frac{1}{2^{-(n_1+n_2)}} \left(\int_{G^2} \max_{1 \leq 2^s < 2^{n_1}} |\sigma_{(2^{n_1}+2^s, [\alpha(2^{n_1}+2^s)]+1)} f_{\overline{n_1}}(x^1, x^2)|^{1/2} d\mu(x^1, x^2) \right)^2.
\end{aligned}$$

Now, we investigate the integral
$I := \int_{G^2} \max_{1\le 2^s<2^{n_1}} |\sigma_{(2^{n_1}+2^s,[\alpha(2^{n_1}+2^s)])} f_{\overline{n_1}}(x^1,x^2)|^{1/2} d\mu(x^1,x^2)$, we decompose the set G as the following disjoint union

$$G = I_A \cup \bigcup_{t=0}^{A-1} (I_t \backslash I_{t+1}).$$

Since, α is strictly monotone increasing function we have that $K_{[\alpha(2^{n_1}+2^t)]+1-2^{n_2}} \ne K_0 = 0$ for all t $(0 < t < n_1)$.

We set $k \in \mathbf{P}$ and we fix it. We choose a big number n_1^k, such that $(N_1^{k,s}, N_2^{k,s}) \in L$ for all s with the condition $n_1^k - 1 \ge s \ge n_1^k - k > 0$, where $N_1^{k,s} := 2^{n_1^k} + 2^s$ and $N_2^{k,s} := [\alpha(2^{n_1^k} + 2^s)] + 1$. We note that if $\beta > 1$ is small, then n_1^k have to be large enough.

If there exists a number s such that $n_1^k - 1 \ge s \ge n_1^k - k$ and $N_2^{k,s} \ge 2^{n_2^k+1}$ holds, then we set $s^* := \min\{s : N_2^{k,s} \ge 2^{n_2^k+1},\ n_1^k - 1 \ge s \ge n_1^k - k\}$. We set $s* := n_1^k$ if for all s (with $n_1^k - 1 \ge s \ge n_1^k - k$) the inequality $N_2^{k,s} < 2^{n_2^k+1}$ holds.

Therefore,

$$\begin{aligned}
I &= \int_{G^2} \max_{1\le 2^s<2^{n_1^k}} |\sigma_{(2^{n_1^k}+2^s,[\alpha(2^{n_1^k}+2^s)]+1)} f_{\overline{n_1^k}}(x^1,x^2)|^{1/2} d\mu(x^1,x^2) \\
&\ge \sum_{t=1}^{n_1^k-1} \int_{(I_t\backslash I_{t+1})\times G} \max_{1\le 2^s<2^{n_1^k}} |\sigma_{(2^{n_1^k}+2^s,[\alpha(2^{n_1^k}+2^s)]+1)} f_{\overline{n_1^k}}(x^1,x^2)|^{1/2} d\mu(x^1,x^2) \\
&= \sum_{t=1}^{s^*-1} \int_{(I_t\backslash I_{t+1})\times G} \max_{1\le 2^s<2^{n_1^k}} |\sigma_{(2^{n_1^k}+2^s,[\alpha(2^{n_1^k}+2^s)]+1)} f_{\overline{n_1^k}}(x^1,x^2)|^{1/2} d\mu(x^1,x^2) \\
&\quad + \sum_{t=s^*}^{n_1^k-1} \int_{(I_t\backslash I_{t+1})\times G} \max_{1\le 2^s<2^{n_1^k}} |\sigma_{(2^{n_1^k}+2^s,[\alpha(2^{n_1^k}+2^s)]+1)} f_{\overline{n_1^k}}(x^1,x^2)|^{1/2} d\mu(x^1,x^2) \\
&=: \sum_1 + \sum_2 .
\end{aligned}$$

Now, we discuss $\sum_1$.

$$\begin{aligned}
\sum_1 &\ge \sum_{t=1}^{s^*-1} \frac{1}{2^{(n_1^k+n_2^k+2)/2}} \\
&\quad \times \int\limits_{(I_t\backslash I_{t+1})\times G} \max_{1\le 2^s<2^{n_1^k}} |2^s K_{2^s}(x^1)(N_2^{k,s} - 2^{n_2^k}) K_{N_2^{k,s}-2^{n_2^k}}(x^2)|^{1/2} d\mu(x^1,x^2)
\end{aligned}$$

$$\geq \sum_{t=1}^{s^*-1} \frac{1}{2^{(n_1^k+n_2^k+2)/2}} \int\limits_{(I_t\setminus I_{t+1})\times G} |2^t K_{2^t}(x^1)(N_2^{k,t}-2^{n_2^k})K_{N_2^{k,t}-2^{n_2^k}}(x^2)|^{1/2} d\mu(x^1,x^2)$$

$$\geq \sum_{t=n_1^k-k}^{s^*-1} \frac{1}{2^{(n_1^k+n_2^k+2)/2}}$$

$$\times \int\limits_{(I_t\setminus I_{t+1})\times(I_{t^*}\setminus I_{t^*+1})} |2^t K_{2^t}(x^1)(N_2^{k,t}-2^{n_2^k})K_{N_2^{k,t}-2^{n_2^k}}(x^2)|^{1/2} d\mu(x^1,x^2)$$

$$= \sum_{t=n_1^k-k}^{s^*-1} \frac{1}{2^{(n_1^k+n_2^k+2)/2}}$$

$$\times \int\limits_{(I_t\setminus I_{t+1})\times(I_{t^*}\setminus I_{t^*+1})} \left|\frac{2^t(2^t+1)}{2}(N_2^{k,t}-2^{n_2^k})K_{N_2^{k,t}-2^{n_2^k}}(x^2)\right|^{1/2} d\mu(x^1,x^2)$$

$$\geq \frac{1}{2^{(n_1^k+n_2^k+2)/2}} \frac{1}{4} \sum_{t=n_1^k-k}^{s^*-1} \int\limits_{I_{t^*}\setminus I_{t^*+1}} \left|(N_2^{k,t}-2^{n_2^k})K_{N_2^{k,t}-2^{n_2^k}}(x^2)\right|^{1/2} d\mu(x^2)$$

$$\geq \frac{1}{2^{(n_1^k+n_2^k+2)/2}} \sum_{t=n_1^k-k}^{s^*-1} \frac{1}{16} \geq \frac{1}{2^{(n_1^k+n_2^k+2)/2}} \frac{s^*-n_1^k+k}{16},$$

where $t^* := |N_2^{k,t}-2^{n_2^k}| = |[\alpha(2^{n_1^k}+2^t)]+1-2^{n_2^k}|$ (that is, the order of $[\alpha(2^{n_1^k}+2^t)]+1-2^{n_2^k}$).

Now, we discuss $\sum_2$. In this case we have to modify some steps in the estimation of the sum $\sum_1$. Now, we write that

$$\sum_2 \geq \sum_{t=s^*}^{n_1^k-1} \int_{(I_t\setminus I_{t+1})\times G} \max_{1\leq 2^s<2^{n_1^k}} |\sigma_{(2^{n_1^k}+2^s,N_2^{k,s})} f_{\overline{n_1^k}}(x^1,x^2)|^{1/2} d\mu(x^1,x^2)$$

$$\geq \sum_{t=s^*}^{n_1^k-1} \int_{(I_t\setminus I_{t+1})\times(I_{n_2^k-1}\setminus I_{n_2^k})} |\sigma_{(2^{n_1^k}+2^t,N_2^{k,t})} f_{\overline{n_1^k}}(x^1,x^2)|^{1/2} d\mu(x^1,x^2)$$

$$\geq \frac{1}{4} \sum_{t=s^*}^{n_1^k-1} \frac{1}{2^{(n_1^k+1)/2}(N_2^{k,t})^{1/2}} \int_{I_{n_2^k-1}\setminus I_{n_2^k}} |(2^{n_2^k}-1)K_{2^{n_2^k}-1}(x^2)$$

$$+ (N_2^{k,t}-2^{n_2^k+1}+1)D_{2^{n_2^k}}(x^2)|^{1/2} d\mu(x^2).$$

$D_{2^{n_2^k}}(x^2) = 0$ for all $x^2 \in I_{n_2^k-1} \backslash I_{n_2^k}$. Lemma 1 of Gát immediately gives that

$$\begin{aligned}\int_{I_{n_2^k-1}\backslash I_{n_2^k}} |\dots|^{1/2} d\mu(x^2) &\geq \int_{I_{n_2^k-1}\backslash I_{n_2^k}} |(2^{n_2^k}-1)K_{2^{n_2^k}-1}(x^2)|^{1/2} d\mu(x^2)\\ &\geq \int_{I_{n_2^k-1}\backslash I_{n_2^k}} |2^{n_2^k} K_{2^{n_2^k}}(x^2) - D_{2^{n_2^k}}(x^2)|^{1/2} d\mu(x^2)\\ &= \int_{I_{n_2^k-1}\backslash I_{n_2^k}} |2^{n_2^k} K_{2^{n_2^k}}(x^2)|^{1/2} d\mu(x^2)\\ &\geq \int_{I_{n_2^k-1}\backslash I_{n_2^k}} |2^{n_2^k} 2^{n_2^k-2}|^{1/2} d\mu(x^2) \geq \frac{1}{4}.\end{aligned}$$

That is, we showed that

$$\frac{\|\sigma_L^* f_{\overline{n_1}}\|_{1/2}}{\|f_{\overline{n_1}}\|_{H_{1/2}^\alpha}} \geq 2^{n_1^k+n_2^k}\left(\sum_1 + \sum_2\right)^2$$

$$\geq 2^{n_1^k+n_2^k}\left(\frac{1}{2^{(n_1^k+n_2^k+2)/2}}\frac{s^*-n_1^k+k}{16} + \frac{1}{2^{(n_1^k+1)/2}(N_2^{k,n_1^k-1})^{1/2}}\frac{n_1^k-s^*}{16}\right)^2.$$

For N_2^{k,n_1^k-1} we write that

$$N_2^{k,n_1^k-1} = [\alpha(2^{n_1^k}+2^{n_1^k-1})]+1 \leq [\alpha(2\cdot 2^{n_1^k})]+1 \leq [\gamma_2^{\log_\varsigma 2}\alpha(2^{n_1^k})]+1 \leq \gamma_2^{\log_\varsigma 2} 2^{n_2^k+1},$$

where $\gamma_2^{\log_\varsigma 2} > 1$. From this we have that

$$\frac{\|\sigma_L^* f_{\overline{n_1^k}}\|_{1/2}}{\|f_{\overline{n_1^k}}\|_{H_{1/2}^\alpha}} \geq ck^2,$$

where $c = \frac{1}{2^{10}\gamma_2^{2\log_\varsigma 2}}$. This completes the proof of our main Theorem. □

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