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Strong convergence theorem for Vilenkin–Fejér means

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Abstract. As main result we prove strong convergence theorems of Vilenkin–Fejér means when $0 < p \leq 1/2$.

1. Introduction

It is well-known that Vilenkin system does not form basis in the space $L_1(G_m)$. Moreover, there is a function in the Hardy space $H_1(G_m)$, such that the partial sums of f are not bounded in L_1-norm. However, in Gát [7] the following strong convergence result was obtained for all $f \in H_1$:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0,$$

where $S_k f$ denotes the k-th partial sum of the Vilenkin–Fourier series of f. (For the trigonometric analogue see in Smith [17], for the Walsh–Paley system in Simon [15]). Simon [16] (see also [23]) proved that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[p]} n} \sum_{k=1}^{n} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p \leq 1) \tag{1}$$

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for all $f \in H_p$ and $n \in \mathbb{P}_+$, where $[p]$ denotes integer part of p. In [21] it was proved that sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ $(0 < p < 1)$ in (1) are given exactly.

Weisz [27] considered the norm convergence of Fejér means of Walsh–Fourier series and proved the following:

Theorem W1 (Weisz)**.** *Let $p > 1/2$ and $f \in H_p$. Then there exists an absolute constant c_p, depending only on p, such that for all $f \in H_p$ and $k = 1, 2, \dots$*

$$\|\sigma_k f\|_p \le c_p \|f\|_{H_p}.$$

Theorem W1 implies that

$$\frac{1}{n^{2p-1}} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \le c_p \|f\|_{H_p}^p, \quad (1/2 < p < \infty,\ n = 1, 2, \dots).$$

If Theorem W1 holds for $0 < p \le 1/2$, then we would have

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \le c_p \|f\|_{H_p}^p, \quad (0 < p \le 1/2,\ n = 2, 3, \dots). \tag{2}$$

However, in [18] it was proved that the assumption $p > 1/2$ in Theorem W1 is essential. In particular, the following is true:

Theorem T1. *There exists a martingale $f \in H_{1/2}$ such that*

$$\sup_n \|\sigma_n f\|_{1/2} = +\infty.$$

For the Walsh system in [22] it was proved that (2) holds, though Theorem T1 is not true for $0 < p < 1/2$.

As main result we generalize inequality (2) for bounded Vilenkin systems.

The results for summability of Fejér means of Walsh–Fourier series can be found in [3], [4], [5], [8], [9], [10], [11], [12], [13], [14].

2. Definitions and notations

Let $\mathbb{P}_+$ denote the set of the positive integers, $\mathbb{P} := \mathbb{P}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups only, that is

$$\sup_n m_n < \infty.$$

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_k, \dots) \quad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighbourhood of G_m

$$I_0(x) := G_m,$$
$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots,\ y_{n-1} = x_{n-1}\} \quad (x \in G_m,\ n \in \mathbb{P})$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{P}$ and $\overline{I_n} := G_m \backslash I_n$.

Let

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G_m \quad (n \in \mathbb{P}).$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_l \neq 0, x_{l+1}, \dots, x_{N-1}, x_N, x_{N+1}, \dots), \\ \qquad k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_N, x_{N+1}, \dots), \\ l = N. \end{cases}$$

and

$$\overline{I_N} = \left(\bigcup_{K=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \right) \bigcup \left(\bigcup_{K=0}^{N-1} I_N^{k,N} \right). \tag{3}$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{P})$$

then every $n \in \mathbb{P}$ can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j,$$

where $n_j \in Z_{m_j}$ $(j \in \mathbb{P})$ and only a finite number of n_j's differ from zero. Let $|n| := \max\{j \in \mathbb{P};\ n_j \neq 0\}$.

For $n = \sum_{i=1}^{r} s_i M_{n_i}$, where $n_1 > n_2 > \cdots > n_r \geq 0$ and $1 \leq s_i < m_{n_i}$ for all $1 \leq i \leq r$ we denote

$$\mathbb{A}_{0,2} = \left\{ n \in \mathbb{P} : n = M_0 + M_2 + \sum_{i=1}^{r-2} s_i M_{n_i} \right\}.$$

The norm (or quasi-norm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p \, d\mu(x) \right)^{1/p} \quad (0 < p < \infty).$$

The space $L_{p,\infty}(G_m)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}}^p := \sup_{\lambda > 0} \lambda^p \mu\{f > \lambda\} < +\infty.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system.

At first, define the complex valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi \imath x_k / m_k) \quad (\imath^2 = -1,\ x \in G_m,\ k \in \mathbb{P}).$$

It is known that

$$\sum_{k=0}^{m_n - 1} r_n^k(x) = \begin{cases} m_n, & x_n = 0, \\ 0, & x_n \neq 0, \end{cases} \tag{4}$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{P})$ on G_m as:

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{P}).$$

Specially, we call this system the Walsh–Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$, [1], [24].

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f\overline{\psi}_n d\mu, \quad (n \in \mathbb{P}_+)$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \quad (n \in \mathbb{P}_+),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f, \quad (n \in \mathbb{P}_+),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{P}_+),$$

$$K_n := \frac{1}{n}\sum_{k=1}^{n} D_k, \quad (n \in \mathbb{P}_+).$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases}. \tag{5}$$

and

$$D_n = \psi_n \sum_{j=0}^{\infty} D_{M_j} \sum_{p=m_j-n_j}^{m_j-1} r_j^p. \tag{6}$$

It is well-known that

$$\sup_n \int_{G_m} |K_n(x)|\, d\mu(x) \le c < \infty. \tag{7}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\digamma_n$ $(n \in \mathbb{P})$. Denote by $f = (f^{(n)},\ n \in \mathbb{P})$ a martingale with respect to $\digamma_n$ $(n \in \mathbb{P})$ (for details see e.g. [25]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbb{P}} \left|f^{(n)}\right|.$$

In case $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in \mathbb{P})$ is a martingale. Moreover, the maximal functions are also be given by

$$f^*(x) = \sup_{n\in\mathbb{P}} \frac{1}{|I_n(x)|}\left|\int_{I_n(x)} f(u)\,\mu(u)\right|$$

For $0<p<\infty$ the Hardy martingale spaces $H_p\ (G_m)$ consist of all martingales for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

If $f=(f^{(n)},\ n\in\mathbb{P})$ is martingale then the Vilenkin–Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i):=\lim_{k\to\infty}\int_{G_m}f^{(k)}(x)\overline{\psi}_i(x)d\mu(x).$$

The Vilenkin–Fourier coefficients of $f\in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f):n\in\mathbb{P})$ obtained from f.

A bounded measurable function a is p-atom, if there exist a dyadic interval I, such that

$$\int_I a d\mu=0,\quad \|a\|_\infty\leq\mu/(I)^{-1/p},\quad \operatorname{supp}(a)\subset I.$$

3. Formulation of main result

Theorem 1. *Let $0<p\leq 1/2$. Then there exists an absolute constant $c_p>0$, depending only on p, such that for all $f\in H_p$ and $n=2,3,\dots$*

$$\frac{1}{\log^{[1/2+p]}n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p,$$

where $[x]$ denotes integer part of x.

Corollary 1. *Let $f\in H_{1/2}$. Then*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\sigma_k f-f\|_{1/2}^{1/2}}{k}\to 0,\quad \text{as } n\to\infty.$$

Theorem 2. *Let $0<p<1/2$ and $\Phi:\mathbb{P}_+\to[1,\infty)$ be any non-decreasing function, satisfying the conditions $\Phi(n)\uparrow\infty$ and*

$$\overline{\lim_{n\to\infty}}\frac{n^{2-2p}}{\Phi(n)}=\infty. \tag{8}$$

Then there exists a martingale $f\in H_p$, such that

$$\sum_{k=1}^{\infty}\frac{\|\sigma_k f\|_{L_{p,\infty}}^p}{\Phi(k)}=\infty.$$

4. Auxiliary propositions

Lemma 1 ([26] (see also [25])). *A martingale $f=(f^{(n)},n\in\mathbb{P})$ is in $H_p(0<p\leq 1)$ if and only if there exist a sequence $(a_k,k\in\mathbb{P})$ of p-atoms and a sequence $(\mu_k,k\in\mathbb{P})$ of a real numbers such that for every $n\in\mathbb{P}$*

$$\sum_{k=0}^{\infty}\mu_k S_{M_n}a_k=f^{(n)},\qquad \sum_{k=0}^{\infty}|\mu_k|^p<\infty. \tag{9}$$

Moreover, $\|f\|_{H_p}\backsim \inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decomposition of f of the form (9).

Lemma 2 ([6]). *Let $n>t$, $t,n\in\mathbb{P}$, $x\in I_t\backslash I_{t+1}$. Then*

$$K_{M_n}(x)=\begin{cases}0, & \text{if } x-x_te_t\notin I_n,\\ \dfrac{M_t}{1-r_t(x)}, & \text{if } x-x_te_t\in I_n.\end{cases}$$

Lemma 3 ([19], [20]). *Let $x\in I_N^{k,l}$, $k=0,\dots,N-2$, $l=k+1,\dots,N-1$. Then*

$$\int_{I_N}|K_n(x-t)|\,d\mu(t)\leq\frac{cM_lM_k}{nM_N},\quad \text{when } n\geq M_N.$$

Let $x\in I_N^{k,N}$, $k=0,\dots,N-1$. Then

$$\int_{I_N}|K_n(x-t)|\,d\mu(t)\leq\frac{cM_k}{M_N},\quad \text{when } n\geq M_N.$$

Lemma 4. *Let $n=\sum_{i=1}^{r}s_iM_{n_i}$, where $n_1>n_2>\dots>n_r\geq 0$ and $1\leq s_i<m_{n_i}$ for all $1\leq i\leq r$ as well as $n^{(k)}=n-\sum_{i=1}^{k}s_iM_{n_i}$, where $0<k\leq r$. Then*

$$nK_n=\sum_{k=1}^{r}\left(\prod_{j=1}^{k-1}r_{n_j}^{s_j}\right)s_kM_{n_k}K_{s_kM_{n_k}}+\sum_{k=1}^{r-1}\left(\prod_{j=1}^{k-1}r_{n_j}^{s_j}\right)n^{(k)}D_{s_kM_{n_k}}.$$

Proof. It is easy to see that if $k,s,n\in\mathbb{P}$, $0\leq k<M_n$, then

$$D_{k+sM_n}=D_{sM_n}+\sum_{i=sM_n}^{sM_n+k-1}\psi_i=D_{sM_n}+\sum_{i=0}^{k-1}\psi_{i+sM_n}=D_{sM_n}+r_n^sD_k.$$

With help of this fact we get

$$\begin{aligned} nK_n &= \sum_{k=1}^{n} D_k = \sum_{k=1}^{s_1M_{n_1}} D_k + \sum_{k=s_1M_{n_1}+1}^{n} D_k = s_1M_{n_1}K_{s_1M_{n_1}} + \sum_{k=1}^{n^{(1)}} D_{k+s_1M_{n_1}} \\ &= s_1M_{n_1}K_{s_1M_{n_1}} + \sum_{k=1}^{n^{(1)}} \left(D_{s_1M_{n_1}} + r_{n_1}^{s_1}D_k\right) \\ &= s_1M_{n_1}K_{s_1M_{n_1}} + n^{(1)}D_{s_1M_{n_1}} + r_{n_1}^{s_1}n^{(1)}K_{n^{(1)}}. \end{aligned}$$

If we unfold $n^{(1)}K_{n^{(1)}}$ in similar way, we have

$$n^{(1)}K_{n^{(1)}} = s_2M_{n_2}K_{s_2M_{n_2}} + n^{(2)}D_{s_2M_{n_2}} + r_{n_2}^{s_2}n^{(2)}K_{n^{(2)}},$$

so

$$\begin{aligned} nK_n = s_1M_{n_1}K_{s_1M_{n_1}} &+ r_{n_1}^{s_1}s_2M_{n_2}K_{s_2M_{n_2}} + r_{n_1}^{s_1}r_{n_2}^{s_2}n^{(2)}K_{n^{(2)}} \\ &+ n^{(1)}D_{s_1M_{n_1}} + r_{n_1}^{s_1}n^{(2)}D_{s_2M_{n_2}}. \end{aligned}$$

Using this method with $n^{(2)}K_{n^{(2)}}, \dots, n^{(r-1)}K_{n^{(r-1)}}$, we obtain

$$\begin{aligned} nK_n = \sum_{k=1}^{r} &\left(\prod_{j=1}^{k-1} r_{n_j}^{s_j}\right) s_kM_{n_k}K_{s_kM_{n_k}} + \left(\prod_{j=1}^{r} r_{n_j}^{s_j}\right) n^{(r)}K_{n^{(r)}} \\ &+ \sum_{k=1}^{r-1}\left(\prod_{j=1}^{k-1} r_{n_j}^{s_j}\right) n^{(k)}D_{s_kM_{n_k}}. \end{aligned}$$

According to $n^{(r)} = 0$ it yields the statement of the Lemma 4. □

Lemma 5 ([2])**.** *Let $s, n \in \mathbb{P}$. Then*

$$D_{sM_n} = D_{M_n}\sum_{k=0}^{s-1}\psi_{kM_n} = D_{M_n}\sum_{k=0}^{s-1} r_n^k.$$

Lemma 6. *Let $s, t, n \in \mathbb{N},\ n > t,\ s < m_n,\ x \in I_t \backslash I_{t+1}$. If $x - x_te_t \notin I_n$, then*

$$K_{sM_n}(x) = 0.$$

Proof. In [6] G. Gát proved similar statement to $K_{M_n}(x) = 0$. We will use his method. Let $x \in I_t\backslash I_{t+1}$. Using (5) and (6) we have

$$sM_nK_{sM_n}(x) = \sum_{k=1}^{sM_n} D_k(x) = \sum_{k=1}^{sM_n} \psi_k(x)\left(\sum_{j=0}^{t-1} k_jM_j + M_t\sum_{i=m_t-k_t}^{m_t-1} r_t^i(x)\right)$$

$$= \sum_{k=1}^{sM_n} \psi_k(x) \sum_{j=0}^{t-1} k_j M_j + \sum_{k=1}^{sM_n} \psi_k(x) M_t \sum_{i=m_t-k_t}^{m_t-1} r_t^i(x) = J_1 + J_2.$$

Let $k := \sum_{j=0}^{n} k_j M_j$. Applying (4) we get $\sum_{k_t=0}^{m_t-1} r_t^{k_t}(x) = 0$, for $x \in I_t \backslash I_{t+1}$. It follows that

$$J_1 = \sum_{k_0=0}^{m_0-1} \cdots \sum_{k_{t-1}=0}^{m_{t-1}-1} \sum_{k_{t+1}=0}^{m_{t+1}-1} \cdots \sum_{k_{n-1}=0}^{m_{n-1}-1} \sum_{k_n=0}^{s-1} \left(\prod_{\substack{l=0 \\ l\neq t}}^{n} r_l^{k_l}(x) \right) \sum_{j=0}^{t-1} k_j M_j \sum_{k_t=0}^{m_t-1} r_t^{k_t}(x) = 0.$$

On the other hand

$$J_2 = \sum_{k_0=0}^{m_0-1} \cdots \sum_{k_{t-1}=0}^{m_{t-1}-1} \sum_{k_{t+1}=0}^{m_{t+1}-1} \cdots \sum_{k_{n-1}=0}^{m_{n-1}-1} \sum_{k_n=0}^{s-1} \left(\prod_{\substack{l=0 \\ l\neq t}}^{n} r_l^{k_l}(x) \right) M_t \sum_{i=0}^{k_t-1} r_t^i(x)$$

$$= \prod_{\substack{l=0 \\ l\neq t}}^{n-1} \left(\sum_{k_l=0}^{m_l-1} r_l^{k_l}(x) \right) \left(\sum_{k_p=0}^{s} r_p^{k_p}(x) \right) M_t \sum_{i=0}^{k_t-1} r_t^i(x).$$

Since $x - x_t e_t \notin I_n$, at least one of $\sum_{k_l=0}^{m_l-1} r_l^{k_l}(x)$ will be zero, if $l = p \neq t$ and $0 \leq p \leq n-1$, that is $J_2 = 0$. □

5. Proof of the theorems

Proof of Theorem 1. By Lemma 1, the proof of Theorem 1 will be complete, if we show that with a constant c_p

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k a\|_p^p}{k^{2-2p}} \leq c_p < \infty \quad (n = 2, 3, \dots).$$

for every p-atom a, where $[1/2+p]$ denotes the integers part of $1/2+p$. We may assume that a be an arbitrary p-atom with support I, $\mu(I) = M_N^{-1}$ and $I = I_N$. It is easy to see that $\sigma_n(a) = 0$, when $n \leq M_N$. Therefore we can suppose that $n > M_N$.

Let $x \in I_N$. Since σ_n is bounded from L_∞ to L_∞ (the boundedness follows from (7)) and $\|a\|_\infty \leq c M_N^{1/p}$ we obtain

$$\int_{I_N} |\sigma_m a(x)|^p \, d\mu(x) \leq c \, \|a\|_\infty^p / M_N \leq c_p < \infty, \quad 0 < p \leq 1/2.$$

Hence

$$\frac{1}{\log^{[1/2+p]} n}\sum_{m=1}^{n}\frac{\int_{I_N}|\sigma_m a(x)|^p d\mu(x)}{m^{2-2p}}\leq\frac{c}{\log^{[1/2+p]} n}\sum_{m=1}^{n}\frac{1}{m^{2-2p}}\leq c_p<\infty. \quad (10)$$

It is easy to show that

$$\begin{aligned}|\sigma_m a(x)| &\leq \int_{I_N}|a(t)|\,|K_m(x-t)|\,d\mu(t)\\ &\leq \|a\|_\infty\int_{I_N}|K_m(x-t)|\,d\mu(t)\leq cM_N^{1/p}\int_{I_N}|K_m(x-t)|\,d\mu(t).\end{aligned}$$

Let $x\in I_N^{k,l}$, $0\leq k<l<N$. Then from Lemma 3 we get

$$|\sigma_m a(x)|\leq\frac{cM_lM_kM_N^{1/p-1}}{m}. \quad (11)$$

Let $x\in I_N^{k,N}$ $0\leq k<N$. Then from Lemma 3 we have

$$|\sigma_m a(x)|\leq cM_kM_N^{1/p-1}. \quad (12)$$

Since

$$\sum_{k=0}^{N-2}1/M_k^{1-2p}\leq N^{[1/2+p]},\quad \text{for } 0<p\leq 1/2$$

by combining (3) and (11–12) we obtain

$$\begin{aligned}\int_{\overline{I_N}}|\sigma_m a(x)|^p d\mu(x) &= \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0,j\in\{l+1,\dots,N-1\}}^{m_{j-1}}\int_{I_N^{k,l}}|\sigma_m a(x)|^p d\mu(x)\\ &\quad+\sum_{k=0}^{N-1}\int_{I_N^{k,N}}|\sigma_m a(x)|^p d\mu(x)\\ &\leq c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{m_{l+1}\dots m_{N-1}}{M_N}\frac{(M_lM_k)^pM_N^{1-p}}{m^p}+\sum_{k=0}^{N-1}\frac{1}{M_N}M_k^pM_N^{1-p}\\ &\leq\frac{cM_N^{1-p}}{m^p}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{(M_lM_k)^p}{M_l}+\sum_{k=0}^{N-1}\frac{M_k^p}{M_N^p}\\ &=\frac{cM_N^{1-p}}{m^p}\sum_{k=0}^{N-2}\frac{1}{M_k^{1-2p}}\sum_{l=k+1}^{N-1}\frac{M_k^{1-p}}{M_l^{1-p}}+\sum_{k=0}^{N-1}\frac{M_k^p}{M_N^p}\\ &\leq\frac{cM_N^{1-p}N^{[1/2+p]}}{m^p}+c_p.\end{aligned} \quad (13)$$

It is easy to show that

$$\sum_{m=M_N+1}^{n} \frac{1}{m^{2-p}} \leq \frac{c}{M_N^{1-p}}, \quad \text{for } 0 < p \leq 1/2.$$

By applying (10) and (13) we get

$$\begin{aligned}
\frac{1}{\log^{[1/2+p]} n} \sum_{m=1}^{n} \frac{\|\sigma_m a\|_p^p}{m^{2-2p}} &\leq \frac{1}{\log^{[1/2+p]} n} \sum_{m=M_N+1}^{n} \frac{\int_{\overline{I_N}} |\sigma_m a(x)|^p d\mu(x)}{m^{2-2p}} \\
&+ \frac{1}{\log^{[1/2+p]} n} \sum_{m=M_N+1}^{n} \frac{\int_{I_N} |\sigma_m a(x)|^p d\mu(x)}{m^{2-2p}} \\
&\leq \frac{1}{\log^{[1/2+p]} n} \sum_{m=M_N+1}^{n} \left(\frac{c_p M_N^{1-p} N^{[1/2+p]}}{m^{2-p}} + \frac{c_p}{m^{2-p}} \right) + c_p \\
&\leq \frac{c_p M_N^{1-p} N^{[1/2+p]}}{\log^{[1/2+p]} n} \sum_{m=M_N+1}^{n} \frac{1}{m^{2-p}} + \frac{1}{\log^{[1/2+p]} n} \sum_{m=M_N+1}^{n} \frac{1}{m^{2-p}} + c_p \\
&\leq c_p < \infty.
\end{aligned}$$

which completes the proof of Theorem 1. □

Proof of Theorem 2. Under condition (8) there exists a sequence of increasing numbers $\{n_k : k \geq 0\}$, such that

$$\lim_{k\to\infty} \frac{c n_k^{2-2p}}{\Phi(n_k)} = \infty.$$

It is evident that for every n_k there exists a positive integer λ_k such that

$$M_{|\lambda_k|+1} \leq n_k < M_{|\lambda_k|+2} \leq \lambda M_{|n_k|+1},$$

where $\lambda = \sup_n m_n$. Since $\Phi(n)$ is a nondecreasing function we have

$$\overline{\lim}_{k\to\infty} \frac{M_{|\lambda_k|+1}^{2-2p}}{\Phi(M_{|\lambda_k|+1})} \geq \lim_{k\to\infty} \frac{c n_k^{2-2p}}{\Phi(n_k)} = \infty. \tag{14}$$

Applying (14) there exists a sequence $\{\alpha_k : k \geq 0\} \subset \{\lambda_k : k \geq 0\}$ such that

$$|\alpha_k| \geq 2, \quad \text{for } k \in \mathbb{P}, \tag{15}$$

$$\lim_{k\to\infty} \frac{M_{|\alpha_k|}^{1-p}}{\Phi^{1/2}(M_{|\alpha_k|+1})} = \infty \tag{16}$$

and

$$\sum_{\eta=0}^{\infty}\frac{\Phi^{1/2}(M_{|\alpha_\eta|+1})}{M_{|\alpha_\eta|}^{1-p}}=m_{|\alpha_\eta|}^{1-p}\sum_{\eta=0}^{\infty}\frac{\Phi^{1/2}(M_{|\alpha_\eta|+1})}{M_{|\alpha_\eta|+1}^{1-p}}<c<\infty. \tag{17}$$

Let

$$f_A=\sum_{\{k:|\alpha_k|<A\}}\lambda_k a_k,$$

where

$$\lambda_k=\lambda\cdot\frac{\Phi^{1/2p}(M_{|\alpha_k|+1})}{M_{|\alpha_k|}^{1/p-1}}$$

and

$$a_k=\frac{M_{|\alpha_k|}^{1/p-1}}{\lambda}\left(D_{M_{|\alpha_k|+1}}-D_{M_{|\alpha_k|}}\right),$$

where $\lambda:=\sup_{n\in\mathbb{P}}m_n$. Since

$$S_{M_n}a_k=\begin{cases}a_k, & |\alpha_k|<n,\\ 0, & |\alpha_k|\geq n,\end{cases}$$

and

$$\operatorname{supp}(a_k)=I_{|\alpha_k|},\quad \int_{I_{|\alpha_k|}}a_k d\mu=0,\quad \|a_k\|_\infty\leq M_{|\alpha_k|}^{1/p}=(\operatorname{supp}a_k)^{-1/p}$$

if we apply Lemma 1 and (17) we conclude that $f\in H_p$.

It is easy to show that

$$\widehat{f}(j)$$

$$=\begin{cases}\Phi^{1/2p}\left(M_{|\alpha_k|+1}\right), & \text{if } j\in\left\{M_{|\alpha_k|},\ldots,\ M_{|\alpha_k|+1}-\right\},\ k=0,1,2,\ldots,\\ 0, & \text{if } j\notin\bigcup_{k=0}^{\infty}\left\{M_{|\alpha_k|},\ldots,\ M_{|\alpha_k|+1}-1\right\}.\end{cases} \tag{18}$$

By using (18) we can write that

$$\sigma_{\alpha_k}f=\frac{1}{\alpha_k}\sum_{j=1}^{M_{|\alpha_k|}}S_jf+\frac{1}{\alpha_k}\sum_{j=M_{|\alpha_k|}+1}^{\alpha_k}S_jf=I+II. \tag{19}$$

It is simple to show that

$$S_jf=\begin{cases}\Phi^{1/2p}\left(M_{|\alpha_0|+1}\right), & \text{if } M_{|\alpha_0|}<j\leq M_{|\alpha_0|+1}\\ 0, & \text{if } 0\leq j\leq M_{|\alpha_0|}.\end{cases}$$

Suppose that $M_{|\alpha_s|} < j \leq M_{|\alpha_s|+1}$, for some $s = 1, 2, \ldots, k$. Then by applying (18) we have that

$$
\begin{aligned}
S_j f &= \sum_{v=0}^{M_{|\alpha_{s-1}|}} \widehat{f}(v) w_v + \sum_{v=M_{|\alpha_s|}+1}^{j-1} \widehat{f}(v) w_v \\
&= \sum_{\eta=0}^{s-1} \sum_{v=M_{|\alpha_\eta|}}^{M_{|\alpha_\eta|+1}-1} \widehat{f}(v) w_v + \sum_{v=M_{|\alpha_s|}+1}^{j-1} \widehat{f}(v) w_v \\
&= \sum_{\eta=0}^{s-1} \sum_{v=M_{|\alpha_\eta|}}^{M_{|\alpha_\eta|+1}-1} \Phi^{1/2p}(M_{|\alpha_\eta|+1}) w_v + \Phi^{1/2p}(M_{|\alpha_s|+1}) \sum_{v=M_{|\alpha_s|}+1}^{j-1} w_v \\
&= \sum_{\eta=0}^{s-1} \Phi^{1/2p}(M_{|\alpha_\eta|+1}) \big(D_{M_{|\alpha_\eta|+1}} - D_{M_{|\alpha_\eta|}}\big) \\
&\quad + \Phi^{1/2p}(M_{|\alpha_s|+1}) \big(D_j - D_{M_{|\alpha_s|}}\big). \qquad (20)
\end{aligned}
$$

Let $M_{|\alpha_s|+1} < j \leq M_{|\alpha_{s+1}|}$, for some $s = 1, 2, \ldots, k$. Analogously to (20) we get that

$$
S_j f = \sum_{v=0}^{M_{|\alpha_s|+1}} \widehat{f}(v) w_v = \sum_{\eta=0}^{s} \Phi^{1/2p}(M_{|\alpha_\eta|+1}) \big(D_{M_{|\alpha_\eta|+1}} - D_{M_{|\alpha_\eta|}}\big). \qquad (21)
$$

Let $x \in I_2^{0,1} = (x_0 = 1,\ x_1 = 1, x_2, \ldots)$. Since (see (5) and Lemma 2)

$$
K_{M_n}(x) = D_{M_n}(x) = 0, \quad for\ n \geq 2 \qquad (22)
$$

from (15) and (20)–(21) we obtain that

$$
\begin{aligned}
I &= \frac{1}{n} \sum_{\eta=0}^{k-1} \Phi^{1/2p} \left(M_{|\alpha_\eta|+1}\right) \sum_{v=M_{|\alpha_\eta|}+1}^{M_{|\alpha_\eta|+1}} D_v \\
&= \frac{1}{n} \sum_{\eta=0}^{k-1} \Phi^{1/2p} \left(M_{|\alpha_\eta|+1}\right) \Big(M_{|\alpha_\eta|+1} K_{M_{|\alpha_\eta|+1}}(x) - M_{|\alpha_\eta|} K_{M_{|\alpha_\eta|}}(x)\Big) = 0. \qquad (23)
\end{aligned}
$$

By applying (20), when $s = k$ in II we get that

$$
\begin{aligned}
II = {} & \frac{\alpha_k - M_{|n_k|}}{\alpha_k} \sum_{\eta=0}^{k-1} \Phi^{1/2p} \left(M_{|\alpha_\eta|+1}\right) \left(D_{M_{|\alpha_\eta|+1}} - D_{M_{|\alpha_\eta|}}\right) \\
& + \frac{\Phi^{1/2p} \left(M_{|n_k|+1}\right)}{\alpha_k} \sum_{j=M_{|n_k|}+1}^{\alpha_k} \left(D_j - D_{M_{|n_k|}}\right) = II_1 + II_2. \qquad (24)
\end{aligned}
$$

By using (22) we have that

$$II_1 = 0, \quad \text{for } x \in I_2^{0,1}. \tag{25}$$

Let $\alpha_k \in \mathbb{A}_{0,2}$ and $x \in I_2^{0,1}$. Since $\alpha_k - M_{|\alpha_k|} \in \mathbb{A}_{0,2}$ and

$$D_{j+M_{|\alpha_k|}} = D_{M_{|\alpha_k|}} + w_{M_{|\alpha_k|}} D_j, \quad \text{when} \quad j < M_{|\alpha_k|}$$

By combining (5) Lemmas 4 and 6 we obtain that

$$\begin{aligned}
|II_2| &= \frac{\Phi^{1/2p}(M_{|\alpha_k|+1})}{\alpha_k} \left| \sum_{j=1}^{\alpha_k - M_{|\alpha_k|}} \left(D_{j+M_{|\alpha_k|}}(x) - D_{M_{|\alpha_k|}}(x)\right) \right| \\
&= \frac{\Phi^{1/2p}(M_{|\alpha_k|+1})}{\alpha_k} \left| \sum_{j=1}^{\alpha_k - M_{|\alpha_k|}} D_j(x) \right| \\
&= \frac{\Phi^{1/2p}(M_{|\alpha_k|+1})}{\alpha_k} \left| (\alpha_k - M_{|\alpha_k|}) K_{\alpha_k - M_{|\alpha_k|}}(x) \right| \\
&= \frac{\Phi^{1/2p}(M_{|\alpha_k|+1})}{\alpha_k} |M_0 K_{M_0}| \geq \frac{\Phi^{1/2p}(M_{|\alpha_k|+1})}{\alpha_k}.
\end{aligned} \tag{26}$$

Let $0 < p < 1/2$, $n \in \mathbb{A}_{0,2}$ and $M_{|\alpha_k|} < n < M_{|\alpha_k|+1}$. By combining (19–26) we have that

$$\begin{aligned}
\|\sigma_n f\|_{L_{p,\infty}}^p &\geq \frac{c\Phi^{1/2}(M_{|\alpha_k|+1})}{\alpha_k^p} \mu\left\{ x \in I_2^{0,1} : |II_2| \geq \frac{c\Phi^{1/2p}(M_{|\alpha_k|+1})}{\alpha_k} \right\} \\
&\geq \frac{c\Phi^{1/2}(M_{|\alpha_k|+1})}{\alpha_k^p} \mu\left\{ I_2^{0,1} \right\} \geq \frac{c\Phi^{1/2}(M_{|\alpha_k|+1})}{M_{|\alpha_k|+1}^p}.
\end{aligned}$$

By using (16) we get that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\|\sigma_n f\|_{L_{p,\infty}}^p}{\Phi(n)} &\geq \sum_{\{n \in \mathbb{A}_{0,2} : M_{|\alpha_k|} < n < M_{|\alpha_k|+1}\}} \frac{\|\sigma_n f\|_{L_{p,\infty}}^p}{\Phi(n)} \\
&\geq \frac{1}{\Phi^{1/2}(M_{|\alpha_k|+1})} \sum_{\left\{n \in \mathbb{A}_{0,2} : M_{|\alpha_k|} < n < M_{|\alpha_k|+1}\right\}} \frac{1}{M_{|\alpha_k|+1}^p} \\
&\geq \frac{cM_{|\alpha_k|}^{1-p}}{\Phi^{1/2}(M_{|\alpha_k|+1})} \to \infty, \quad \text{when } k \to \infty.
\end{aligned}$$

Theorem 2 is proved. □

References

[1] G. N. Agaev, N. Ya. Vilenkin, G. M. Dzhafarly and A. I. Rubinshtein, Multiplicative systems of functions and harmonic analysis on zero-dimensional groups, *Baku, Ehim* (1981) (in *in Russian*).

[2] I. Blahota, Relation between Dirichlet kernels with respect to Vilenkin-like systems, *Acta Acad. Paedagog. Agriensis, Sect. Mat. (N.S.)* **22** (1994), 109–114.

[3] I. Blahota, G. Gát and U. Goginava, Maximal operators of Fejér means of double Vilenkin–Fourier series, *Colloq. Math.* **107** (2007), 287–296.

[4] I. Blahota, G. Gát and U. Goginava, Maximal operators of Fejér means of Vilenkin-Fourier series, *J. Inequal. Pure Appl. Math.* **7** (2006), 1–7.

[5] N. J. Fujii, A maximal inequality for H_1 functions on the generalized Walsh–Paley group, *Proc. Amer. Math. Soc.* **77** (1979), 111–116.

[6] G. Gát, Cesàro means of integrable functions with respect to unbounded Vilenkin systems, *J. Approx. Theory* **124** (2003), 25–43.

[7] G. Gát, Investigations of certain operators with respect to the Vilenkin system, *Acta Math. Hung.* **61** (1993), 131–149.

[8] U. Goginava, Maximal operators of Fejér means of double Walsh–Fourier series, *Acta Math. Hungar.* **115** (2007), 333–340.

[9] U. Goginava, The maximal operator of the Fejér means of the character system of the p-series field in the Kaczmarz rearrangement, *Publ. Math. Debrecen* **71** (2007), 43–55.

[10] U. Goginava and K. Nagy, On the maximal operator of Walsh–Kaczmarz–Fejér means, *Czechoslovak Math. J.* **61** (2011), 673–686.

[11] J. Pál and P. Simon, On a generalization of the concept of derivate, *Acta Math. Hung.* **29** (1977), 155–164.

[12] F. Schipp, Certain rearrangements of series in the Walsh series, *Mat. Zametki* **18** (1975), 193–201.

[13] P. Simon and F. Weisz, Strong convergence theorem for two-parameter Vilenkin–Fourier series, *Acta Math. Hungar.* **86** (2000), 17–38.

[14] P. Simon, Investigations with respect to the Vilenkin system, *Ann. Univ. Sci. Budapest Eötv., Sect. Math.* **28** (1985), 87–101.

[15] P. Simon, Strong convergence of certain means with respect to the Wals-Fourier series, *Acta Math. Hung.* **49** (1987), 425–431.

[16] P. Simon, Strong Convergence Theorem for Vilenkin–Fourier Series., *J. Math. Anal. Appl.* **245** (2000), 52–68.

[17] B. Smith, A strong Convergence Theorem for $H^1(T)$, in: Lecture Notes in Math., 995, *Springer, Berlin*, 1994, 169–173.

[18] G. Tephnadze, Fejér means of Vilenkin–Fourier series, *Stud. Sci. Math. Hung* **49** (2012), 79–90.

[19] G. Tephnadze, On the maximal operator of Vilenkin–Fejér means, *Turk. J. Math.* **37** (2013), 308–318.

[20] G. Tephnadze, On the maximal operators of Vilenkin–Fejér means on Hardy spaces, *Math. Inequal. Appl.* **16** (2013), 301–312.

[21] G. Tephnadze, A note on the Fourier coefficients and partial sums of Vilenkin–Fourier series, *Acta Math. Acad. Paed. Nyíregyháza* **28** (2012), 167–176.

[22] G. Tephnadze, Strong convergence theorems of Walsh–Fejér means, *Acta Math. Hungar.* **142** (2014), 244–259.

[23] G. Tephnadze, On the partial sums of Vilenkin–Fourier series, *J. Contemp. Math. Anal.* **49** (2014), 23–32.

[24] N. Ya. Vilenkin, On a class of complete orthonormal systems, *Izv. Akad. Nauk. U.S.S.R., Ser. Mat.* **11** (1947), 363–400.

[25] F. Weisz, Martingale Hardy Spaces and their Applications in Fourier Analysis, *Springer, Berlin – Heideiberg – New York*, 1994.

[26] F. Weisz, Hardy spaces and Cesàro means of two-dimensional Fourier series, *Bolyai Soc. Math. Studies* (1996), 353–367.

[27] F. Weisz, Cesàro summability of one and two-dimensional Fourier series, *Anal. Math.* **5** (1996), 353–367.

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