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Left invariant Randers metrics on the 3-dimensional Heisenberg group

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Dedicated to Professor Lajos Tamássy on the occassion of his ninetieth birthday

Abstract. In the present paper we give a complete description of the Chern–Rund connection defined by a left invariant Randers metric on the 3 dimensional Heisenberg group.

1. Introduction

Randers metric is a Finsler metric which is defined as the sum of a Riemannian metric and a 1-form. It is an object that shows strong non-Riemannian characters. The history of Randers metric goes back to G. Randers' research on general relativity [16]. Since then it has been widely applied in many areas, including electron optics and biology. (A more detailed account can be found in [1].) Randers metric can be naturally deduced as the solution of the famous Zermelo navigation problem [3].

In Chapter 11 of [2] the authors give six reasons to study Randers metric. Number 5 is that Randers metrics are computable and this may lead to a better understanding of Finsler metrics. Our strategy in this paper was the same, we specialized our original problem of left invariant Finsler metrics on two-step nilpotent groups (presented in [17]) to Randers metric on Heisenberg group. The

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straight motivation of the original study was P. Eberlein's comprehensive work [7] for the Riemannian case.

In the previous paper [17] we computed some geometric quantities such as curvature and flag curvature for a general left invariant Finsler metric on a two-step nilpotent group. In the first step we gave an explicit formula for the Chern–Rund connection. That paper had limitations, the reference vector for the Chern–Rund connection was chosen from the center of the respective Lie algebra. In the present paper we give a complete description of the Chern–Rund connection defined by a left invariant Randers metric on the 3-dimensional Heisenberg group. The Randers perturbation vector lies in the center of the Lie algebra in this paper.

2. Conventions

2.1. Finsler metrics and the Chern–Rund connection. Through this paper we use [2] as a basic reference for foundations of Finsler geometry. We consider metric structures on a differentiable manifold N and 'differentiable' means C^∞-differentiable. The module of tangent vector fields over N is denoted by $\mathfrak{X}(N)$.

Definition 2.1. A *Finsler manifold* (N, F) is a differentiable manifold N equipped with a Finsler metric F. A *Finsler metric* on N is a continuous map, $F\colon TN \to \mathbf{R}$ differentiable outside the zero section and satisfying three conditions:

(1) F is positively homogeneous,

(2) if $F(X) = 0$ then $X = 0$,

(3) F is strong convex.

In the sequel we fix a nowhere vanishing vector field $W \in \mathfrak{X}(N)$, the so called reference vector field. Generally such a vector field does not exist globally and we arrange that all objects live on an open subset $\mathcal{U} \subset N$, where the reference vector field exists.

Definition 2.2. The *osculating Riemann metric* $\langle\,,\rangle_W$ is determined by the Finslerian fundamental function F and by the reference vector field $W \in \mathfrak{X}(N)$ in the following way:

$$\langle X_p, Y_p\rangle_W = \frac{1}{2}\left.\frac{\partial^2 F^2(W_p + sX_p + tY_p)}{\partial s \partial t}\right|_{s,t=0}, \quad p \in N,\ X, Y \in \mathfrak{X}(N). \tag{1}$$

Definition 2.3. For $X, Y, Z \in \mathfrak{X}(N)$,

$$\mathcal{C}_W(X_p, Y_p, Z_p) = \frac{1}{4}\left.\frac{\partial^3}{\partial r \partial s \partial t}\right|_{r,s,t=0} F^2\left(W_p + rX_p + sY_p + tZ_p\right)$$

is the *(osculating) Cartan tensor*. Its $(1,2)$-type version is defined by

$$\mathcal{C}^2_W\colon \mathfrak{X}(N)\times\mathfrak{X}(N)\to\mathfrak{X}(N),\quad \left\langle \mathcal{C}^2_W(X,Y),Z\right\rangle_W=\mathcal{C}_W(X,Y,Z).$$

For the Cartan tensor we have

$$\mathcal{C}_W(W,X,Y)=\mathcal{C}_W(X,W,Y)=\mathcal{C}_W(X,Y,W)=0. \tag{2}$$

Theorem 2.4 ([15])**.** *The Chern–Rund connection* $\nabla^W\colon \mathfrak{X}(N)\times\mathfrak{X}(N)\to\mathfrak{X}(N)$ *w.r.t. the reference vector field* W *satisfies*

$$\begin{aligned}2\left\langle \nabla^W_X Y,Z\right\rangle_W &= X\left\langle Y,Z\right\rangle_W+Y\left\langle Z,X\right\rangle_W-Z\left\langle X,Y\right\rangle_W+\left\langle [X,Y],Z\right\rangle_W\\ &\quad-\left\langle [Y,Z],X\right\rangle_W+\left\langle [Z,X],Y\right\rangle_W-2\mathcal{C}_W(\nabla^W_X W,Y,Z)\\ &\quad-2\mathcal{C}_W(\nabla^W_Y W,Z,X)+2\mathcal{C}_W(\nabla^W_Z W,X,Y).\end{aligned} \tag{3}$$

The Chern–Rund connection is torsion-free, that is,

$$\nabla^W_X Y-\nabla^W_Y X-[X,Y]=0, \tag{4}$$

and almost metric, that is,

$$X\left\langle Y,Z\right\rangle_W=\left\langle \nabla^W_X Y,Z\right\rangle_W+\left\langle Y,\nabla^W_X Z\right\rangle_W+2\mathcal{C}_W(\nabla^W_X W,Y,Z).$$

In order to get all the local components of the Chern–Rund connection w.r.t. a local base, it is sufficient to show that it can be eliminated from the right hand side of (3). We can do it with the following simple algorithm.

Algorithm 2.5 ('Local strategy')**.** *Let* (E_i) *be an orthonormal base w.r.t.* $\langle\,,\rangle_W$.

1. *Choose* $X,Y\in\{W,E_i\}$ *such that all the terms in the right hand side of* (3) *are explicitly known while computing for* $\left\langle \nabla^W_X Y,E_i\right\rangle_W$.
2. *Set*
$$\nabla^W_X Y=\sum_i\left\langle \nabla^W_X Y,E_i\right\rangle_W E_i.$$
3. *Repeat the previous steps until all the local components of the Chern–Rund connection are known.*

We give further details. For the first six terms of the right hand side of (3) we use the abbreviation $\mathcal{A}_W(X,Y,Z)$. In these terms the Chern–Rund connection does not occur.

1a. Considering (2), equation (3) implies that

$$2\left\langle \nabla_W^W W, E_i \right\rangle_W = \mathcal{A}_W(W, W, E_i),$$

i.e. $\nabla_W^W W$ is explicitly known:

$$2\nabla_W^W W = \sum_i \mathcal{A}_W(W, W, E_i)E_i.$$

1b. Let $S \in \{E_i\}$. From equation (3) we have

$$2\left\langle \nabla_S^W W, E_i \right\rangle_W = \mathcal{A}_W(S, W, E_i) - 2\mathcal{C}_W(\nabla_W^W W, E_i, S). \tag{5}$$

Here $\nabla_W^W W$ is known from the previous step, and we get $\nabla_S^W W$.

1c. Let $S, T \in \{E_i\}$.

$$\begin{aligned} 2\left\langle \nabla_S^W T, E_i \right\rangle_W &= \mathcal{A}_W(S, T, E_i) - 2\mathcal{C}_W(\nabla_S^W W, T, E_i) \\ &\quad - 2\mathcal{C}_W(\nabla_T^W W, E_i, S) + 2\mathcal{C}_W(\nabla_{E_i}^W W, S, T). \end{aligned} \tag{6}$$

Here all the terms in the right hand side are known from 1b.

2.2. Left invariant Randers metrics on 3-dimensional Heisenberg group.

Definition 2.6. Let $\mathcal{Z} = \operatorname{span} Z$ be a 1-dimensional vector space spanned by the element Z. Let (X, Y) be any basis of $\mathbf{R}^2$. Define $[X, Y] = -[Y, X] = Z$ with all other brackets zero. The Lie algebra $\mathcal{N} = \mathcal{Z} \oplus \mathbf{R}^2$ is the 3-dimensional *Heisenberg algebra*. Moreover, let $\langle\, ,\rangle$ denote the positive definite inner product on $\mathcal{N}$ for which (X, Y, Z) is an orthonormal base. Thus $\operatorname{span}(X, Y)$ is the orthogonal complement of $\mathcal{Z}$ for which we use the notation $\mathcal{Z}^\perp$.

Let $\{N, \langle\, ,\rangle\}$ denote the three-dimensional Heisenberg group, i.e. N is a simply connected 2-step nilpotent group with Lie algebra $\mathcal{N}$ and $\langle\, ,\rangle$ is the left invariant Riemannian metric induced by left translations from the original metric given on $\mathcal{N}$. In this paper we shall regard the elements of $\mathcal{N}$ as left invariant vector fields on N determined by their values at the identity of N. We remark that the first three terms of the right hand side of (3) vanish for left invariant vector fields.

Left invariant Cartan tensor and Chern–Rund connection can be derived from a left invariant Finsler metric. To be more precise let $W \in \mathcal{N}$ and we may regard ∇^W as a bilinear mapping from $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$. Similarly the trilinear function $\mathcal{C}_W$ lives on $\mathcal{N}$, too:

$$\mathcal{C}_W : \mathcal{N} \times \mathcal{N} \times \mathcal{N} \to \mathbf{R}$$

It is well-known that for $X_0 \in \mathcal{N}$ with property $\|X_0\| < 1$ the function

$$f : \mathcal{N} \to \mathbf{R},\ X \mapsto f(X) = \sqrt{\langle X, X\rangle} + \langle X_0, X\rangle \tag{7}$$

defines a Minkowski functional on $\mathcal{N}$, therefore it can be extended to a *left-invariant Randers type Finsler metric* F on the Lie group N by left translations. Excluding the case $X_0 = 0$, the remaining Randers metrics are non-Riemannian [2, p. 283]. By a direct computation we can express $\langle\, ,\rangle_W$ and $\mathcal{C}_W(\, ,\, ,)$ in terms of the Riemannian metric.

Proposition 2.7 ([8], [10]). *Let $W \in \mathcal{N}$ and $\langle W, W\rangle = 1$. Then*

$$\begin{aligned}\langle U, V\rangle_W = \langle U, V\rangle + \langle X_0, U\rangle\langle X_0, V\rangle - \langle X_0, W\rangle\langle W, U\rangle\langle W, V\rangle \\ + \langle X_0, U\rangle\langle W, V\rangle + \langle X_0, W\rangle\langle U, V\rangle + \langle X_0, V\rangle\langle W, U\rangle\end{aligned} \tag{8}$$

and

$$\begin{aligned}\mathcal{C}_W(U, V, X) = \frac{1}{2}\sum_{[U,V,X]}\{\langle X_0, W\rangle\langle W, U\rangle\langle W, V\rangle\langle W, X\rangle \\ - \langle X_0, W\rangle\langle X, V\rangle\langle U, W\rangle - \langle X_0, X\rangle\langle W, V\rangle\langle W, U\rangle + \langle X_0, U\rangle\langle X, V\rangle\}.\end{aligned} \tag{9}$$

where $\sum_{[U,V,X]}$ refers to the cyclic sum with respect to U, V, X.

3. Determination of the Chern–Rund connection

In the present paper we assume that the Randers-type Minkowski functional (7) on the three-dimensional Heisenberg algebra (= span(X, Y, Z), as in Definition 2.6) is determined by $X_0 = \xi Z \in \mathcal{Z}$, $(0 < \xi < 1)$ i.e. it is distinguished algebraically by the one-dimensional center of the Lie algebra. The reference vector W is supposed to be normalized w.r.t. $\langle\, ,\rangle$ in the sequel.

We use the so called Berwald–Moór frame ([11], [14]) for computation.

3.1. The Berwald-Moór frame. A. Moór used in the paper [14] a special orthonormal frame which was a generalization of the Berwald frame of two-dimensional Finsler spaces. We adapt the original definition to our context. The first base vector is the normalized reference vector W:

$$E_1 = \frac{1}{\sqrt{\langle W, W\rangle_W}} W.$$

The second base vector is the normalized Cartan vector.

Definition 3.1 (c.f. [12]). The *Cartan vector w.r.t.* W is the unique vector $C_W \in \mathcal{N}$ such that

$$\forall S \in \mathcal{N} : \langle S, C_W\rangle_W = (\operatorname{trace} \mathcal{C}^2_W)(S) = \operatorname{trace}(U \mapsto \mathcal{C}^2_W(S, U)). \tag{10}$$

It follows directly from the definition that

$$\langle W, C_W\rangle_W = \operatorname{trace}(U \mapsto \mathcal{C}^2_W(W, U)) = \operatorname{trace}(U \mapsto 0) = 0,$$

i.e. the Cartan vector w.r.t W is always orthogonal to W. Deicke's classical theorem states that $\forall W : C_W = 0$ if and only if the metric is Euclidean [4]. However, $C_W = 0$ is possible for some W in the non-Euclidean case.

Proposition 3.2. *If* $W \notin \mathcal{Z}$ *then* $C_W \neq 0$.

Proof. Let $(X_1 = W, X_2, X_3)$ be an orthonormal base w.r.t. $\langle\,,\rangle$, $g_{ij} = \langle X_i, X_j\rangle_W$, $(g^{ij}) = (g_{ij})^{-1}$ and $S = \langle S, X_l\rangle X_l = g^{kl}\langle S, X_k\rangle_W X_l$ an arbitrary vector. From the definition of the trace operator it follows that

$$\begin{aligned}\operatorname{trace}\{U \mapsto \mathcal{C}^2_W(S, U)\} &= \sum_{i=1}^{3}\left\langle X_i, \mathcal{C}^2_W(S, X_i)\right\rangle = g^{ij}\left\langle \mathcal{C}^2_W(S, X_i), X_j\right\rangle_W \\ &= g^{ij}\mathcal{C}_W(S, X_i, X_j).\end{aligned} \tag{11}$$

Equation (9) gives

$$\begin{aligned}\mathcal{C}_W(X_2, X_2, X_2) &= \frac{3}{2}\langle X_0, X_2\rangle, \quad \mathcal{C}_W(X_2, X_2, X_3) = \frac{1}{2}\langle X_0, X_3\rangle, \\ \mathcal{C}_W(X_3, X_3, X_2) &= \frac{1}{2}\langle X_0, X_2\rangle, \quad \mathcal{C}_W(X_3, X_3, X_3) = \frac{3}{2}\langle X_0, X_3\rangle,\end{aligned} \tag{12}$$

and all the terms $\mathcal{C}_W(X_1, X_i, X_j)$ vanish. Substituting (12) into (11) we have

$$(\operatorname{trace} C^2_W)(S) = \frac{3}{2}(g^{22} + g^{33})\left(\langle X_0, S\rangle - \langle X_0, W\rangle\langle S, W\rangle\right). \tag{13}$$

Substitute Z for S:

$$\begin{aligned}\langle C_W, Z\rangle_W &= (\operatorname{trace}\mathcal{C}_W^2)(Z) = \frac{3}{2}(g^{22}+g^{33})\left(\langle X_0, Z\rangle - \langle X_0, W\rangle\langle Z, W\rangle\right)\\ &= \frac{3}{2}(g^{22}+g^{33})\xi(1-\langle Z, W\rangle^2).\end{aligned} \tag{14}$$

$g^{22}, g^{33} > 0$ because (g^{ij}) is positive definite. $\xi \neq 0$ because the space is non-Riemannian. From the Cauchy–Schwarz inequality we have $1-\langle Z, W\rangle^2 \geq 0$, and equality holds if and only if $W = \pm Z$. □

In the paper [17] the case of $W \in \mathcal{Z}$ was completely described. The following proposition shows that this case has a Riemannian flavour for the Randers metric.

Proposition 3.3.

$$\begin{aligned}&\langle X, X\rangle_Z = \xi+1, \quad \langle X, Y\rangle_Z = 0, \quad \langle Y, Y\rangle_Z = \xi+1,\\ &\langle X, Z\rangle_Z = 0, \quad \langle Y, Z\rangle_Z = 0, \quad \langle Z, Z\rangle_Z = (1+\xi)^2,\end{aligned} \tag{15}$$

and all the local components of the Cartan tensor $\mathcal{C}_Z$ are zero.

The next Proposition specializes Proposition 8 of [17] for the Randers metric.

Proposition 3.4. *The local components of the Chern–Rund connection ∇^Z w.r.t. base (X, Y, Z) are*

$$\begin{aligned}&\nabla^Z_X X = 0, \quad \nabla^Z_X Y = \frac{1}{2}Z, \quad \nabla^Z_Y X = -\frac{1}{2}Z, \quad \nabla^Z_Y Y = 0,\\ &\nabla^Z_Z X = \nabla^Z_X Z = -\frac{\xi+1}{2}Y, \quad \nabla^Z_Z Y = \nabla^Z_Y Z = \frac{\xi+1}{2}X,\\ &\nabla^Z_Z Z = 0.\end{aligned}$$

From here we suppose that $W \notin \mathcal{Z}$ and the second base vector is given by

$$E_2 = \frac{1}{\sqrt{\langle C_W, C_W\rangle_W}}C_W.$$

E_3 completes (E_1, E_2) such that (E_1, E_2, E_3) is orthonormal w.r.t. $\langle\,,\rangle_W$. (Later we fix the orientation of the triplet.)

Lemma 3.5. *If the Randers-type Minkowski functional on the three-dimensional Heisenberg algebra is determined by $X_0 = \xi Z \in \mathcal{Z}$, then $[C_W, W] = 0$.*

Proof. It is enough to see that if a vector S satisfies $\langle S, C_W \rangle_W = 0$ and $\langle S, W \rangle_W = 0$ then $\langle S, X_0 \rangle_W = 0$, i.e. $C_W \in \operatorname{span}(W, Z)$.

We prove that if $\langle S, C_W \rangle_W = 0$ and $\langle S, W \rangle_W = 0$ then $\langle S, X_0 \rangle = 0$. By (8)

$$\langle S, W \rangle_W = (1 + \langle X_0, W \rangle)(\langle S, W \rangle + \langle X_0, S \rangle). \tag{16}$$

Using the Cauchy-Schwarz inequality

$$\langle X_0, W \rangle^2 \leq \langle X_0, X_0 \rangle \langle W, W \rangle = \xi^2 < 1, \tag{17}$$

from which it follows that $1 + \langle X_0, W \rangle \neq 0$.

By (13) and (16) conditions $\langle S, C_W \rangle_W = \langle S, W \rangle_W = 0$ imply that

$$\langle X_0, S \rangle - \langle X_0, W \rangle \langle S, W \rangle = 0 \tag{18}$$

$$\langle X_0, S \rangle + \langle S, W \rangle = 0, \tag{19}$$

from which it follows

$$\langle S, W \rangle (1 + \langle X_0, W \rangle) = 0.$$

Again, by (17) we have

$$\langle S, W \rangle = 0, \tag{20}$$

and (19) implies that

$$\langle X_0, S \rangle = 0. \tag{21}$$

Substituting (20) and (21) into (8) we have $\langle S, X_0 \rangle_W = 0$. □

The proof of the following statement has already been shown previously, but we formulate the result separately for future reference.

Corollary 3.6. *If a vector S satisfies $\langle S, C_W \rangle_W = 0$ and $\langle S, W \rangle_W = 0$ then $\langle S, X_0 \rangle_W = 0$ and $\langle S, X_0 \rangle = \langle S, W \rangle = 0$. In particular,*

$$\langle E_3, X_0 \rangle_W = 0 \quad \text{and} \quad \langle E_3, X_0 \rangle = \langle E_3, W \rangle = 0. \tag{22}$$

3.2. The case of $W \notin \mathcal{Z}$. To compute the Cartan tensor, we require a simple technical lemma.

Lemma 3.7. *If the Randers-type Minkowski functional on the three-dimensional Heisenberg algebra is determined by $X_0 = \xi Z \in \mathcal{Z}$, then for the Berwald–Moór frame we have*

$$\langle W, E_2 \rangle + \langle X_0, E_2 \rangle = 0 \tag{23}$$

$$\langle E_3, E_2 \rangle = 0 \tag{24}$$

$$(1 + \langle X_0, W \rangle)\left(\langle E_2, E_2 \rangle - \langle X_0, E_2 \rangle^2\right) = 1 \tag{25}$$

$$\langle E_3, E_3 \rangle (1 + \langle X_0, W \rangle) = 1. \tag{26}$$

Proof. All statements follow directly from Proposition 2.7. In more detail, using (8) we have

$$\begin{aligned}0 = \langle W, E_2\rangle_W &= \langle W, E_2\rangle + \langle X_0, W\rangle \langle X_0, E_2\rangle - \langle X_0, W\rangle \langle W, W\rangle \langle W, E_2\rangle \\ &\quad + \langle X_0, W\rangle \langle W, E_2\rangle + \langle X_0, W\rangle \langle W, E_2\rangle + \langle X_0, E_2\rangle \langle W, W\rangle \\ &= (1 + \langle X_0, W\rangle)(\langle W, E_2\rangle + \langle X_0, E_2\rangle)\end{aligned}$$

by the fact that $\langle W, W\rangle = 1$. $1 + \langle X_0, W\rangle \neq 0$ by (17), which gives (23).

Similarly,

$$\begin{aligned}0 = \langle E_3, E_2\rangle_W &= \langle E_3, E_2\rangle + \langle X_0, E_3\rangle \langle X_0, E_2\rangle - \langle X_0, W\rangle \langle W, E_3\rangle \langle W, E_2\rangle \\ &\quad + \langle X_0, E_3\rangle \langle W, E_2\rangle + \langle X_0, W\rangle \langle E_3, E_2\rangle + \langle X_0, E_2\rangle \langle W, E_3\rangle ,\end{aligned}$$

but since $\langle X_0, E_3\rangle = 0$, and $\langle W, E_3\rangle = 0$ (cf. Corollary 3.6)

$$= (1 + \langle X_0, W\rangle) \langle E_3, E_2\rangle .$$

which, since $1 + \langle X_0, W\rangle \neq 0$, yields (24).

We next prove (25).

$$\begin{aligned}1 = \langle E_2, E_2\rangle_W &= \langle E_2, E_2\rangle + \langle X_0, E_2\rangle^2 - \langle X_0, W\rangle \langle W, E_2\rangle^2 \\ &\quad + \langle X_0, E_2\rangle \langle W, E_2\rangle + \langle X_0, W\rangle \langle E_2, E_2\rangle + \langle X_0, E_2\rangle \langle W, E_2\rangle ,\end{aligned}$$

because of $\langle X_0, E_2\rangle = -\langle W, E_2\rangle$

$$\begin{aligned}&= \langle E_2, E_2\rangle - \langle X_0, E_2\rangle^2 - \langle X_0, W\rangle \langle X_0, E_2\rangle^2 + \langle X_0, W\rangle \langle E_2, E_2\rangle \\ &= (1 + \langle X_0, W\rangle) \left(\langle E_2, E_2\rangle - \langle X_0, E_2\rangle^2\right).\end{aligned}$$

Finally, we prove (26).

$$\begin{aligned}1 = \langle E_3, E_3\rangle_W &= \langle E_3, E_3\rangle + \langle X_0, E_3\rangle^2 - \langle X_0, W\rangle \langle W, E_3\rangle^2 \\ &\quad + \langle X_0, E_3\rangle \langle W, E_3\rangle + \langle X_0, W\rangle \langle E_3, E_3\rangle + \langle X_0, E_3\rangle \langle W, E_3\rangle ,\end{aligned}$$

but since $\langle X_0, E_3\rangle = 0$, (cf. (22))

$$= \langle E_3, E_3\rangle (1 + \langle X_0, W\rangle) .$$

□

Corollary 3.8. *With the notations and hypotheses above,*

$$\langle Z, E_2\rangle^2 = \frac{\xi^2 - (w-1)^2}{\xi^2 w^3}, \quad \langle E_2, E_2\rangle = \frac{\xi^2 + 2w - 1}{w^3}, \tag{27}$$

where $w = \|W\|_W = 1 + \langle X_0, W\rangle$*. Moreover,* $\langle Z, E_2\rangle > 0$.

PROOF. From (8) we get $\langle W, W\rangle_W^2 = (1+\langle X_0, W\rangle)^2$. Since (26), $1+\langle X_0, W\rangle > 0$, and we proved that $w = 1+\langle X_0, W\rangle$.

Substituting Z and C_W for S in (13) we get

$$\begin{aligned}\langle C_W, Z\rangle &= \frac{3}{2}\left(g^{22}+g^{33}\right)\left(\langle X_0, Z\rangle - \langle X_0, W\rangle\langle Z, W\rangle\right)\\ &= \frac{3}{2}\left(g^{22}+g^{33}\right)\xi\left(1-\langle Z, W\rangle^2\right)\end{aligned}$$

and

$$\begin{aligned}\langle C_W, C_W\rangle &= \frac{3}{2}\left(g^{22}+g^{33}\right)\left(\langle X_0, C_W\rangle - \langle X_0, W\rangle\langle C_W, W\rangle\right)\\ &= \frac{3}{2}\left(g^{22}+g^{33}\right)\langle X_0, C_W\rangle\left(1+\langle X_0, W\rangle\right), \quad \text{by (22)}.\end{aligned}$$

It follows that

$$\frac{\langle C_W, Z\rangle_W \langle X_0, C_W\rangle}{\langle C_W, C_W\rangle} = \frac{\xi\left(1-\langle Z, W\rangle^2\right)}{w};$$

i.e.

$$\langle E_2, Z\rangle_W \langle E_2, X_0\rangle = \frac{\xi\left(1-\langle Z, W\rangle^2\right)}{w}.$$

Applying (8) again, we obtain

$$\langle Z, E_2\rangle_W = w^2\langle Z, E_2\rangle = \frac{w^2}{\xi}\langle X_0, E_2\rangle.$$

Thus

$$\langle E_2, X_0\rangle^2 = \frac{\xi^2(1-\langle Z, W\rangle^2)}{w^3} = \frac{\xi^2-(w-1)^2}{w^3}.$$

The second statement is a straightforward consequence of this result and (25). By (14), ξ and $\langle Z, E_2\rangle_W = w^2\langle Z, E_2\rangle$ have the same sign. □

To proceed further, we need to know the local components of the Cartan tensor.

Proposition 3.9.

$$\mathcal{C}_W(E_2, E_2, E_2) = \frac{3}{2}\langle X_0, E_2\rangle, \tag{28}$$

$$\mathcal{C}_W(E_2, E_2, E_3) = 0, \tag{29}$$

$$\mathcal{C}_W(E_3, E_3, E_3) = 0, \tag{30}$$

$$\mathcal{C}_W(E_3, E_3, E_2) = \frac{1}{2}\langle X_0, E_2\rangle. \tag{31}$$

PROOF. Equation (9) implies that for arbitrary U

$$\begin{aligned} 2\mathcal{C}_W(E_2,E_2,U) = 3\langle X_0,W\rangle\langle W,E_2\rangle^2\langle W,U\rangle \\ -2\langle X_0,W\rangle\langle W,E_2\rangle\langle E_2,U\rangle - \langle X_0,W\rangle\langle W,U\rangle\langle E_2,E_2\rangle \\ -\langle X_0,U\rangle\langle W,E_2\rangle^2 - 2\langle X_0,E_2\rangle\langle W,U\rangle\langle W,E_2\rangle \\ +2\langle X_0,E_2\rangle\langle E_2,U\rangle + \langle X_0,U\rangle\langle E_2,E_2\rangle. \end{aligned} \tag{32}$$

Let $U=E_2$. (23) gives

$$\begin{aligned} 2\mathcal{C}_W(E_2,E_2,E_2) &= 3\langle X_0,W\rangle\langle W,E_2\rangle^3 - 3\langle X_0,W\rangle\langle W,E_2\rangle\langle E_2,E_2\rangle \\ &\quad -3\langle X_0,E_2\rangle\langle W,E_2\rangle^2 + 3\langle X_0,E_2\rangle\langle E_2,E_2\rangle \\ &= 3\langle X_0,W\rangle\langle W,E_2\rangle^3 - 3\langle X_0,W\rangle\langle W,E_2\rangle\langle E_2,E_2\rangle \\ &\quad -3\langle X_0,E_2\rangle^3 + 3\langle X_0,E_2\rangle\langle E_2,E_2\rangle \\ &= 3(1+\langle X_0,W\rangle)\langle X_0,E_2\rangle(\langle E_2,E_2\rangle - \langle X_0,E_2\rangle^2) \\ &= 3\langle X_0,E_2\rangle, \quad \text{by (25).} \end{aligned}$$

Thus, (28) holds.

Similarly, let $U=E_3$ in (32).

$$\begin{aligned} 2\mathcal{C}_W(E_2,E_2,E_3) &= 3\langle X_0,W\rangle\langle W,E_2\rangle^2\langle W,E_3\rangle \\ &\quad -2\langle X_0,W\rangle\langle W,E_2\rangle\langle E_2,E_3\rangle - \langle X_0,W\rangle\langle W,E_3\rangle\langle E_2,E_2\rangle \\ &\quad -\langle X_0,E_3\rangle\langle W,E_2\rangle^2 - 2\langle X_0,E_2\rangle\langle W,E_3\rangle\langle W,E_2\rangle \\ &\quad +2\langle X_0,E_2\rangle\langle E_2,E_3\rangle + \langle X_0,E_3\rangle\langle E_2,E_2\rangle \\ &= 2\langle X_0,W\rangle\langle X_0,E_2\rangle\langle E_2,E_3\rangle + 2\langle X_0,E_2\rangle\langle E_2,E_3\rangle, \quad \text{by (23)} \\ &= 0, \quad \text{by (24),} \end{aligned}$$

which provides (29).

Again, from (9) we have

$$\begin{aligned} 2\mathcal{C}_W(E_3,E_3,U) = 3\langle X_0,W\rangle\langle W,E_3\rangle^2\langle W,U\rangle \\ -\langle X_0,W\rangle\langle E_3,U\rangle\langle E_3,W\rangle - \langle X_0,W\rangle\langle E_3,E_3\rangle\langle U,W\rangle \\ -\langle X_0,U\rangle\langle W,E_3\rangle^2 - 2\langle X_0,E_3\rangle\langle W,U\rangle\langle W,E_3\rangle \\ +\langle X_0,U\rangle\langle E_3,E_3\rangle + 2\langle X_0,E_3\rangle\langle U,E_3\rangle. \end{aligned} \tag{33}$$

Set $U=E_3$.

$$\begin{aligned} 2\mathcal{C}_W(E_3,E_3,U) &= 3\langle X_0,W\rangle\langle W,E_3\rangle^3 - 3\langle X_0,W\rangle\langle E_3,E_3\rangle\langle E_3,W\rangle \\ &\quad -3\langle X_0,E_3\rangle\langle W,E_3\rangle^2 + 3\langle X_0,E_3\rangle\langle E_3,E_3\rangle. \end{aligned}$$

By Corollary 3.6 we have (30).

Finally, substitute $U = E_2$ in (33).

$$\begin{aligned}2\mathcal{C}_W(E_3, E_3, E_2) &= 3\langle X_0, W\rangle \langle W, E_3\rangle^2 \langle W, E_2\rangle \\ &\quad - 2\langle X_0, W\rangle \langle E_3, E_2\rangle \langle E_2, W\rangle - \langle X_0, W\rangle \langle E_3, E_3\rangle \langle E_2, W\rangle \\ &\quad - \langle X_0, E_2\rangle \langle W, E_3\rangle^2 - 2\langle X_0, E_3\rangle \langle W, E_2\rangle \langle W, E_3\rangle \\ &\quad + \langle X_0, E_2\rangle \langle E_3, E_3\rangle + 2\langle X_0, E_3\rangle \langle E_2, E_3\rangle \\ &= \langle E_3, E_3\rangle \langle X_0, E_2\rangle (1 + \langle X_0, W\rangle), \quad \text{by Corollary 3.6} \\ &= \langle X_0, E_2\rangle, \quad \text{by (26).}\end{aligned}$$

□

Now, we give an explicit formula for E_3.Since we are in a three-dimensional setting, E_3 can be constructed as a cross product of E_1 and E_2, where cross product is determined by the scalar product $\langle\,,\rangle_W$. However, $\langle E_3, W\rangle = \langle E_3, E_2\rangle = 0$, thus E_3 should be parallel to $W \times E_2$ where the cross product $\times$ now refers to the scalar product $\langle\,,\rangle$. In fact, $E_3 = \pm W \times E_2$, as we see from the following statement.

Proposition 3.10. $\|W \times E_2\|_W = 1$ *where* $\times$ *denotes the cross product w.r.t. the scalar product* $\langle\,,\rangle$.

Proof. Substituting into (8) we have

$$\|W \times E_2\|_W^2 = (1 + \langle X_0, W\rangle)\|W \times E_2\|^2.$$

Now we calculate $\|W \times E_2\|$ separately. From (23) and (25) we find that

$$\begin{aligned}\|W \times E_2\|^2 &= \|W\|^2 \cdot \|E_2\|^2 - \langle W, E_2\rangle^2 \\ &= \langle E_2, E_2\rangle - \langle X_0, E_2\rangle^2 = \frac{1}{1 + \langle X_0, W\rangle}.\end{aligned}$$

□

We fix now the the direction of E_3 by $E_3 = W \times E_2$.

Lemma 3.11. *With the notations and hypotheses above,*

$$[E_3, E_1] = \langle E_2, Z\rangle Z \tag{34}$$

$$[E_2, E_3] = \Big(\langle E_2, E_2\rangle \langle W, Z\rangle + \xi \langle Z, E_2\rangle^2\Big) Z. \tag{35}$$

Proof. In view of the relation $[U, V] = \langle U \times V, Z\rangle Z$, we obtain from the triple product identity that

$$\begin{aligned}[E_3, E_1] &= \langle (W \times E_2) \times E_1, Z\rangle Z = (\langle W, E_1\rangle \langle E_2, Z\rangle - \langle E_2, E_1\rangle \langle W, Z\rangle) Z \\ &= \frac{1}{w} \langle E_2, Z\rangle (1 + \langle W, X_0\rangle) Z = \langle E_2, Z\rangle Z.\end{aligned}$$

A similar computation shows (35). □

Proposition 3.12. *If the Randers-type Minkowski functional on the three-dimensional Heisenberg algebra is determined by* $X_0 = \xi Z \in \mathcal{Z}$ $(0 < \xi < 1)$*, the reference vector* $W \notin \mathcal{Z}$ *and* $\|W\| = 1$ *then*

$$\nabla^W_{E_1} W = f_1 E_3, \quad \nabla^W_{E_2} W = f_2 E_3, \quad \nabla^W_{E_3} W = f_3 E_2 \tag{36}$$

where f_1, f_2, f_3 *are functions of* $w = \|W\|_W$. *Explicitly, we have*

$$\begin{aligned} f_1 &= \langle [E_3, E_1], W \rangle_W \,, \\ f_2 &= \frac{1}{2}\big(\langle [E_3, W], E_2 \rangle_W + \langle [E_3, E_2], W \rangle_W - \langle [E_3, W], W \rangle_W \langle X_0, E_2 \rangle \big), \\ f_3 &= \frac{1}{2}\big(\langle [E_3, W], E_2 \rangle_W + \langle [E_2, E_3], W \rangle_W - \langle [E_3, W], W \rangle_W \langle X_0, E_2 \rangle \big). \end{aligned}$$

Proof. We follow the 'local strategy'. From (3) we determine the coordinates of $\nabla^W_W W$ w.r.t. the Berwald–Moór frame. Since

$$2 \left\langle \nabla^W_W W, E_i \right\rangle_W = - \langle [W, E_i], W \rangle_W + \langle [E_i, W], W \rangle_W = -2 \langle [W, E_i], W \rangle_W \,,$$

it follows that

$$\left\langle \nabla^W_W W, E_1 \right\rangle_W = 0, \quad \left\langle \nabla^W_W W, E_2 \right\rangle_W = 0, \quad \left\langle \nabla^W_W W, E_3 \right\rangle_W = \langle [E_3, W], W \rangle_W \,,$$

which formulae lead to $\nabla^W_W W = \langle [E_3, W], W \rangle_W E_3$, thus

$$\nabla^W_{E_1} E_1 = \langle [E_3, E_1], E_1 \rangle_W E_3 \quad \text{and} \quad \nabla^W_{E_1} W = \langle [E_3, E_1], W \rangle_W E_3 = f_1 E_3.$$

Similarly, (3) yields

$$\begin{aligned} 2 \left\langle \nabla^W_{E_2} W, U \right\rangle_W = &- \langle [W, U], E_2 \rangle_W + \langle [U, E_2], W \rangle_W \\ &- 2 \langle [E_3, W], W \rangle_W \, \mathcal{C}_W(E_3, U, E_2). \end{aligned}$$

Thus

$$\left\langle \nabla^W_{E_2} W, E_1 \right\rangle_W = 0, \quad \left\langle \nabla^W_{E_2} W, E_2 \right\rangle_W = 0$$

and from (31)

$$\begin{aligned} 2 \left\langle \nabla^W_{E_2} W, E_3 \right\rangle_W = &- \langle [W, E_3], E_2 \rangle_W + \langle [E_3, E_2], W \rangle_W \\ &- \langle [E_3, W], W \rangle_W \langle X_0, E_2 \rangle = 2 f_2, \end{aligned}$$

and this implies that $\nabla^W_{E_2} W = f_2 E_3$.

Finally,

$$2\left\langle\nabla^W_{E_3}W,U\right\rangle_W=\langle[E_3,W],U\rangle_W-\langle[W,U],E_3\rangle_W+\langle[U,E_3],W\rangle_W\\ -2\langle[E_3,W],W\rangle_W\,\mathcal{C}_W(E_3,U,E_3),$$

which yields to

$$\left\langle\nabla^W_{E_3}W,E_1\right\rangle_W=0,\quad\left\langle\nabla^W_{E_3}W,E_3\right\rangle_W=0$$

and

$$\left\langle\nabla^W_{E_3},E_2\right\rangle_W=\langle[E_3,W],E_2\rangle_W+\langle[E_2,E_3],W\rangle_W-\langle[E_3,W],W\rangle_W\langle X_0,E_2\rangle=2f_3,$$

which gives the last statement of (36).

We show that f_1 depends only on w. Combining (34) with (8) we obtain

$$f_1=\langle E_2,Z\rangle\langle Z,W\rangle_W=w(\xi+\langle Z,W\rangle)\langle E_2,Z\rangle.$$

In Corollary 3.8 we computed $\langle E_2,Z\rangle$ directly from w, moreover we have

$$\langle Z,W\rangle=\frac{w-1}{\xi}.$$

Thus

$$f_1=\sqrt{\frac{\xi^2-(w-1)^2}{\xi^2w^3}}w\left(\xi+\frac{w-1}{\xi}\right)=\frac{1}{\xi}\sqrt{\frac{\xi^2-(w-1)^2}{w}}\left(\xi+\frac{w-1}{\xi}\right).$$

Argument similar to that of the previous statement shows that

$$\langle[E_3,W],E_2\rangle_W=w^3\langle Z,E_2\rangle^2,$$

and

$$\langle[E_2,E_3],W\rangle_W-\langle[E_3,W],W\rangle_W\langle X_0,E_2\rangle=\langle Z,W\rangle^2+\langle X_0,W\rangle.$$

Combining these relations yields $f_3=\frac{1}{2}w$.

Finally,

$$f_2=f_3+\langle[E_3,E_2],W\rangle_W=\frac{w}{2}-\frac{\xi^2+w-1}{w}\left(1+\frac{w-1}{\xi^2}\right).\qquad\square$$

Proposition 3.13. *If the Randers-type Minkowski functional on the three-dimensional Heisenberg algebra is determined by $X_0 = \xi Z \in \mathcal{Z}$ $(0 < \xi < 1)$, the reference vector $W \notin \mathcal{Z}$ and $\|W\| = 1$ then the local components of the Chern–Rund connection ∇^W w.r.t. Berwald-Moór frame are*

$$\nabla^W_{E_1} E_1 = \frac{f_1}{w} E_3, \quad \nabla^W_{E_1} E_2 = \nabla^W_{E_2} E_1 = \frac{f_2}{w} E_3, \quad \nabla^W_{E_2} E_2 = f_4 E_3,$$

$$\nabla^W_{E_3} E_1 = \nabla^W_{E_1} E_3 + [E_3, E_1] = \frac{f_3}{w} E_2,$$

$$\nabla^W_{E_2} E_3 = \nabla^W_{E_3} E_2 + [E_2, E_3] = -\frac{f_2}{w} E_1 + f_5 E_2,$$

$$\nabla^W_{E_3} E_3 = -\frac{f_3}{2} \langle X_0, E_2 \rangle E_3,$$

where $w = \|W\|_W$, f_1, f_2, f_3 are defined in Proposition 3.12, and

$$f_4 = -\frac{f_1}{w} - f_2 \langle X_0, E_2 \rangle + \frac{3w}{4} \langle X_0, E_2 \rangle$$

$$f_5 = \langle [E_2, E_3], E_2 \rangle_W - \frac{3}{2} f_3 \langle X_0, E_2 \rangle .$$

Proof. The proof is similar to that given in the Proposition 3.12. □

4. The flag curvature of left invariant Randers metrics on 3-dimensional Heisenberg group

The geometry of any Lie group with left invariant Riemannian metric reflects strongly the algebraic structure of the corresponding Lie algebra. Papers e.g. by J. Wolf, J. Milnor ([13], [18]) serve many evidence for this statement. As an example we recall P. Eberlein's result for 2-step nilpotent groups with left invariant Riemannian metric.

Theorem (Eberlein, [7])**.** $\Pi = \operatorname{span}(X, Y) \subseteq \mathcal{N}$, *where (X, Y) is orthonormal pair. Let $K(\Pi) = K(X, Y)$ is the sectional curvature map. Then*

$$K(X, Y) = -\frac{3}{4} \|[X, Y]\|^2, \quad X, Y \in \mathcal{V}$$

$$K(X, Z) = \frac{1}{4} \|j(Z)X\|^2, \quad X \in \mathcal{V}, Z \in \mathcal{Z}$$

$$K(Z, Z^*) = 0, \quad Z, Z^* \in \mathcal{Z}$$

where $\mathcal{Z}$ is the center of the Lie algebra, $\mathcal{V}$ is the orthogonal complement of the center w.r.t. the left invariant Riemannian metric and $j(Z)\colon \mathcal{V} \to \mathcal{V}$ is defined by $\langle j(Z)X, Y\rangle = \langle [X,Y], Z\rangle$ for all $X, Y \in \mathcal{V}$.

The purpose of this section is to generalise this result to left invariant Randers metrics on the 3-dimensional Heisenberg group. We note that S. Deng and Z. Hu have recently obtained some remarkable results for curvatures of homogeneous Randers metrics ([5], [6], [9]).

The flag curvature of the Finsler manifold N is determined by a basepoint $p \in N$, the flagpole $W \in T_pN$ and an edge (transverse vector) $U \in T_pN$ by the formula

$$K(\Pi, W) = K(\Pi) = \frac{\langle R(U,W)W, U\rangle_W}{\langle U,U\rangle_W \langle W,W\rangle_W - \langle U,W\rangle_W^2} \tag{37}$$

where $\Pi = \operatorname{span}(U, W)$ and R is the affine curvature tensor of the Chern–Rund connection (see e.g. [2, Section 3.9.]).

Theorem 4.1. *If the Randers-type Minkowski functional on the three-dimensional Heisenberg algebra (=* $\operatorname{span}(X,Y,Z)$*), as in Definition 2.6 is determined by $X_0 = \xi Z \in \mathcal{Z}$ $(0 < \xi < 1)$ and $W = Z$, then the flag curvature of the Chern–Rund connection is*

$$K(\Pi) = \frac{1}{4} \quad \text{for all } U \in \operatorname{span}(X,Y).$$

Proof. Let $U = \alpha X + \beta Y$. From relations in Proposition 3.4 we get

$$R(U,W)W = \frac{\alpha(\xi+1)^2}{4}X + \frac{\beta(\xi+1)^2}{4}Y,$$

and an easy calculation gives the statement. □

In what follows $W \in \operatorname{span}(X,Y)$ and we get special case of Proposition 3.13.

Proposition 4.2 (With the notations from Proposition 3.13)**.**
If $W \in \operatorname{span}(X,Y)$ then $\operatorname{span}(X,Y) = \operatorname{span}(E_1, E_3)$ and $E_2 = -\xi E_1 + Z$. Moreover, $f_1 = \xi$, $f_2 = \frac{1}{2} - \xi^2$, $f_4 = \xi^3 - \frac{3\xi}{4}$, $f_5 = \frac{\xi}{4}$, and the local components of the Chern–Rund connection w.r.t. Berwald–Moór frame are

$$\nabla^W_{E_1} E_1 = \xi E_3, \quad \nabla^W_{E_1} E_2 = \nabla^W_{E_2} E_1 = \left(\frac{1}{2} - \xi^2\right) E_3, \quad \nabla^W_{E_2} E_2 = \xi\left(\xi^2 - \frac{3}{4}\right) E_3,$$

$$\nabla^W_{E_3} E_1 = \frac{1}{2}E_2, \quad \nabla^W_{E_1} E_3 = -\xi E_1 - \frac{1}{2}E_2,$$

$$\nabla^W_{E_2} E_3 = \left(\xi^2 - \frac{1}{2}\right) E_1 + \frac{1}{4}\xi E_2, \quad \nabla^W_{E_3} E_2 = -\frac{1}{2}E_1 - \frac{3}{4}\xi E_2,$$

$$\nabla^W_{E_3} E_3 = -\frac{1}{4}\xi E_3.$$

Proof. E_3 is always orthogonal to the centrum (in both senses, i.e. with respect to the euclidean scalar product $\langle\,,\rangle$ and osculating scalar product $\langle\,,\rangle_W$, see Corollary 3.6), which means that $E_3 \in \operatorname{span}(X, Y)$.

Substituting into (8) we have

$$\langle E_1, Z\rangle_W = \xi \tag{38}$$

$$\langle W, W\rangle_W = 1 \tag{39}$$

$$\langle Z, Z\rangle_W = 1 + \xi^2. \tag{40}$$

It follows that $E_1 = W$ and

$$\langle E_1, -\xi E_1 + Z\rangle_W = 0 \quad \langle -\xi E_1 + Z, -\xi E_1 + Z\rangle_W = 1 \quad \langle E_3, -\xi E_1 + Z\rangle_W = 0.$$

Thus $E_2 = -\xi E_1 + Z$ or $E_2 = \xi E_1 - Z$. From Corollary 3.8 we know that $\operatorname{sgn}\langle Z, E_2\rangle = \operatorname{sgn}\xi > 0$ and this fact implies $E_2 = -\xi E_1 + Z$. Now, the statements are simple consequences of Proposition 3.13. □

Theorem 4.3. *If the Randers-type Minkowski functional on the three-dimensional Heisenberg algebra is determined by $X_0 = \xi Z \in \mathcal{Z}$ $(0 < \xi < 1)$ and $W \in \operatorname{span}(X, Y)$, then the the flag curvature of the Chern–Rund connection is*

a) $K(\Pi) = \frac{\xi^2 - 3}{4} < 0$ *for all* $U \in \operatorname{span}(X, Y)$

b) $K(\Pi) = \frac{1 - \xi^2}{4} > 0$ *for* $U = Z$.

Proof. Let $U = \alpha E_1 + \beta E_3$. From Proposition 4.2 we have

$$R(U, W)W = \beta\frac{\xi^2 - 3}{4}E_3,$$

so

$$\langle R(U, W)W, U\rangle_W = \beta\frac{\xi^2 - 3}{4}\langle E_3, \alpha E_1 + \beta E_3\rangle_W = \beta^2\frac{\xi^2 - 3}{4}.$$

Applying (38)–(40), the denominator of (37) is β^2 and statement *4.3)* holds.

Let $U = Z$. A simple substitution into Proposition 4.2 gives

$$R(Z, W)W = \frac{1 - \xi^2}{4}E_2,$$

and we have

$$\langle R(Z,W)W,Z\rangle_W = \frac{1-\xi^2}{4}\langle E_2, \xi W + E_2\rangle_W = \frac{1-\xi^2}{4}.$$

Moreover, from (38)–(40) we get

$$\langle U,U\rangle_W \langle W,W\rangle_W - \langle U,W\rangle_W^2 = 1,$$

then we obtain statement *4.3*. □

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