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Products of three factorials

By ANDREJ DUJELLA (Zagreb), FILIP NAJMAN (Zagreb), N. SARADHA (Mumbai) and TARLOK N. SHOREY (Mumbai)

Abstract. We study a question of Erdős and Graham on products of three factorials being a square.

1. Introduction

For any integer $n > 1$ denote by $P(n)$ and $Q(n)$ the greatest prime factor and the square free part of n and put $P(1) = Q(1) = 1$. For any prime p, let $\mathrm{ord}_p(n)$ denote the largest exponent of p dividing n. For any set A of positive integers let

$$m(A) = \prod_{a \in A} a!$$

and $M(A)$ denote the largest integer in A. In 1975, Erdős and Selfridge [7] proved a remarkable result that a product of two or more consecutive positive integers is never a square or a higher power. Continuing this line of investigation, Erdős and Graham [9] asked if a product of two or more disjoint blocks of consecutive integers can be a square or a higher power. It was noted by Ulas [19] that if all the blocks are of length exactly 4 and if the number of such blocks is large enough, then the product takes on square values infinitely often. In subsequent papers Bauer and Bennett [1] and Bennett and Luijk [2] gave further possibilities when such a product is a square infinitely often. Skałba [17] obtained bounds for the number of solutions and the smallest solution. Luca and

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Walsh [13] studied a special case of this problem where products of two blocks of length 4 led to quartic Diophantine equations.

In this paper, we consider a product of factorials being a square i.e., we search for positive rational integral solutions to the Diophantine equation

$$\prod_{k=1}^{t} a_k! = y^2 \tag{1}$$

with $a_1 \geq a_2 \geq \cdots \geq a_t > 1$ and $y > 1$. By canceling out an even number of equal factorials on the left hand side, we may assume that

$$a_1 > a_2 > \cdots > a_t > 1.$$

In [8], the above equation was studied when a_1 is given and t is minimal such that (1) has a solution. It was shown in [8] that the set of possible values of the left hand side of (1) is sparse. Hence it may very well be that (1) has no or very few solutions. Define $F_0 = \emptyset$ and for $k \geq 1$, let

$$F_k = \{n : \text{there exists some set } A \text{ with } |A| \leq k, \\ M(A) = n \text{ and } m(A) \text{ is a square}\}$$

and

$$D_k = F_k \setminus F_{k-1}.$$

If the equation has a solution $a_1, \ldots, a_t$ then $a_1 \in F_t$. Thus in order to study the equation (1) Erdős and Graham investigated the sets F_k and D_k.

1.1. Properties of the sets F_k and D_k.

(i) No prime belongs to any F_k.

(ii) $D_1 = F_1 = \{1\}$ since $n!$ for $n \geq 2$ is never a square.

(iii) Let $A = \{a^2, a^2 - 1\}$ with $a > 1$. Then $t = 2$, $a_1 = a^2$, $a_2 = a_1 - 1$ and $m(A)$ is a square. Hence $a^2 \in F_2$. By [7], it follows that $D_2 = \{a^2, a > 1\}$.

(iv) Let $a > 1, b > 1$ and $A = \{a^2Q(b!), a^2Q(b!) - 1, b\}$. Then $t = 3$, $a_1 = a^2Q(b!)$, $a_2 = a_1 - 1$, $a_3 = b$ and $m(A)$ is a square. Hence $a^2Q(b!) \in D_3$.

(v) Let x, y be solutions of the Pell's equation

$$ux^2 - vy^2 = 1$$

where $uv = Q(a!)$ for some $a > 1$ and $\gcd(u, v) = 1$. Let $A = \{ux^2, vy^2 - 1, a\}$. Then $t = 3$, $a_1 = ux^2$, $a_2 = a_1 - 2$, $a_3 = a$ and $m(A)$ is a square. Hence $ux^2 \in D_3$ when u is not a square. This example is slightly different from the one given in [7] where uv is taken to be equal to $a!$.

Erdős and Graham predict that there are perhaps only finitely many elements of D_3 apart from the two classes of integers given in (iv) and (v). This problem is still *open*. On the other hand, they showed that D_3 is sparse by proving

$$D_3(X) = o(X)$$

where $D_3(X)$ denotes the number of elements of D_3 which do not exceed X. The above result was improved by Luca, Saradha and Shorey [12] to

$$D_3(X) = O\left(\frac{X}{\exp(c_0(\log X)^{1/4}(\log\log X)^{1/2})}\right)$$

for some constant $c_0 > 0$. Their work depends on an application of Jutila's result on exponential sums and known estimates for the Dickman's function which counts integers with small prime factors.

1.2. About D_3. In the two classes of examples in (iv) and (v) we have $a_2 \geq a_1 - 2$. There are examples where $a_2 = a_1 - 3$. For instance,

$$10!7!6!;\ 50!47!3!;\ 50!47!4! \tag{2}$$

are all squares. Erdős and Graham asked if there are other examples, i.e., does the equation

$$a_1!(a_1-3)!a_3! = y^2 \tag{3}$$

have any other solution apart from the three examples given above? We prove

Theorem 1.1. *Let $a_3 \leq 100$. Then all the solutions of* (3) *are given by*

$$(a_1, a_3) \in \{(10,6),(50,3),(50,4),(324,26),(352,13),(442,18),(2738,26)\}.$$

Note that (3) can be re-written as

$$a_1(a_1-1)(a_1-2)a_3! = y^2. \tag{4}$$

The left hand side of (4) is a product of two non-overlapping blocks, one of length 3 and the length of the other is not fixed. These cases are not covered in [1] and [2]. For $x \in \mathbb{R}$, by $\lceil x \rceil$ we denote the ceiling function of x, i.e. the smallest integer not less than x. By $\pi(x)$ we denote the number of prime numbers not greater than x. We show

Theorem 1.2. *Let $h = \pi(a_3) - \pi(a_3/2)$. Then*

(a) $a_1 > (a_3/2)^{\lceil h/3 \rceil}$.

(b) *No term $N \in \{a_1, a_1-1, a_1-2\}$ is of the form αN_1 where $N_1 = P_1^{\theta_1} \dots P_s^{\theta_s} > 1$ with P_i's primes, θ_i odd, $\Omega(N_1) \leq \lceil h/3 \rceil$ and $\alpha \leq \frac{a_3^{\lceil h/3 \rceil - \Omega(N_1)}}{2^{\lceil h/3 \rceil}} - \frac{2}{a_3^{\Omega(N_1)}}$, where $\Omega(N_1)$ denotes the number of prime factors of N_1 counted with multiplicity.*

(c) *No term N is of the form αN_1 where $N_1 = P_1, P_1^3, P_1P_2$, $P_1P_2P_3$ and $\alpha \leq 64393, 6, 637, 6$, respectively, except when*

$$(a_1, a_3) \in \{(10, 6), (50, 3), (50, 4), (324, 26), (352, 13), (442, 18), (2738, 26)\}.$$

2. Proof of Theorem 1.1

Equation (3) can be re-written as

$$a_1(a_1 - 1)(a_1 - 2) = Q(a_3!)Y^2. \tag{5}$$

Putting $a_1 - 1 = X'$ and multiplying by $(Q(a_3!))^3$ the equation is transformed to the elliptic equation

$$X(X^2 - b^2) = Y^2 \tag{6}$$

where $X = Q(a_3!)X', Y = (Q(a_3!))^2Y'$ and $b = Q(a_3!)$. Actually, (6) is the so called congruent number elliptic curve. For $a_3 \leq 53$, $a_3 \notin \{32, 38, 40, 47, 51\}$, we were able to find all integral points (X, Y) on the elliptic curve (6) by using the function `IntegralPoints` in Magma [3]. Integral points on elliptic curves are computed by first computing the generators of the Mordell–Weil group, and after that by using linear forms in elliptic logarithms to bound the sizes of the coordinates of integral points [10], [18].

However, we are interested only in those integer points for which X is divisible by b. The integer points satisfying this additional condition give exactly the solutions given in Theorem 1.1.

For the remaining values of a_3 we were not able to find all integral points on (6), mainly because we were unable to find the generators of the Mordell–Weil group of the curve (6). Therefore, in these cases we take a different approach by transforming our problem into a family of systems of Pellian equations. Since, $\gcd(a_1, a_1 - 1) = \gcd(a_1 - 1, a_2 - 1) = 1$ and $\gcd(a_1, a_1 - 2) \in \{1, 2\}$, from (5) we get:

$$a_1 = b_1x^2, \quad a_1 - 1 = b_2y^2, \quad a_1 - 2 = b_3z^2,$$

or

$$a_1 = 2b_1x^2, \quad a_1 - 1 = b_2y^2, \quad a_1 - 2 = 2b_3z^2,$$

where $b_1b_2b_3 = b$. These relations induce the system of Pellian equations

$$b_1x^2 - b_2y^2 = 1, \quad b_1x^2 - b_3z^2 = 2,$$

and

$$2b_1x^2 - b_2y^2 = 1, \quad b_1x^2 - b_3z^2 = 1,$$

respectively.

The number of such systems can be very large ($3^{\omega(b)}$, where $\omega(b)$ denotes the number of prime factors of b). However, we expect that most of them have no solutions. We will follow an approach from [5], [6]. So we start by eliminating those systems which are not locally solvable. Indeed, almost all system can be shown to be non-solvable by considering solvability modulo 8 and modulo p for primes p dividing b (the solvability conditions can be written in terms of Legendre symbols, e.g. for the first system we have $\left(\frac{-b_2}{p_1}\right) = 1$ and $\left(\frac{-2b_3}{p_1}\right) = 1$ for all primes p_1 dividing b_1, and similar conditions for primes dividing b_2 and b_3). For each value of a_3 in the considered range, there remains only a few locally solvable systems. Note that we always have the system

$$2x^2 - y^2 = 1, \quad x^2 - b_3z^2 = 1, \quad y^2 - 2b_3z^2 = 1, \tag{7}$$

in which each particular equation is certainly solvable in positive integers. However, from the system (7) we get $y^2 + 1 = 2x^2$, $y^2 - 1 = 2b_3z^2$ and hence

$$y^4 - b_3(2xz)^2 = 1. \tag{8}$$

Now we can apply a result due to Cohn [4] which says that the only possible solutions of the equation $X^4 - DY^2 = 1$ in positive integers are given by $X^2 = u_0$ and $X^2 = 2u_0^2 - 1$, where $u_0 + v_0\sqrt{D}$ is the fundamental solution of Pell's equation $u^2 - Dv^2 = 1$. For $a_3 \leq 88$, we were able to compute the fundamental solution by using the function `quadunit` in Pari [15]. For $89 \leq a_3 \leq 100$, the fundamental units become too large to be computed by standard methods, as they grow exponentially in the size of the discriminant of the quadratic field. Because of this, we used compact representations of fundamental units and the algorithm for modular arithmetic on such representations described in [14].

We find that for $a_3 \leq 100$ the equation (8) has a solution in positive integers only for $a_3 = 3, 4, 18$, and these cases were already handled by the elliptic curve approach.

The remaining Pellian equations, not eliminated by the previously explained methods, have the shape $\alpha x^2 - \beta y^2 = 1$ or 2, where $\alpha > 1$, $\beta > 0$. The criteria for solvability of such equations were given by Grelak and Grytczuk [11] in terms of the fundamental solution (u_0, v_0) of the equation $u^2 - \alpha\beta v^2 = 1$. For the equation $\alpha x^2 - \beta y^2 = 1$, the criterion is that $2\alpha|(u_0 + 1)$ and $2\beta|(u_0 - 1)$, while for $\alpha x^2 - \beta y^2 = 2$, $\alpha|(u_0 + 1)$ and $\beta|(u_0 - 1)$.

After an application of these criteria, only three systems remain unsolved. We list them explicitly. Each of them can be solved by using some divisibility properties of the corresponding fundamental units.

For $a_3 = 32$, we have to consider the system

$$x^2 - 1964315y^2 = 1, \quad x^2 - 34z^2 = 2.$$

The second equation clearly implies that $3|x$. Since the fundamental solution (x_0, y_0) of the first equation has the property $3|y_0$, we conclude that also $3|y$. But it is clear that there is no solution of the first equation such that both x and y are divisible by 3.

For $a_3 = 54$, the remaining system is

$$63017x^2 - y^2 = 1, \quad 63017x^2 - 39407479z^2 = 2.$$

From the fundamental solution of the first equation we get that $37|x$. But this contradicts the second equation since also $37|39407479$.

For $a_3 = 84$, the remaining system is

$$163373405489x^2 - 17755y^2 = 1, \quad 17755y^2 - 16077666z^2 = 1.$$

Now we get contradiction since $7|163373405489$, while from the fundamental solution of the second equation it follows that also $7|y$. □

3. Proof of Theorem 1.2

We make the following observations:

Let $S = \{a_1, a_1 - 1, a_1 - 2\}$.

1) None of the elements of S is a prime since $a_1 - 3 = a_2 > a_3$.
2) The gcd of any two of the elements of S is either 1 or 2.
3) $a_3 \geq 101$ by Theorem 1.1. Then $k \geq 11$. This can be checked for $a_3 \geq 250$ by estimates for $\pi(X)$ (see [16]) and for $101 \leq a_3 < 250$ by the exact values of $\pi(X)$.

Proof of (a).

All the primes dividing $a_3!$ to an odd power must divide the terms in S. This is true in particular for the primes in the range $(a_3/2, a_3]$. Since there are k primes in $(a_3/2, a_3]$, there is a term $N \in S$ which is divisible by at least $\lceil h/3 \rceil$ primes. Thus

$$a_1 \geq N > (a_3/2)^{\lceil h/3 \rceil}. \tag{9}$$

Proof of (b).

Suppose a term $N \in S$ satisfies the given conditions of (b). By (4), each of the primes occurring in N is $\leq a_3$, so we get

$$a_1 \leq N + 2 \leq (a_3/2)^{\lceil h/3 \rceil}.$$

This is a contradiction to (9), proving (b).

Proof of (c).

By Theorem 1.1 we may assume that $a_3 \geq 101$. Then $h \geq 11$. It follows by (9) that $a_1 > a_3^4/16$. By (b), there is no term $N \in S$ of the form αN_1 with $N_1 \in S_1 = \{P_1, P_1^3, P_1P_2, P_1P_2P_3\}$ and $\alpha \leq \frac{a_3^{4-\Omega(N_1)}}{16} - \frac{2}{a_3^{\Omega(N_1)}}$. Since $a_3 \geq 101$, it follows that there is no term $N = \alpha N_1$ with $N_1 \in S_1$ and $\alpha \leq \frac{101^{4-\Omega(N_1)}}{16} - \frac{2}{101^{\Omega(N_1)}}$. Thus we get $\alpha \leq 64393, 6, 637, 6$ according as $N_1 = P_1, P_1^3, P_1P_2, P_1P_2P_3$, respectively. This proves (c). □

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ANDREJ DUJELLA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ZAGREB
BIJENIČKA CESTA 30
10000 ZAGREB
CROATIA
E-mail: `duje@math.hr`

FILIP NAJMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ZAGREB
BIJENIČKA CESTA 30
10000 ZAGREB
CROATIA
E-mail: `fnajman@math.hr`

N. SARADHA
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
MUMBAI - 400 005
INDIA
E-mail: `saradha@math.tifr.res.in`

TARLOK N. SHOREY
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY
POWAI, MUMBAI - 400 076
INDIA
E-mail: `shorey@math.iitb.ac.in`

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