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Two ultrafilter properties for vector lattices of real-valued functions

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Abstract. Let E be a vector lattice of real-valued functions defined on a set X, and $\mathcal{H}(E) := \{\{f \geq 1\} : f \in E\}$. X is said to be E-compact [resp., $\mathcal{H}(E)$-complete] if every E-stable $\mathcal{H}(E)$-ultrafilter [resp., every $\mathcal{H}(E)$-ultrafilter with the cip] has nonvoid intersection. We study the relations between these two concepts and give several characterizations of E-compact spaces. In case E is the ring of all [bounded] continuous functions on a completely regular space, we obtain some new measure-theoretic characterizations of realcompactness [compactness].

1. Introduction

Consider a ring E satisfying $C^b(X) \subset E \subset C(X)$ where $C(X)$ [$C^b(X)$] denotes the family of all continuous [and bounded] real-valued functions on a completely regular space X. Then E is a vector lattice, and the collection $\mathcal{H}(E) := \{\{f \geq 1\} : f \in E\}$ coincides with the family $\mathfrak{Z}(X)$ of all zero-sets in X. Byun and Watson ([6]) call X E-compact if every real maximal ideal in E is fixed, and they prove that X is E-compact iff every E-stable $\mathcal{H}(E)$-ultrafilter has nonempty intersection. Since the latter condition does not depend on the ring structure of E, we do use this condition to define E-compactness of an arbitrary set X for any vector lattice E of real-valued functions on X.

In section 2 we compare E-compact spaces with $\mathcal{H}(E)$-complete spaces in the sense of [1]. In particular, we present several measure-theoretic conditions that are intermediate between E-compactness and $\mathcal{H}(E)$-completeness.

In section 3 we consider certain rings E of $\mathcal{L}$-continuous functions with 'bounded support' ($\mathcal{L}$ being a δ-lattice of subsets of X) and characterize E-compactness by several algebraic properties. As a special case we obtain the characterization theorem of Byun and Watson mentioned above.

Section 4 [Section 5] is concerned with different characterizations of E-compact spaces where E is the ring of all $\mathcal{L}$-continuous [and bounded] functions with 'bounded support'. In particular, if X is a completely regular space and $E = C(X)$ [resp., $E = C^b(X)$] we obtain some new measure-theoretic characterizations of realcompactness [resp., compactness].

Now we fix the notation. $\mathbb{N}$ denotes the set of positive integers. The set $\mathbb{R}$ of real numbers is always assumed to be equipped with the Euclidean topology.

Let X be an arbitrary set and $\mathcal{L}$ a family of subsets of X. We write $\sigma(\mathcal{L})$ for the σ-algebra in X generated by $\mathcal{L}$. $\mathcal{L}$ is said

(i) to have the *finite* [resp., *countable*] *intersection property* (fip) [resp., (cip)] if every finite [resp., countable] subfamily of $\mathcal{L}$ has nonvoid intersection;

(ii) to be a *compact class* if every subfamily of $\mathcal{L}$ with the fip has nonvoid intersection;

(iii) to be a *lattice* [resp., *δ-lattice*] if $\mathcal{L}$ is closed under finite unions and finite [resp., countable] intersections.

Let $\mathcal{L}$ be a lattice of subsets of X. A nonempty subfamily $\mathcal{D}$ of $\mathcal{L}$ is called an *$\mathcal{L}$-filter* [resp., *$\mathcal{L}$-ultrafilter*] if $\mathcal{D}$ satisfies the following conditions (1)–(3) [resp., (1)–(4)]:

(1) $\phi \notin \mathcal{D}$;
(2) $\mathcal{D}$ is closed under finite intersections;
(3) $D \in \mathcal{D}$, $L \in \mathcal{L}$ and $D \subset L$ imply $L \in \mathcal{D}$;
(4) $L \in \mathcal{L}$ and $D \cap L \neq \phi$ for all $D \in \mathcal{D}$ imply $L \in \mathcal{D}$.

It is a consequence of Zorn's lemma that every subfamily of $\mathcal{L}$ with the fip is contained in an $\mathcal{L}$-ultrafilter.

Let $E \subset \mathbb{R}^X$ be a vector lattice (with respect to pointwise operations). Define $E_+ := \{f \in E : f \geq 0\}$. We write $\sigma(E)$ for the smallest σ-algebra in X making all functions $f \in E$ measurable. Similarly $\tau(E)$ denotes the coarsest topology in X making all functions $f \in E$ continuous.

For $x \in X$ we denote by I_x the evaluation functional on E associated to the point x, i.e. $I_x(f) = f(x)$ for $f \in E$.

For $f \in \mathbb{R}^X$ and $\alpha \in \mathbb{R}$, we write $\{f \geq \alpha\}$ for the set $\{x \in X : f(x) \geq \alpha\}$. In the same way, we use the abbreviations $\{f = \alpha\}$, $\{f \neq \alpha\}$, $\{f < \alpha\}$, $\{f > \alpha\}$ and $\{f \leq \alpha\}$.

By a measure we always understand a $[0, \infty]$-valued σ-additive set function defined on a σ-algebra and vanishing at ϕ. For a measure μ we denote by μ^* [μ_*] the outer [inner] measure associated to μ. We write δ_x for the Dirac measure pertaining to the point $x \in X$.

If $(X, \mathcal{A}, \mu)$ is a measure space and $\mathcal{L} \subset \mathcal{A}$, then μ is called

(i) *$\mathcal{L}$-regular* if $\mu(A) = \sup\{\mu(L) : L \in \mathcal{L}, L \subset A\}$ for $A \in \mathcal{A}$;

(ii) *$\mathcal{L}$-τ-smooth* if $\inf_\alpha \mu(L_\alpha) = 0$ for every net (L_α) in $\mathcal{L}$ with $L_\alpha \downarrow \phi$.

Observe that μ is $\mathcal{L}$-τ-smooth iff $\inf\{\mu(F) : F \in \mathcal{F}\} = 0$ for every $\mathcal{L}$-filter $\mathcal{F}$ having empty intersection.

Now let X be a topological space. $C(X)$ $[C^b(X)]$ denotes the ring of all continuous [and bounded] real-valued functions on X. A *zero-set* in X is a set of the form $\{f = 0\}$ with $f \in C(X)$. We write $\mathfrak{Z}(X)$, $\mathcal{F}(X)$, $\mathcal{K}(X)$ for the family of all zero-, closed, compact sets in X, respectively.

$\mathcal{B}_0(X) := \sigma(\mathfrak{Z}(X)) = \sigma(C(X)) = \sigma(C^b(X))$ $[\mathcal{B}(X) := \sigma(\mathcal{F}(X))]$ denotes the *Baire* [*Borel*] σ-algebra in X.

A finite measure defined on $\mathcal{B}_0(X)$ is called a *Baire measure* on X. We write $\mathcal{M}_0(X)$ for the collection of all Baire measures on X. For $\mu \in \mathcal{M}_0(X)$ the set $\operatorname{supp}(\mu) := \bigcap\{Z \in \mathfrak{Z}(X) : \mu(Z) = \mu(X)\}$ is called the *support* of μ. Following the terminology of [12] we call $\mu \in \mathcal{M}_0(X)$

(i) *tight* if $\mu(X) = \sup\{\mu^*(K) : K \in \mathcal{K}(X)\}$;

(ii) τ-*smooth* if μ is $\mathfrak{Z}(X)$-τ-smooth.

Since every Baire measure is $\mathfrak{Z}(X)$-regular ([12], Part I, Theorem 18), $\mu \in \mathcal{M}_0(X)$ is tight iff $\mu(B) = \sup\{\mu^*(K) : K \in \mathcal{K}(X),\ K \subset B\}$ for all $B \in \mathcal{B}_0(X)$. Furthermore, for every $\mu \in \mathcal{M}_0(X)$ the implications

$$\mu \ \text{ tight } \quad \Longrightarrow \ \mu\ \tau\text{-smooth} \quad \Longrightarrow \ \mu^*(\operatorname{supp}(\mu)) = \mu(X)$$

are valid.

A completely regular space is always assumed to be Hausdorff.

2. On the relations between E-compact and $\mathcal{H}(E)$-complete spaces

In this section we consider a vector lattice E of real-valued functions defined on an arbitrary nonvoid set X. We assume that E satisfies the following three conditions:

(*C1*) For every $f \in E$ there is some $g \in E$ such that $\{f{\neq}0\} \subset \{g{\geq}1\}$.

(*C2*) $\min(1, f) \in E$ for every $f \in E$ (Stone's condition).

(*C3*) For every $x \in X$ there is some $f \in E$ with $f(x) \neq 0$.

Note that these conditions are fulfilled if E contains the constant functions.

Define $\mathcal{H}(E) := \{\{f \geq 1\} : f \in E\}$ and $\mathcal{G}(E) := \{\{f > 1\} : f \in E\}$. $\mathcal{H}(E)$ and $\mathcal{G}(E)$ are lattices of subsets of X satisfying $H \setminus G \in \mathcal{H}(E)$, $G \setminus H \in \mathcal{G}(E)$ for $G \in \mathcal{G}(E)$, $H \in \mathcal{H}(E)$ (see [2], 1.1). Furthermore, we write $\mathcal{M}(E)$ for the family of all measures μ on $\sigma(E)$ satisfying $E \subset \mathcal{L}_\infty(X, \sigma(E), \mu)$ and $\mu(H) < \infty$ for all $H \in \mathcal{H}(E)$.

A family $\mathcal{C}$ of subsets of X is said to be E-*stable* ([6]) if every function $f \in E$ is bounded on some $\mathcal{C}$-set.

We can now introduce the two ultrafilter properties announced in the title of this paper.

Definition. a) X is said to be E-*compact* if every E-stable $\mathcal{H}(E)$-ultrafilter has nonempty intersection.

b) X is said to be $\mathcal{H}(E)$-*complete* ([1]) if every $\mathcal{H}(E)$-ultrafilter with the cip has nonempty intersection.

The following fact is obvious.

2.1 Lemma. *If every $f \in E$ is bounded, then X is E-compact iff $\mathcal{H}(E)$ is a compact class.*

It is easy to see that every $\mathcal{H}(E)$-ultrafilter with the cip is E-stable. Thus E-compactness implies $\mathcal{H}(E)$-completeness. The following main result of this section states some measure-theoretic properties that are intermediate between E-compactness and $\mathcal{H}(E)$-completeness.

2.2 Theorem. *Consider the following five statements*

(S1) *X is E-compact.*

(S2) *For $\mu \in \mathcal{M}(E)$ and $H \in \mathcal{H}(E)$, the set $S(\mu, H) := \bigcap\{H \setminus G : G \in \mathcal{G}(E)$ with $\mu(G) = 0\}$ is $\tau(E)$-compact and $\mu^*(S(\mu, H)) = \mu(H)$.*

(S3) *For $\mu \in \mathcal{M}(E)$ and $H \in \mathcal{H}(E)$, we have $\mu(H) = \sup\{\mu^*(K) : H \supset K$ $\tau(E)$-compact$\}$.*

(S4) *Every $\mu \in \mathcal{M}(E)$ is $\mathcal{H}(E)$-τ-smooth.*

(S5) *X is $\mathcal{H}(E)$-complete.*

Then each of these statements implies its successor. If, in addition, the following condition (C4) *is satisfied, then all five statements are equivalent.*

(C4) *For every sequence $(H_n) \subset \mathcal{H}(E)$ with $H_n \downarrow \phi$ there is a function $h \in E$ such that $h \mid H_n \geq n$ for all $n \in \mathbb{N}$.*

Proof. (S1) $\Longrightarrow$ (S2) Let $\mu \in \mathcal{M}(E)$ and $H \in \mathcal{H}(E)$ be given.

(α) We first show the $\tau(E)$-compactness of the set $S(\mu, H)$. For this purpose let $\{F_i : i \in I\}$ be a family of $\tau(E)$-closed subsets of X such that $\bigcap\{S(\mu, H) \cap F_i : i \in I\} = \phi$. Assume that $\bigcap\{S(\mu, H) \cap F_i : i \in \tilde{I}\} \neq \phi$ for every finite subset $\tilde{I}$ of I. Then the collection $\mathfrak{N} := \{S \in \mathcal{H}(E) : S(\mu, H) \cap F_i \subset S$ for some $i \in I\}$ has the fip. Consequently there exists an $\mathcal{H}(E)$-ultrafilter $\mathfrak{M}$ with $\mathfrak{N} \subset \mathfrak{M}$.

Let $f \in E$ be given. Since $f \in \mathcal{L}_\infty(\mu)$, we have $\mu(\{|f| > r\}) = 0$ for some $r \in (0, \infty)$. Then $G := \{|f| > r\} \in \mathcal{G}(E)$ and $S(\mu, H) \subset H \setminus G$, hence $H \setminus G \in \mathfrak{N} \subset \mathfrak{M}$. As f is bounded on $H \setminus G$, we have shown that $\mathfrak{M}$ is E-stable.

For every $i \in I$, there exists by [2], 3.2, a subfamily E_i of E such that $S(\mu, H) \cap F_i = \bigcap\{\{f \geq 1\} : f \in E_i\}$. Then $\{f \geq 1\} \in \mathfrak{N}$ for all $f \in E_i$, $i \in I$. Consequently, $\phi = \bigcap\{S(\mu, H) \cap F_i : i \in I\} = \bigcap\{\{f \geq 1\} : f \in E_i, i \in I\}$ which implies $\bigcap \mathfrak{N} = \phi$ and, in particular, $\bigcap \mathfrak{M} = \phi$. Since $\mathfrak{M}$ is E-stable, we obtain a contradiction to (S1). Thus $\bigcap\{S(\mu, H) \cap F_i : i \in \tilde{I}\} = \phi$ must hold for some finite subset $\tilde{I}$ of I, and so $S(\mu, H)$ is $\tau(E)$-compact.

(β) To prove $\mu(H) = \mu^*(S(\mu, H))$ we can assume $\mu(H) > 0$. Then $\mathcal{D} := \{D \in \mathcal{H}(E) : \mu(D \cap H) = \mu(H)\}$ is an $\mathcal{H}(E)$-filter, and for $F := \bigcap \mathcal{D}$ we have $F \cap H \subset S(\mu, H)$. So $F \cap H$ is $\tau(E)$-compact, too. In addition,

$$\mu(H) = \mu^*(F \cap H). \tag{2.1}$$

Assume that $\mu^*(F \cap H) < \mu(H)$. By [2], 1.7 and 1.8, there exists an increasing sequence $(G_n) \subset \mathcal{G}(E)$ such that $F \cap H \subset \bigcup_{n=1}^{\infty} G_n$ and $\mu\left(\bigcup_{n=1}^{\infty} G_n\right) < \mu(H)$. The compactness of $F \cap H$ implies $F \cap H \subset G_{n_0}$ for some $n_0 \in \mathbb{N}$. Now for every $D \in \mathcal{D}$, we have $\mu(H) = \mu(D \cap H) = \mu((D \cap H) \setminus G_{n_0}) + \mu(D \cap H \cap G_{n_0})$, hence $\mu((D \cap H) \setminus G_{n_0}) = \mu(H) - \mu(D \cap H \cap G_{n_0}) \geq \mu(H) - \mu(G_{n_0}) > 0$. It follows that $\tilde{\mathfrak{N}} := \{(D \cap H) \setminus G_{n_0} : D \in \mathcal{D}\} \subset \mathcal{H}(E)$ has the fip. Thus there is an $\mathcal{H}(E)$-ultrafilter $\tilde{\mathfrak{M}} \supset \tilde{\mathfrak{N}}$.

For any $f \in E$, we have $\mu(\{|f| > r\}) = 0$ for some $r \in (0, \infty)$. Then $\tilde{D} := H \cap \{|f| \leq r\} \in \mathcal{D}$ and so $\tilde{S} := (\tilde{D} \cap H) \setminus G_{n_0} \in \tilde{\mathfrak{N}} \subset \tilde{\mathfrak{M}}$. As f is bounded on $\tilde{S}$, we have shown that $\tilde{\mathfrak{M}}$ is E-stable.

By (S1) we have $\bigcap \tilde{\mathfrak{M}} \neq \phi$. This implies $\phi \neq \bigcap \tilde{\mathfrak{N}} = \bigcap\{(D \cap H) \setminus G_{n_0} : D \in \mathcal{D}\} = (F \cap H) \setminus G_{n_0}$ which contradicts $F \cap H \subset G_{n_0}$. So (2.1) holds, and from the inclusion $F \cap H \subset S(\mu, H)$ we infer the equality $\mu(H) = \mu^*(S(\mu, H))$.

(S2) $\Longrightarrow$ (S3) We have $\mu(H) \geq \sup\{\mu^*(K) : K \subset H,\ K\ \tau(E)\text{-compact}\} \geq \mu^*(S(\mu, H)) = \mu(H)$ for $\mu \in \mathcal{M}(E)$ and $H \in \mathcal{H}(E)$.

(S3) $\Longrightarrow$ (S4) Let $\mu \in \mathcal{M}(E)$ and $\{H_\alpha : \alpha \in A\} \subset \mathcal{H}(E)$ with $H_\alpha \downarrow \phi$. Fix some $\tilde{\alpha} \in A$. For a given $\beta > 0$ choose a $\tau(E)$-compact set $K \subset H_{\tilde{\alpha}}$ such that $\mu(H_{\tilde{\alpha}}) < \mu^*(K) + \beta$ holds. Then there is an index $\hat{\alpha} \in A$ with $H_{\hat{\alpha}} \subset H_{\tilde{\alpha}}$ and $K \cap H_{\hat{\alpha}} = \phi$ which implies $\mu(H_{\hat{\alpha}}) \leq \mu_*(H_{\tilde{\alpha}} \setminus K) = \mu(H_{\tilde{\alpha}}) - \mu^*(K) < \beta$. Thus $\inf\{\mu(H_\alpha) : \alpha \in A\} = 0$.

(S4) $\Longrightarrow$ (S5) can be proved in the same way as the implication (1) $\Longrightarrow$ (3) of Theorem 2.1 in [3].

Now we assume that condition (C4) is satisfied.

(S5) $\Longrightarrow$ (S1) Let $\mathfrak{M}$ be an E-stable $\mathcal{H}(E)$-ultrafilter. To prove $\bigcap \mathfrak{M} \neq \phi$ it suffices to show that $\mathfrak{M}$ has the cip. For this purpose consider a decreasing sequence $(H_n) \subset \mathfrak{M}$ and suppose $H_n \downarrow \phi$. By (C4) there is a function $h \in E$ satisfying $h \mid H_n \geq n$ for every $n \in \mathbb{N}$. Since $\mathfrak{M}$ is E-stable we can find a set $U \in \mathfrak{M}$ and an index n_0 such that $h \mid U \leq n_0$. This leads to the contradiction $\phi = U \cap H_{n_0+1} \in \mathfrak{M}$.

2.3 Remarks. a) An analysis of the proof of 2.2 reveals that the properties (C1)–(C3) of E are only needed to prove the implication (S1) $\Longrightarrow$ (S2).

b) If $1 \in E$ (and hence $X \in \mathcal{H}(E)$) then the statement (S2) is equivalent to

(S2*) For every $\mu \in \mathcal{M}(E)$, the set $S(\mu) := \bigcap\{H \in \mathcal{H}(E) : \mu(H) = \mu(X)\}$ is $\tau(E)$-compact and $\mu^*(S(\mu)) = \mu(X)$.

c) In section 5 we will show that the condition (C4) is essential for the validity of the equivalence of the statements (S1)–(S5).

3. E-compactness for certain function rings E

In the sequel we consider a δ-lattice $\mathcal{L}$ of subsets of X such that $\phi, X \in \mathcal{L}$. We denote by $C(\mathcal{L}) := \{f \in \mathbb{R}^X : f^{-1}(F) \in \mathcal{L}$ for all closed subsets F of $\mathbb{R}\}$ the family of so-called $\mathcal{L}$-continuous functions. Furthermore let $C^b(\mathcal{L}) := \{f \in C(\mathcal{L}) : f \text{ bounded}\}$. $C(\mathcal{L})$ and $C^b(\mathcal{L})$ are vector lattices and algebras containing the constant functions ([4]). In addition, let $\mathcal{B}$ be an $\mathcal{L}$-bounding system, i.e. $\mathcal{B}$ is a nonvoid family of subsets of X satisfying the following two conditions:

(i) $\mathcal{B} \uparrow X$;

(ii) for every $B \in \mathcal{B}$ there exist $f \in C(\mathcal{L})$ and $\tilde{B} \in \mathcal{B}$ such that $f \mid B = 1$ and $\{f \neq 0\} \subset \tilde{B}$.

Define $C(\mathcal{L}, \mathcal{B}) := \{f \in C(\mathcal{L}) : \{f \neq 0\} \subset B$ for some $B \in \mathcal{B}\}$ and $C^b(\mathcal{L}, \mathcal{B}) := \{f \in C(\mathcal{L}, \mathcal{B}) : f \text{ bounded}\}$. $C(\mathcal{L}, \mathcal{B})$ and $C^b(\mathcal{L}, \mathcal{B})$ are also vector lattices and algebras.

Note that $X \in \mathcal{B}$ iff $1 \in C(\mathcal{L}, \mathcal{B})$ iff $C(\mathcal{L}, \mathcal{B}) = C(\mathcal{L})$.

Throughout this section E denotes a ring such that $C^b(\mathcal{L}, \mathcal{B}) \subset E \subset C(\mathcal{L}, \mathcal{B})$.

3.1 Proposition. *E is a vector lattice satisfying the conditions* (C1)–(C3).

Proof. Let $f \in E$ and $a \in \mathbb{R}$ be given. Choose $B \in \mathcal{B}$ and $g \in C(\mathcal{L}, \mathcal{B})$ such that $\{f \neq 0\} \subset B$ and $g \mid B = 1$. W.l.o.g. we can assume $0 \leq g \leq 1$. Then $g \in C^b(\mathcal{L}, \mathcal{B})$, hence $ag \in C^b(\mathcal{L}, \mathcal{B})$ and so $ag \in E$ which implies $af = agf \in E$. Now define $F := \{f \geq 1\}, G := \{f \leq -1\}$ and $h := \max(-g, \min(f, g))$. Then $h \in C^b(\mathcal{L}, \mathcal{B})$, $-1 \leq h \leq 1$, $h \mid F = 1$ and $h \mid G = -1$. Let $\ell := hf - |f|$.

We have $\ell(x) = 0$ for $x \in F \cup G$ and $|\ell(x)| \leq |h(x)| \cdot |f(x)| + |f(x)| \leq 2|f(x)| \leq 2$ for $x \notin F \cup G$. Thus $\ell \in C^b(\mathcal{L}, \mathcal{B}) \subset E$ and so $|f| = hf - \ell \in E$. This shows that E is a vector lattice. It is easy to see that E satisfies the conditions (C1)–(C3).

3.2 Lemma. *The topology $\tau(E)$ is independent of E, i.e. $\tau(E) = \tau(C^b(\mathcal{L}, \mathcal{B})) = \tau(C(\mathcal{L}, \mathcal{B}))$.*

Proof. The inclusions $\tau(C^b(\mathcal{L}, \mathcal{B})) \subset \tau(E) \subset \tau(C(\mathcal{L}, \mathcal{B}))$ are obvious. If $f \in C(\mathcal{L}, \mathcal{B})$ then $g := \min(2, f^+) \in C^b(\mathcal{L}, \mathcal{B})$ and $\{f>1\} =$

$\{g>1\} \in \tau(C^b(\mathcal{L},\mathcal{B}))$. Since the family $\mathcal{G}(C(\mathcal{L},\mathcal{B})) = \{\{f > 1\} : f \in C(\mathcal{L},\mathcal{B})\}$ is a basis for the topology $\tau(C(\mathcal{L},\mathcal{B}))$ (cf. [2], 3.1), we obtain $\tau(C(\mathcal{L},\mathcal{B})) = \tau(C^b(\mathcal{L},\mathcal{B}))$.

In the same way one can show that the set systems $\mathcal{G}(E)$, $\mathcal{H}(E)$ and $\sigma(E)$ do not depend on E.

3.3 Lemma. *For any $f \in E$ there exists $g \in E$ such that $0 \leq g \leq 1$ and $fg = f$.*

Proof. For a given $f \in E$ choose $B \in \mathcal{B}$ and $\hat{g} \in C(\mathcal{L},\mathcal{B})$ such that $\{f \neq 0\} \subset B$ and $\hat{g} \mid B = 1$. Then $g := \max(0, \min(1, \hat{g})) \in C^b(\mathcal{L},\mathcal{B}) \subset E$ and $fg = f$.

If $\mathcal{J}$ is a maximal ideal in E then it is a simple consequence of 3.3 that $E/\mathcal{J}$, the residue-class ring generated by $\mathcal{J}$, is even a field.

3.4 Lemma. *Consider a maximal ideal $\mathcal{J}$ in E and let $f^* \in E$ be such that $[f^*]$ is the unit element of the residue-class field $E/\mathcal{J}$.*

a) There exists a function $g \in E$ such that $0 \leq g \leq 1$ and $[f^*] = [g]$.

b) $[rf^*] \neq [0]$ for all $r \in \mathbb{R} \setminus \{0\}$ (and hence $[rf^*] \neq [sf^*]$ for $r \neq s$).

Proof. a) By 3.3, we can find $g \in E$ satisfying $0 \leq g \leq 1$ and $f^* = gf^*$. Then $[f^*] = [gf^*] = [g][f^*] = [g]$.

b) Assume that $[rf^*] = [0]$ for some $r \neq 0$. Then $[f^*] = [r^{-1}f^*][rf^*] = [r^{-1}f^*][0] = [0]$ which is impossible.

Definition. A maximal ideal $\mathcal{J}$ in E is called

(i) *real* if $E/\mathcal{J} = \{[rf^*] : r \in \mathbb{R}\}$;

(ii) *fixed* if $\mathcal{J} = \{f \in E : f(y) = 0\}$ for some $y \in X$.

We can now prove the main result of this section.

3.5 Theorem. *The following statements are equivalent:*

(S1) *X is E-compact.*

(S6) *Every real maximal ideal in E is fixed.*

(S7) *Every nonzero multiplicative linear functional on E is an evaluation*

Proof. (S1) $\Longrightarrow$ (S6) Let $\mathcal{J}$ be a real maximal ideal in E and let $f^* \in E$ be such that $[f^*]$ is the unit element of the field $E/\mathcal{J}$. Choose a function $g \in E$ such that $0 \leq g \leq 1$ and $\{f^* \neq 0\} \subset \{g = 1\}$.

Define $\mathfrak{Z}(\mathcal{J}) := \{\{f \leq 1\} \cap \{g = 1\} : f \in \mathcal{J}\}$. It is obvious that $\phi \neq \mathfrak{Z}(\mathcal{J}) \subset \mathcal{H}(E)$.

In addition, we have

$$\phi \notin \mathfrak{Z}(\mathcal{J}). \tag{3.1}$$

Assume that $\{f \leq 1\} \cap \{g = 1\} = \phi$ for some $f \in \mathcal{J}$. W.l.o.g. we can assume $f \geq 0$. Then $\{f^* \neq 0\} \subset \{g = 1\} \subset \{f > 1\}$. Define $h(x) := f(x)$

or $f(x)^{-1}$ according as $f(x) \leq 1$ or $f(x) \geq 1$. It is easy to see that $h \in C^b(\mathcal{L}, \mathcal{B})$ and so $h \in E$. Since $f^* = gf^* = fhf^*$ and $f \in \mathcal{J}$, we conclude $f^* \in \mathcal{J}$ and hence $[f^*] = [0]$ which is impossible.

Furthermore,

$$\mathfrak{Z}(\mathcal{J}) \text{ has the fip.} \tag{3.2}$$

Assuming the contrary, there exist functions $f_1, \ldots, f_n \in \mathcal{J}$ such that $\bigcap\{\{f_i \leq 1\} \cap \{g = 1\} : i = 1, \ldots, n\} = \phi$. Then $f := \sum_{i=1}^{n} f_i^2 \in \mathcal{J}$ and so $\phi = \{f \leq 1\} \cap \{g = 1\} \in \mathfrak{Z}(\mathcal{J})$ which contradicts (3.1).

It follows from (3.2) that $\mathfrak{Z}(\mathcal{J})$ is contained in some $\mathcal{H}(E)$-ultrafilter $\mathfrak{M}$. Next we show that

$$\mathfrak{M} \text{ is } E\text{-stable.} \tag{3.3}$$

For this purpose let $f \in E$ be given. As $\mathcal{J}$ is real, $[|f|] = [rf^*] = [rgf^*] = [rg]$ for some $r \in \mathbb{R}$. Then $h := |f| - rg \in \mathcal{J}$ and hence $Z := \{h \leq 1\} \cap \{g = 1\} \in \mathfrak{Z}(\mathcal{J}) \subset \mathfrak{M}$. Since f is bounded on Z, (3.3) follows.

According to (S1) we have $y \in \bigcap \mathfrak{M}$ for some $y \in X$. This implies $f(y) = 0$ for all $f \in \mathcal{J}$ and so $\mathcal{J} = \{f \in E : f(y) = 0\}$, i.e. $\mathcal{J}$ is fixed.

(S6) $\Longrightarrow$ (S7) Let $\Phi \neq 0$ be a multiplicative linear functional on E and define $\mathcal{J} := \{f \in E : \Phi(f) = 0\}$. It is standard to show that $\mathcal{J}$ is a real maximal ideal in E. According to (S6), we have $\mathcal{J} = \{f \in E : f(y) = 0\}$ for some $y \in X$ which implies $\Phi = I_y$.

(S7) $\Longrightarrow$ (S1) Let $\mathfrak{M}$ be an E-stable $\mathcal{H}(E)$-ultrafilter and define $\Phi(f) := \sup\{t \in [0, \infty) : \{f \geq t\} \in \mathfrak{M} \cup \{X\}\}$ for $f \in E_+$. Then $\Phi(f) \in [0, \infty)$ for all $f \in E_+$ and $\Phi(f) > 0$ for some $f \in E_+$.

It is not hard to verify that Φ is additive, positively homogeneous and multiplicative. Defining $\Phi(f) := \Phi(f^+) - \Phi(f^-)$ for all $f \in E$, we obtain a multiplicative linear functional $\Phi \neq 0$ on E. By (S7) we have $\Phi = I_y$ for some $y \in X$ which implies $y \in \bigcap \mathfrak{M}$.

We will close this section with two examples of rings E that are properly included between $C^b(\mathcal{L}, \mathcal{B})$ and $C(\mathcal{L}, \mathcal{B})$.

3.6 Examples. a) Let X be a topological space, $\mathcal{B} = \{X\}$ and $\mathcal{L} = \mathcal{F}(X)$. Then $C^b(\mathcal{L}, \mathcal{B}) = C^b(X)$ and $C(\mathcal{L}, \mathcal{B}) = C(X)$. In [5] there is given an example of a ring E strictly between $C^b(X)$ and $C(X)$.

b) For any measure space $(X, \mathcal{A}, \mu)$, $E := \mathcal{L}_\infty(X, \mathcal{A}, \mu)$ is a ring satisfying $C^b(\mathcal{A}) \subset E \subset C(\mathcal{A})$. In the special case $X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$ and $\mu =$ Lebesgue measure, E is strictly between $C^b(\mathcal{A})$ and $C(\mathcal{A})$.

4. The ring $C(\mathcal{L}, \mathcal{B})$

The following result is shown within the proof of Theorem 3.1. in [3].

4.1 Proposition. *$E := C(\mathcal{L}, \mathcal{B})$ satisfies the condition* (C4), *and $\mathcal{M}(E)$ consists exactly of those measures that integrate every function $f \in E$, i.e. for a measure μ on $\sigma(E)$ we have $\mu \in \mathcal{M}(E)$ iff $E \subset \mathcal{L}_1(\mu)$.*

The following characterization theorem is an immediate consequence of 2.2, 3.5 and 4.1.

4.2 Theorem. *For $E := C(\mathcal{L}, \mathcal{B})$ the statements* (S1)–(S7) *are equivalent.*

We now apply 4.2 to several special cases. First we consider a measurable space $(X, \mathcal{A})$. Then $C(\mathcal{A})$ is the ring of all $\mathcal{A}$-measurable real-valued functions on X. Since $\mathcal{H}(C(\mathcal{A})) = \mathcal{A}$, we infer from 4.2 and 4.1

4.3 Corollary. *For a measurable space $(X, \mathcal{A})$, the following statements are equivalent:*

(1) *X is $C(\mathcal{A})$-compact.*
(2) *For every measure μ on $\mathcal{A}$ with $C(\mathcal{A}) = \mathcal{L}_1(\mu)$, the set $S(\mu) := \bigcap\{A \in \mathcal{A} : \mu(A) = \mu(X)\}$ is $\tau(C(\mathcal{A}))$-compact and $\mu^*(S(\mu)) = \mu(X)$.*
(3) *For every measure μ on $\mathcal{A}$ with $C(\mathcal{A}) = \mathcal{L}_1(\mu)$, we have $\mu(A) = \sup\{\mu^*(K) : K \subset A,\ K\ \tau(C(\mathcal{A}))\text{-compact}\}$ for $A \in \mathcal{A}$.*
(4) *Every measure μ on $\mathcal{A}$ with $C(\mathcal{A}) = \mathcal{L}_1(\mu)$ is $\mathcal{A}$-τ-smooth.*
(5) *X is $\mathcal{A}$-complete.*
(6) *Every real maximal ideal in $C(\mathcal{A})$ is fixed.*
(7) *Every nonzero multiplicative linear functional on $C(\mathcal{A})$ is an evaluation.*

4.4 Remark. Using the fact that the σ-algebra $\mathcal{A}$ is a basis for the $\tau(C(\mathcal{A}))$-open sets, it is not hard to see that statement (2) of 4.3 is equivalent to

(2^*) Every measure μ on $\mathcal{A}$ with $C(\mathcal{A}) = \mathcal{L}_1(\mu)$ is a finite linear combination of Dirac measures.

In the remaining part of this section let X be a topological space. First we consider the ring $E = C(X)$ of all continuous real-valued functions on X. Then $E = C(\mathcal{F}(X)) = C(\mathfrak{Z}(X))$, $\mathcal{H}(E) = \mathfrak{Z}(X)$ and $\mathcal{M}(E) = \{\mu \in \mathcal{M}_0(X) : C(X) \subset \mathcal{L}_1(\mu)\}$. If X is completely regular, then $\tau(C(X))$ is the given topology on X and, by [1], Corollary 2.3, both $\mathfrak{Z}(X)$-completeness and $\mathcal{B}_0(X)$-completeness are properties equivalent to realcompactness. Thus we infer from 4.2, 4.3 and 4.4

4.5 Corollary. *For a completely regular space X the following statements are equivalent:*

(1) *X is $C(X)$-compact.*

(2) *For every* $\mu \in \mathcal{M}_0(X)$ *with* $C(X) \subset \mathcal{L}_1(\mu)$, $\mathrm{supp}(\mu) \in \mathcal{K}(X)$ *and* $\mu^*(\mathrm{supp}(\mu)) = \mu(X)$.
(3) *Every* $\mu \in \mathcal{M}_0(X)$ *with* $C(X) \subset \mathcal{L}_1(\mu)$ *is tight.*
(4) *Every* $\mu \in \mathcal{M}_0(X)$ *with* $C(X) \subset \mathcal{L}_1(\mu)$ *is* τ*-smooth.*
(5) X *is realcompact.*
(6) X *is* $C(\mathcal{B}_0(X))$*-compact.*
(7) *Every* $\mu \in \mathcal{M}_0(X)$ *with* $C(\mathcal{B}_0(X)) = \mathcal{L}_1(\mu)$ *is a finite linear combination of Dirac measures.*
(8) *Every real maximal ideal in* $C(\mathcal{B}_0(X))$ *is fixed.*
(9) *Every nonzero multiplicative linear functional on* $C(\mathcal{B}_0(X))$ *is an evaluation.*

Remark. In view of 4.5, Theorem 3.5 (for $E = C(X)$ and X completely regular) is a well-known characterization of realcompactness ([7]).

In accordance with [9], $\mathcal{B}(X)$-complete topological spaces X are called *Borel-complete*. By means of 4.3, these spaces can be characterized in the following way.

4.6 Corollary. *A topological space* X *is Borel-complete iff it is* $C(\mathcal{B}(X))$*-compact.*

Our final application of 4.3 is concerned with the algebra $\mathcal{E}(X)$ of clopen subsets of X. It is well-known (see [3], p. 176) that a zero-dimensional Hausdorff space X is $\mathbb{N}$-compact (i.e. homeomorphic to a closed subset of some power of $\mathbb{N}$) iff X is $\sigma(\mathcal{E}(X))$-complete. Thus we obtain from 4.3

4.7 Corollary. *For a zero-dimensional Hausdorff space* X, $\mathbb{N}$*-compactness and* $C(\sigma(\mathcal{E}(X)))$*-compactness are equivalent properties.*

5. The ring $C^b(\mathcal{L}, \mathcal{B})$

Here is the main result of this section.

5.1 Theorem. *For* $E := C^b(\mathcal{L}, \mathcal{B})$ *the following statements are equivalent:*
(S1) X *is* E*-compact.*
(S2) *For* $\mu \in \mathcal{M}(E)$ *and* $H \in \mathcal{H}(E)$*, the set* $S(\mu, H) := \bigcap\{H \setminus G : G \in \mathcal{G}(E)$ *with* $\mu(G) = 0\}$ *is* $\tau(E)$*-compact and* $\mu^*(S(\mu, H)) = \mu(H)$.
(S6*) *Every maximal ideal in* E *is fixed.*
(S7) *Every nonzero multiplicative linear functional on* E *is an evaluation.*
(S8) X *is* $\mathcal{H}(E)$*-complete and* $E = C(\mathcal{L}, \mathcal{B})$.
(S9) *Every set* $H \in \mathcal{H}(E)$ *is* $\tau(E)$*-compact.*

Proof. Since every maximal ideal in E is real ([8]), the equivalences (S1) $\Longleftrightarrow$ (S6*) $\Longleftrightarrow$ (S7) are an immediate consequence of 3.5.

(S1) $\Longrightarrow$ (S2) holds by 2.2.

(S2) $\Longrightarrow$ (S8) By 2.2, X is $\mathcal{H}(E)$-complete. To prove $E = C(\mathcal{L}, \mathcal{B})$ we assume the contrary. Then there exists a function $f \in C(\mathcal{L}, \mathcal{B}) \setminus E$. Choose $g \in E$ such that $\{f \neq 0\} \subset \{g \geq 1\}$ and put $H := \{g \geq 1\} \in \mathcal{H}(E)$. For every $n \in \mathbb{N}$, there is a point $x_n \in X$ such that $|f(x_n)| > n$. Then $\mu := \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n} \in \mathcal{M}(E)$ and $\{x_1, x_2, \dots\} \subset S(\mu, H)$. As $S(\mu, H)$ is $\tau(E)$-compact, the sequence (x_n) has a cluster point $\hat{x} \in S(\mu, H)$. Then $\hat{U} := \{x \in X : |f(x) - f(\hat{x})| < 1\}$ is a neighborhood of $\hat{x}$ (cf. 3.2). Thus there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \in \hat{U}$ for all $k \in \mathbb{N}$. This implies $|f(\hat{x})| \geq |f(x_{n_k})| - |f(x_{n_k}) - f(\hat{x})| > |f(x_{n_k})| - 1 > n_k - 1$ for all $k \in \mathbb{N}$ which, however, is impossible since f is real-valued. This contradiction proves the claim $E = C(\mathcal{L}, \mathcal{B})$.

(S8) $\Longrightarrow$ (S1) follows from 4.2.

(S1) $\Longrightarrow$ (S9) By 2.1, $\mathcal{H}(E)$ is a compact class. Therefore the proof of the $\tau(E)$-compactness of an $\mathcal{H}(E)$-set H is similar to that of (S1) $\Longrightarrow$ (S2), part (α).

(S9) $\Longrightarrow$ (S8) Let $f \in C(\mathcal{L}, \mathcal{B})$ be given. Choose an $g \in E$ such that $\{f \neq 0\} \subset \{g \geq 1\}$. Since $\{g \geq 1\}$ is $\tau(E)$-compact and f is $\tau(E)$-continuous (cf. 3.2), f is bounded. Hence $E = C(\mathcal{L}, \mathcal{B})$. Furthermore, it follows from (S9) that every $\mathcal{H}(E)$-ultrafilter has nonvoid intersection. In particular, X is $\mathcal{H}(E)$-complete.

Next we will give a topological application of 5.1. For this purpose we consider a completely regular space X together with the ring $E = C^b(X)$. Then $E = C^b(\mathcal{F}(X)) = C^b(\mathfrak{Z}(X))$, $\mathcal{H}(E) = \mathfrak{Z}(X)$, $\mathcal{M}(E) = \mathcal{M}_0(X)$, and $\tau(E)$ equals the given topology on X. In this case the statements (S5), (S4) and (S3) mean realcompactness, measure-compactness and strong measure-compactness, respectively (cf. [11]). Since there exist both realcompact spaces X that are not measure-compact and measure-compact spaces X that are not strongly measure-compact ([10, 11]), we see that the implications (S5) $\Longrightarrow$ (S4) $\Longrightarrow$ (S3) are not true, in general. On the other hand, the subsequent Theorem 5.2 states that (for $E = C^b(X)$ and X completely regular) each of the statements (S1) and (S2) is equivalent to compactness. Since a Polish space is strongly measure-compact, we thus infer from 5.2 that for every non-compact Polish space X (and $E = C^b(X)$) the implication (S3) $\Longrightarrow$ (S2) ($\Longleftrightarrow$ (S2*)) is false.

5.2 Theorem. *For a completely regular space X, the following statements are equivalent:*

(1) *X is $C^b(X)$-compact.*

(2) *For every $\mu \in \mathcal{M}_0(X)$, we have $\operatorname{supp}(\mu) \in \mathcal{K}(X)$ and $\mu^*(\operatorname{supp}(\mu)) = \mu(X)$.*

(2') *For every $\mu \in \mathcal{M}_0(X)$, there is some $K \in \mathcal{K}(X)$ with $\mu^*(K) = \mu(X)$.*

(2") *For every $\mu \in \mathcal{M}_0(X) \setminus \{0\}$, we have $\operatorname{supp}(\mu) \in \mathcal{K}(X) \setminus \{\phi\}$.*

(3) *Every maximal ideal in $C^b(X)$ is fixed.*

(4) *Every nonzero multiplicative linear functional on $C^b(X)$ is an evaluation.*

(5) *X is realcompact and pseudocompact.*

(6) *X is compact.*

Proof. Since $\mathfrak{Z}(X)$-completeness is the same as realcompactness, the equivalence of the statements (i), $i = 1, \ldots, 6$, is a direct consequence of 5.1 (with $\mathcal{L} \in \{\mathcal{F}(X), \mathfrak{Z}(X)\}, \mathcal{B} = \{X\}$) and 2.3b).

(2) $\Longrightarrow$ (2') is trivial.

(2') $\Longrightarrow$ (2") Let $\mu \in \mathcal{M}_0(X) \setminus \{0\}$. By assumption, there is a set $K \in \mathcal{K}(X)$ with $\mu^*(K) = \mu(X)$. So μ is tight and hence $\operatorname{supp}(\mu) \neq \phi$. On the other hand, we have $\operatorname{supp}(\mu) \subset K$ which implies $\operatorname{supp}(\mu) \in \mathcal{K}(X)$.

(2") $\Longrightarrow$ (2) By [10], 2.1, X is measure-compact. Thus, for every $\mu \in \mathcal{M}_0(X)$, we have $\mu^*(\operatorname{supp}(\mu)) = \mu(X)$ and, of course, $\operatorname{supp}(\mu) \in \mathcal{K}(X)$.

Remark. Whereas the statements (3), (4) and (5) of 5.2 are well-known characterizations of compactness (see [7]), the equivalence of the statements (2), (2'), (2") and (6) seems to be a new result in topological measure theory.

References

[1] W. Adamski, Complete spaces and zero-one measures, *Manuscripta Math.* **18** (1976), 343–352.

[2] W. Adamski, An abstract approach to weak topologies in spaces of measures, *Bull. Soc. Math. Grèce (N.S.)* **18** (1977), 28–68.

[3] W. Adamski, On measures integrating all functions of a given vector lattice, *Monatshefte Math.* **103** (1987), 169–176.

[4] G. Bachman and A. Sultan, Regular lattice measures: mappings and spaces, *Pacific Journal Math.* **67** (1976), 291–321.

[5] R. M. Brooks, A ring of analytic functions, *Studia Math.* **24** (1964), 191–210.

[6] H. L. Byun and S. Watson, Prime and maximal ideals in subrings of $C(X)$, *Topology and its Appl.* **40** (1991), 45–62.

[7] L. Gillman and M. Jerison, Rings of continuous functions, *Van Nostrand, Princeton, N.J.*, 1960.

[8] G. G. Gould, A Stone–Čech–Alexandroff–type compactification and its application to measure theory, *Proc. London Math. Soc. (3)* **14** (1964), 221–244.

[9] A. W. Hager, G. D. Reynolds and M. D. Rice, Borel-complete topological spaces, *Fund. Math.* **75** (1972), 135–143.

[10] W. Moran, The additivity of measures on completely regular spaces, *Journal London Math. Soc.* **43** (1968), 633–639.

[11] W. Moran, Measures and mappings on topological spaces, *Proc. London Math. Soc. (3)* **19** (1969), 493–508.

[12] V. S. Varadarajan, Measures on topological spaces, *Amer. Math. Soc. Transl. (2)* **48** (1965), 161–228.

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