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Quasi Ahlfors–David regularity of Moran sets

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Abstract. For Moran fractals in Euclidean spaces, under suitable assumption, we obtain their quasi Ahlfors–David regularity and quasi-Lipschitz equivalence.

1. Introduction

1.1. Moran set. Recall some notions in fractals. A map $S:\mathbb{R}^n \to \mathbb{R}^n$ is called a similitude with ratio r, if $|S(x)-S(y)| = r|x-y|$ for all $x,y\in\mathbb{R}^n$. For two subsets A and B of $\mathbb{R}^n$, we say that A is geometrically similar to B with ratio r, if there is a similitude S of ratio r such that $S(A)=B$. For subset A, let $|A|$ be the diameter of the set, and $\bar{A}$ the closure of A.

The notion of Moran set is introduced by Wen in [16].

Given integers $\{n_k\}_{k\geq 1}$ with $n_k \geq 2$ and ratios $\{c_{k,i}\}_{k\geq 1,\ 1\leq i\leq n_k} \subset (0,1)$, let $\tilde{\Sigma}^* = \bigcup_{k=0}^{\infty}\prod_{i=1}^{k}\{1,\dots,n_i\}$ be the collection of finite words with k-th letter in $\{1,\dots,n_k\}$ for each k, suppose $\{V_{i_1\cdots i_k}\}_{i_1\cdots i_k\in\tilde{\Sigma}^*}$ are non-empty open sets satisfying

$$V_{i_1\cdots i_{k-1}i_k} \subset V_{i_1\cdots i_{k-1}} \text{ and } V_{i_1\cdots i_{k-1}i_k}\cap V_{i_1\cdots i_{k-1}j_k} = \varnothing \quad \text{if } i_k \neq j_k, \tag{1.1}$$

$V_{i_1\cdots i_{k-1}i_k}$ is geometrically similar to $V_{i_1\cdots i_{k-1}}$ with ratio

$$|V_{i_1\cdots i_{k-1}i_k}|/|V_{i_1\cdots i_{k-1}}| = c_{k,i_k}. \tag{1.2}$$

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A *Moran set* F is defined by

$$F = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k \in \tilde{\Sigma}^*} \bar{V}_{i_1 \cdots i_k}. \tag{1.3}$$

For convenience, we write $d_k = \min_{1 \le i \le n_k} c_{k,i}$, $D_k = \max_{1 \le i \le n_k} c_{k,i}$ and suppose s_k satisfies $\prod_{j=1}^{k} \left(\sum_{i=1}^{n_j} (c_{j,i})^{s_k} \right) = 1$. Let $s_* = \underline{\lim}_{k \to \infty} s_k$ and $s^* = \overline{\lim}_{k \to \infty} s_k$.

Remark 1. The self-similar set satisfying the open set condition is a Moran set. In fact, suppose $E = \cup_{i=1}^{m} S_i(E)$ is a self-similar set (see [7]), where $\{S_i\}_{i=1}^{m}$ are contracting similitudes with ratios $\{r_i\}_{i=1}^{m}$ such that the open set condition holds, i.e., there is a non-empty open set U such that

$$\cup_i S_i(U) \subset U \text{ and } S_i(U) \cap S_j(U) = \varnothing \quad \text{for any } i \neq j. \tag{1.4}$$

Let $n_k \equiv m$ and $c_{k,i_k} = r_{i_k}$, $\Sigma^* = \cup_{k=0}^{\infty} \{1, \ldots, m\}^k$ and $\emptyset$ the empty word. We write $U_\emptyset = U$ and $U_{i_1 \cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(U)$ for word $i_1 \cdots i_k \in \Sigma^*$, then (1.4) implies $U_{i_1 \cdots i_{k-1} i_k} \subset U_{i_1 \cdots i_{k-1}}$ and $U_{i_1 \cdots i_{k-1} i_k} \cap U_{i_1 \cdots i_{k-1} j_k} = \varnothing$ for any $i_1 \cdots i_k \in \Sigma^*$ and $i_k \neq j_k$and $U_{i_1 \cdots i_{k-1} i_k}$ is geometrically similar to $U_{i_1 \cdots i_{k-1}}$ with ratio $|U_{i_1 \cdots i_{k-1} i_k}| / |U_{i_1 \cdots i_{k-1}}| = r_{i_k}$. It follows from (1.4) that $\cup_i S_i(\bar{U}) \subset \bar{U}$, hence the self-similar set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k \in \Sigma^*} \bar{U}_{i_1 \cdots i_k}. \tag{1.5}$$

1.2. Ahlfors–David regularity. A compact set E is said to be Ahlfors–David s-regular [11], [12], if there is a Borel measure ν supported on E such that

$$C^{-1} r^s \le \nu(B(x,r)) \le C r^s \tag{1.6}$$

for any $x \in E$ and $r \le |E|$. For self-similar set E satisfying the open set condition as above, as shown in [7], the self-similar measure ν satisfies (1.6) where $s = \dim_H E = \dim_B E$ is the solution of the equation $(r_1)^s + \cdots + (r_m)^s = 1$.

For Ahlfors–David s-regular set E, by Theorem 5.7 of [11], we have

$$\overline{\dim}_B E = \underline{\dim}_B E = \dim_H E = s.$$

It was shown in [16] that, if $c_* = \inf d_k > 0$ for Moran set F as above, then

$$\dim_H F = s_* \quad \text{and} \quad \overline{\dim}_B F = s^*.$$

Therefore, if $s_* < s^*$, then F can not be Ahlfors–David regular.

Example 1. Let $n_k \equiv 2$ and $c_k \in \{1/3, 1/5\}$. Then $c_* > 0$. Take a sequence $\{c_k\}_k$ such that $a = \underline{\lim}_{k\to\infty} q_k < \overline{\lim}_{k\to\infty} q_k = b$, where $q_k = \frac{\#\{i\le k: c_k = 1/3\}}{k}$. Then $\underline{\lim}_{k\to\infty} s_k = \underline{\lim}_{k\to\infty} \frac{k\log 2}{-(\log c_1\cdots c_k)} = \underline{\lim}_{k\to\infty} \frac{\log 2}{q_k\log 3 + (1-q_k)\log 5} = \frac{\log 2}{a\log 3 + (1-a)\log 5}$ and $\overline{\lim}_{k\to\infty} s_k = \frac{\log 2}{b\log 3 + (1-b)\log 5}$, which means $\dim_H F < \overline{\dim}_B F$.

In the following example, $\dim_H F = \dim_B F$ but F is not Ahlfors–David regular.

Example 2. Let $s = \log 3/\log 5$ and $c_k = \frac{1}{5}(1 + \frac{1}{2k})$ for all k, then $\lim_{k\to\infty} c_k = 1/5$ and $\lim_{k\to\infty} 3^k (c_1\cdots c_k)^s = \lim_{k\to\infty} \big[\prod_{i=1}^k \big(1 + \frac{1}{2i}\big)\big]^s = \infty$. Set $n_k \equiv 3$. Fix $I_\phi = [0,1]$ for empty word ϕ. Given an interval $I_{i_1\cdots i_{k-1}} = [c,d]$ with $i_1\cdots i_{k-1} \in \{1,2,3\}^{k-1}$, we put $I_{i_1\cdots i_{k-1}1} = [c, c + c_k(d-c)]$, $I_{i_1\cdots i_{k-1}2} = \big[\frac{c+d}{2} - \frac{c_k(d-c)}{2}, \frac{c+d}{2} + \frac{c_k(d-c)}{2}\big]$ and $I_{i_1\cdots i_{k-1}3} = [d - (d-c)c_k, d]$. Let

$$F = \cap_{k=1}^\infty \cup_{i_1\cdots i_k \in \{1,2,3\}^k} I_{i_1\cdots i_k}.$$

Using result in [16], we have $\dim_H F = \dim_B F = s$. We will show that F is not Ahlfors–David s-regular. On the contrary, there is a constant $\alpha > 0$ such that for the corresponding Ahlfors–David s-regular measure ν, we have $(c_1\cdots c_k)^s \le \alpha\nu(I_{i_1\cdots i_k})$ for all $i_1\cdots i_k \in \{1,2,3\}^k$. Therefore,

$$3^k (c_1\cdots c_k)^s \le \alpha \sum_{i_1\cdots i_k \in \{1,2,3\}^k} \nu(I_{i_1\cdots i_k}) = \alpha\nu([0,1]) < \infty.$$

Letting $k\to\infty$, we obtain a contradiction.

However, the Moran set in Example 2 is quasi Ahlfors–David s-regular.

1.3. Quasi Ahlfors–David regularity. We say that F is quasi Ahlfors–David s-regular [15], if there is a Borel measure μ supported on F such that

$$\frac{\log \mu(B(x,r))}{\log r} \to s \text{ uniformly for } x\in F \text{ as } r\to 0. \tag{1.7}$$

Notice that Ahlfors–David regularity implies quasi Ahlfors–David regularity.

Can we find the conditions for a Moran set to be quasi Ahlfors–David regular? Now, we shall pose some assumptions upon the Moran sets:

(A1): $\lim_{k\to\infty} \frac{\log d_k}{\log(D_1D_2\cdots D_{k-1})} = 0$;

(A2): $\sup_k \frac{\log(n_1n_2\cdots n_k)}{-\log(D_1D_2\cdots D_k)} < \infty$;

(A3): $\lim_{k\to\infty} s_k = s \in (0,\infty)$.

Remark 2. (A1) means $\frac{\log|\bar V_{i_1\cdots i_{k-1}i_k}|}{\log|\bar V_{i_1\cdots i_{k-1}}|} \to 1$ uniformly as $k \to \infty$, this is a natural assumption. We pose (A2) as a technical need.

Remark 3. If $c_* = \inf_k d_k > 0$, then (A1)–(A2) hold since $\sup_k n_k \le 1/(c_*)^n$ and $0 < c_* \le d_k \le D_k \le \sqrt[n]{1-(c_*)^n}$ by (1.1)–(1.2).

Remark 4. (A3) means $\dim_H F = \dim_B F = s$ in some sense as in [16]. If $c_* = \inf_k d_k > 0$, it follows from [16] that $\dim_H F = s_*$ and $s^* = \overline{\dim}_B F$. Using (A3), we have $\dim_H F = \dim_B F = s$.

Remark 5. For self-similar set E satisfying the open set condition as above, we have $D_k \equiv \max_i r_i$, $d_k \equiv \min_i r_i$, $n_k \equiv m$, $c_{k,i} = r_i$ and $s_k \equiv s$ with

$$(r_1)^s + \cdots + (r_m)^s = 1.$$

It is obvious that assumptions (A1)–(A3) hold. The Ahlfors–David regularity of E supports the rationality of these assumptions.

Under these assumption, we get the quasi Ahlfors–David regularity.

Theorem 1. *Suppose F is a Moran set satisfying* (A1)–(A3). *Then F is quasi Ahlfors–David s-regular.*

Besides Example 2, we give the following planar Moran set satisfying (A1)–(A3).

Example 3 ([3]). Consider a plane fractal defined as follows. Given two sequences $\{l_k\}_{k\ge1}$ and $\{n_k\}_{k\ge1}$ with $n_k \le l_k^2$ and

$$\lim_{k\to\infty} \frac{\log(n_1\cdots n_k)}{\log(l_1\cdots l_k)} = s.$$

Take a unit square $[0,1]^2$ as the initial set. In the first step, we divide $[0,1]^2$ into $(l_1)^2$ equal squares with side length $1/l_1$, then we select n_1 ones. By induction, assume that we get a square $Q_{i_1\cdots i_{k-1}}$ ($i_1\cdots i_{k-1} \in \tilde\Sigma^*$) of side length $(l_1\cdots l_{k-1})^{-1}$, we divide it into $(l_k)^2$ equal squares with side length $(l_1\cdots l_{k-1}l_k)^{-1}$, and then we select n_k ones from them, denoted by $\{Q_{i_1\cdots i_{k-1}i_k}\}_{i_k=1}^{n_k}$. Again and again, we get a limit set which is a Moran set, where $c_{k,i} \equiv (l_k)^{-1}$ and $V_{i_i\cdots i_k}$ is the interior of $Q_{i_1\cdots i_k}$. It is easy to check that assumptions (A1)–(A3) hold.

1.4. Quasi Lipschitz equivalence of Moran sets. We say that two subsets A and B of Euclidean spaces are Lipschitz equivalent, if there is a bijection from A to B such that for all $x, y \in A$,

$$C^{-1}|x-y| \le |f(x)-f(y)| \le C|x-y|$$

for some constant $C > 0$.

Even for self-similar sets, their Lipschitz equivalences are very difficult to study, for example see Cooper and Pignataro [1], David and Semmes [3], Deng and He [4], Deng, Wen, Xiong and Xi [5], Falconer and Marsh [6], Lau and Luo [10], Llorente and Mattila [9], Rao, Ruan and Wang [13], Xi and Xiong [17], [19], [20], [21]. As we known, two dust-like self-similar sets with the same dimension may not be Lipschitz equivalent. But they are quasi Lipschitz equivalent.

The notion of quasi-Lipschitz equivalence is introduced by Xi [18].

We say that two subsets A and B of Euclidean spaces are quasi-Lipschitz equivalent, if there is a bijection from A to B such that for all $x, y \in A$,

$$\frac{\log|f(x)-f(y)|}{\log|x-y|} \to 1 \text{ uniformly as } |x-y| \to 0.$$

It is proved in [18] that two self-conformal sets are quasi-Lipschitz equivalent if and only if they have the same dimension. This result was developed by Wang and Xi [14], [15], Xiong and Xi [22].

Li *et al.* [8] studied the quasi-Lipschitz equivalence for some homogeneous Moran sets in line, a special class of Moran sets. For any Moran set in this class, we have $c_{k,1} = \cdots = c_{k,n_k} = c_k$ and an additional assumption $\lim_{k\to\infty} \frac{\log n_1\cdots n_k}{\log c_1\cdots c_k} = s \in (0,1)$. We see that the condition $\lim_{k\to\infty} \frac{\log n_1\cdots n_k}{\log c_1\cdots c_k} = s$ is like (A3). But the condition $s < 1$ plays an important role which is like the separation condition for self-similar sets.

In this paper, we study the general Moran sets under the separation assumption:

(A4): For any $i_1 \cdots i_{k-1} i_k \in \tilde{\Sigma}^*$,

$$\frac{\log d(\bar{V}_{i_1\cdots i_{k-1}i_k}, F\backslash \bar{V}_{i_1\cdots i_{k-1}i_k})}{\log|\bar{V}_{i_1\cdots i_{k-1}}|} \to 1 \text{ uniformly as } k \to \infty.$$

Here $d(A, B)$ is the least distance between compact sets A and B.

For $s > 0$, a class $\mathcal{A}_s$ of Moran sets is defined by

$$\mathcal{A}_s = \begin{cases} \{F : F \text{ is a Moran set satisfying (A1)–(A4)}\} & \text{if } s \ge 1, \\ \{F : F \text{ is a Moran set satisfying (A1)–(A3)}\} & \text{if } s < 1. \end{cases}$$

Theorem 2. *For every $s > 0$, any two Moran sets in $\mathcal{A}_s$ are quasi-Lipschitz equivalent.*

We pose the last assumption:

(A5): There exists a constant $c > 0$ such that for any $i_1 \cdots i_{k-1} i_k \in \tilde{\Sigma}^*$,

$$\frac{|\bar{V}_{i_1 \cdots i_{k-1} i_k}|}{|\bar{V}_{i_1 \cdots i_{k-1}}|} = c_{k,i_k} \geq c \quad \text{and} \quad \frac{\min_{j_k \neq i_k} d(\bar{V}_{i_1 \cdots i_{k-1} i_k}, \bar{V}_{i_1 \cdots i_{k-1} j_k})}{|\bar{V}_{i_1 \cdots i_{k-1}}|} \geq c.$$

Then (A5) implies (A1)–(A2) (see Remark 3) and (A4).

Using Theorem 2, we have

Theorem 3. *Fix $s > 0$. Suppose F_1 and F_2 are Moran sets satisfying* (A3) *(with the same parameter s) and* (A5)*, then they are quasi-Lipschitz equivalent.*

Remark 6. In fact, (A3) and (A5) hold for self-similar sets satisfying the strong separation condition. Then we obtain the result in [18]: two dust-like self-similar sets are quasi-Lipschitz equivalent if and only if they have the same dimension.

The paper is organized as follows. Section 2 is the preliminaries, including the construction of the measure which plays an important role in this paper. Section 3 is the proof of Theorem 1. In Section 4, we get Theorem 2 based on the main result in [15]. In fact, [15] proved that if two compact sets are quasi Ahlfors–David s-regular and quasi uniformly disconnected, then they are quasi-Lipschitz equivalent.

2. Preliminaries

For $\sigma = \sigma_1 \cdots \sigma_k \in \tilde{\Sigma}_*$, let $|\sigma| (= k)$ denote its length. Write

$$(\sigma_1 \cdots \sigma_k) * \sigma_{k+1} = \sigma_1 \cdots \sigma_k \sigma_{k+1}.$$

We say that $\bar{V}_\sigma$ is of rank k, if $|\sigma| = k$. Without loss of generality, we assume the diameter

$$|\bar{V}_\emptyset| = 1.$$

If $\sigma = \sigma_1 \cdots \sigma_k$, then every $\bar{V}_\sigma$ is similar to $|\bar{V}_\emptyset|$ with ratio $c_{1,\sigma_1} c_{2,\sigma_2} \cdots c_{k,\sigma_k}$ and

$$|\bar{V}_\sigma| = c_{1,\sigma_1} c_{2,\sigma_2} \cdots c_{k,\sigma_k}. \tag{2.1}$$

2.1. Construction of measure. Fix $s > 0$. As in [2] by DAI *et al.*, we can define a probability measure supported on F depending on s.

Let $\mu(\bar{V}_\emptyset) = 1$, where $\emptyset$ is the empty word.

By induction, for every $k \geq 1$ and $\bar{V}_\sigma$ of rank $(k-1)$, we define

$$\mu(\bar{V}_{\sigma * i}) = \frac{c_{k,i}^s}{\sum_{j=1}^{n_k} c_{k,j}^s} \mu(\bar{V}_\sigma) \quad \text{for } 1 \leq i \leq n_k.$$

In fact, if $\sigma = \sigma_1 \cdots \sigma_k$, then

$$\mu(\bar{V}_\sigma) = \frac{|\bar{V}_\sigma|^s}{\prod_{i=1}^k \left(\sum_{j=1}^{n_i} c_{i,j}^s \right)}. \tag{2.2}$$

More and more, we get a probability measure supported on F. In fact, we use the condition (1.1) during the above construction of measure, as in the way by HUTCHINSON [7] for the open set condition.

2.2. Disconnectedness and quasi-Lipschitz equivalence. As above, we introduce the Moran set on the analogy of the self-similar sets satisfying the open set condition. Now, we will give some property of disconnectedness on the analogy of the self-similar sets satisfying the strong separation condition (SSC in short), here SSC holds for $E = \cup_{i=1}^m S_i(E)$, if

$$\min_{i \neq j} d(S_i(E), S_j(E)) > 0,$$

where $d(A, B)$ is the distance between A and B.

Definition 1 ([15]). We say that a subset K of Euclidean space is *quasi uniformly disconnected* if there is a function $\eta : (0, +\infty) \to (0, +\infty)$ with $\lim\limits_{t \to 0} \frac{\log \eta(t)}{\log t} = 1$ such that for any $x \in K$ and $r > 0$, there is a subset $B \subset K$ such that

$$K \cap B(x, \eta(r)) \subset B \subset B(x, r) \quad \text{and} \quad d(B, K \setminus B) > \eta(r). \tag{2.3}$$

We notice that if E is a self-similar set satisfying SSC, then E is quasi uniformly disconnected. In fact, we can take

$$\eta(r) = (\min_i r_i) \frac{\min_{i \neq j} d(S_i(E), S_j(E))}{|E|} \cdot r.$$

The following two lemmas come from [15] by WANG and XI.

Lemma 1. *Suppose that compact and quasi uniformly disconnected subsets E_1 and E_2 of Euclidean spaces are quasi Ahlfors–David s -regular. Then E_1 and E_2 are quasi Lipschitz equivalent.*

Lemma 2. *If E is quasi Ahlfors–David s-regular with $s \in (0,1)$, then E is quasi uniformly disconnected.*

In fact,Mattila and Saaranen [12] proved that if E is Ahlfors–David s-regular with $s \in (0,1)$, then E is uniformly disconnected.

3. Quasi Ahlfors–David Regularity

In this section, we will prove Theorem 1. Now, fix

$$s = \lim_{k\to\infty} s_k,$$

where $\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s_k}\right) = 1$.

3.1. Measure of $\bar{V}_\sigma$. We will estimate the measure of $\bar{V}_\sigma$.

Lemma 3. *There is a non-increasing function $\delta : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{k\to\infty}\delta(k) = 0$ and for any basic interval I_σ,*

$$|\bar{V}_\sigma|^{s+\delta(|\sigma|)} \le \mu(\bar{V}_\sigma) \le |\bar{V}_\sigma|^{s-\delta(|\sigma|)}.$$

Proof. Suppose $|\sigma| = k$. By (2.2), we only need to show

$$\frac{\log\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s}\right)}{\log|\bar{V}_\sigma|} \to 0 \text{ as } k\to\infty. \tag{3.1}$$

In fact, using (2.1) and $D_k = \max_i c_{k,i}$, we have

$$\left|\log|\bar{V}_\sigma|\right| \ge |\log(D_1\cdots D_k)|. \tag{3.2}$$

On the other hand, since $\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s_k}\right) = 1$, we have

$$\left|\log\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s}\right)\right|$$

$$= \left|\log\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s}\right) - \log\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s_k}\right)\right| = |g(s)-g(s_k)|,$$

where

$$g(x) = \sum_{i=1}^{k} \log\left(\sum_{j=1}^{n_i} c_{i,j}^{x}\right).$$

By Mean value theorem, there exists ξ_k lying in the interval with endpoints s and s_k such that

$$|g(s) - g(s_k)| = (s_k - s)\sum_{i=1}^{k} \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}}. \tag{3.3}$$

Write $\tilde{c_{i,j}} = \frac{c_{i,j}^{\xi_k}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}}$, then $\sum_{j=1}^{n_i} \tilde{c_{i,j}} = 1$ and

$$\begin{aligned}\sum_{i=1}^{k} \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}} &= \frac{1}{\xi_k}\cdot \sum_{i=1}^{k} \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}^{\xi_k}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}}\\ &= \frac{1}{\xi_k}\cdot\left(\sum_{i=1}^{k}\sum_{j=1}^{n_i} \tilde{c_{i,j}}\log \tilde{c_{i,j}} + \sum_{i=1}^{k}\log\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}\right).\end{aligned} \tag{3.4}$$

Using the convexity of function $f(x) = -x\log x$ on $[0,1]$, we have

$$\Big|\sum_{j=1}^{n_i} \tilde{c_{i,j}}\log \tilde{c_{i,j}}\Big| \le \log n_i \quad \text{and} \quad \prod_{i=1}^{k} D_i^{\xi_k} \le \prod_{i=1}^{k}\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \le \prod_{i=1}^{k} n_i. \tag{3.5}$$

Then it follows from assumption (A2) and (3.3)-(3.5) that

$$\lim_{k\to\infty} \frac{\big|\log \prod_{i=1}^{k}\big(\sum_{j=1}^{n_i} c_{i,j}^{s}\big)\big|}{\log(D_1\cdots D_k)} = 0. \tag{3.6}$$

Therefore, we get (3.1) by (3.2) and (3.6). □

3.2. Proof of Theorem 1. For $x \in F$ and r small enough, now we shall estimate

$$\frac{\log\mu(B(x,r))}{\log r}.$$

Let $N_r = \min\{k : d_1\cdots d_k \le r\} \to \infty$ as $r \to 0$. Let σ^- be the father of σ, that is $\sigma = \sigma^- * i_{|\sigma|}$ for some letter $i_{|\sigma|}$.

(1) *Lower bound*:

Using assumption (A1) $\lim_{k\to\infty} \frac{\log d_k}{\log(D_1 D_2\cdots D_{k-1})} = 0$, we have

$$\lim_{|\sigma|\to\infty} \frac{\log|\bar{V}_\sigma|}{\log|\bar{V}_{\sigma^-}|} = 1, \tag{3.7}$$

then there exists a non-increasing function $\alpha : \mathbb{N} \to (0, \infty)$ with $\lim_{k\to\infty} \alpha(k) = 0$ such that

$$|\bar{V}_{\sigma^-}| \geq |\bar{V}_\sigma| \geq |\bar{V}_{\sigma^-}|^{1+\alpha(|\sigma|)}. \tag{3.8}$$

Take a word σ such that

$$|\bar{V}_\sigma| \leq r \quad \text{and} \quad |\bar{V}_{\sigma^-}| > r, \tag{3.9}$$

Then by Lemma 3, we have

$$\mu(B(x, r)) \geq \mu(\bar{V}_\sigma) \geq |\bar{V}_\sigma|^{s+\delta(|\sigma|)}. \tag{3.10}$$

It follows from (3.8), (3.9) and (3.10) that

$$\mu(B(x, r)) \geq |\bar{V}_{\sigma^-}|^{(s+\delta(|\sigma|))(1+\alpha(|\sigma|))} > r^{(s+\delta(N_r))(1+\alpha(N_r))}, \tag{3.11}$$

where

$$(s + \delta(N_r))(1 + \alpha(N_r)) \to s \text{ as } r \to 0.$$

(2) *Upper bound*:

Let

$$\Omega_{x,r} = \{\sigma : B(x, r) \cap \bar{V}_\sigma \neq \varnothing \quad \text{and} \quad |\bar{V}_\sigma| \leq r, |\bar{V}_{\sigma^-}| > r\},$$

and $l_{x,r} := \min_{\sigma\in\Omega_{x,r}} |\bar{V}_\sigma| = \min_{\sigma\in\Omega_{x,r}} |V_\sigma|$.

It follows from Lemma 3 that for $\sigma \in \Omega_{x,r}$,

$$\mu(\bar{V}_\sigma) \leq r^{s-\delta(N_r)}. \tag{3.12}$$

By (3.8), we have

$$l_{x,r} \geq r^{1+\alpha(N_r)} \text{ with } \alpha(N_r) \to 0 \quad \text{as} \quad r \to 0. \tag{3.13}$$

Denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$. Since $\bigcup_{\sigma\in\Omega_{x,r}} V_\sigma$ is a disjoint union contained in $B(x, 2r)$, we have

$$\begin{aligned}\mathcal{L}^n B(x, 2r) \geq \mathcal{L}^n(\bigcup_{\sigma\in\Omega_{x,r}} V_\sigma) &= \sum_{\sigma\in\Omega_{x,r}} \mathcal{L}^n(V_\sigma) \\ &= \sum_{\sigma\in\Omega_{x,r}} \mathcal{L}^n(V_\emptyset) \cdot \frac{|V_\sigma|^n}{|V_\emptyset|^n} \geq \frac{\mathcal{L}^n(V_\emptyset)}{|V_\emptyset|^n}(\#\Omega_{x,r})(l_{x,r})^n.\end{aligned}$$

This means the cardinality

$$\#\Omega_{x,r} \leq C_1 \frac{r^n}{(l_{x,r})^n} \tag{3.14}$$

for some constant $C_1 > 0$.

For any $y \in F \cap B(x,r)$, we can find a word σ with $y \in \bar{V}_\sigma$ such that $|\bar{V}_\sigma| \le r$ and $|\bar{V}_{\sigma^-}| > r$. That means

$$F \cap B(x,r) \subset \bigcup_{\sigma \in \Omega_{x,r}} \bar{V}_\sigma. \tag{3.15}$$

By (3.12)–(3.15), we have

$$\begin{aligned}\mu(B(x,r)) = \mu(F \cap B(x,r)) &\le \mu\Big(\bigcup_{\sigma \in \Omega_{x,r}} \bar{V}_\sigma\Big) \le \#\Omega_{x,r} \cdot \max_{\sigma \in \Omega_{x,r}} \mu(\bar{V}_\sigma) \\ &\le C_1 \frac{r^n}{(l_{x,r})^n} \max_{\sigma \in \Omega_{x,r}} \mu(\bar{V}_\sigma) \le C_1 \cdot r^{s-\delta(N_r)-n\alpha(N_r)},\end{aligned}$$

where

$$s - \delta(N_r) - n\alpha(N_r) \to s \quad \text{as} \quad r \to 0.$$

Then Theorem 1 follows from the estimates of lower and upper bounds.

4. Quasi-Lipschitz equivalence

In this section, we will prove Theorem 2.

For $E_1, E_2 \in \mathcal{A}_s$, Theorem 1 implies that they are quasi Ahlfors–David s-regular.

By Lemmas 1 and 2, we only need to show that for $s \ge 1$ any set in $F \in \mathcal{A}_s$ is quasi uniformly disconnected by using assumption (A4).

In fact, given $F \in \mathcal{A}_s$ with $|\bar{V}_\emptyset| = 1$, the assumption (A4) shows that for any $\bar{V}_\sigma$,

$$\lim_{|\sigma| \to \infty} \frac{\log d(\bar{V}_\sigma, F \backslash \bar{V}_\sigma))}{\log |\bar{V}_{\sigma^-}|} = 1. \tag{4.1}$$

Using assumption (A1) $\lim_{k\to\infty} \frac{\log d_k}{\log(D_1 D_2 \cdots D_{k-1})} = 0$, we have

$$\lim_{|\sigma| \to \infty} \frac{\log |\bar{V}_\sigma|}{\log |\bar{V}_{\sigma^-}|} = 1. \tag{4.2}$$

By (4.1) and (4.2), we can take a non-increasing function $\varepsilon : \mathbb{N} \to (0, \infty)$ with $\lim_{k\to\infty} \varepsilon(k) = 0$ such that

$$|\bar{V}_{\sigma^-}| \ge d(\bar{V}_\sigma, F \backslash \bar{V}_\sigma)) \ge |\bar{V}_{\sigma^-}|^{1+\varepsilon(|\sigma|)} \tag{4.3}$$

and

$$|\bar{V}_{\sigma^-}| \ge |\bar{V}_\sigma| \ge |\bar{V}_{\sigma^-}|^{1+\varepsilon(|\sigma|)}. \tag{4.4}$$

Take
$$\eta(r) = r^{1+\varepsilon(N_r)},$$
where $N_r = \min\{k : d_1 \cdots d_k \le r\} \to \infty$ as $r \to 0$ which implies
$$\lim_{r\to 0} \frac{\log \eta(r)}{\log r} = 1.$$

For $B(x,r)$ with $x \in F$ and r small enough, take a word σ such that
$$|\bar{V}_\sigma| \le r \quad \text{and} \quad |\bar{V}_{\sigma^-}| > r.$$
Then by (4.3)–(4.4), we have
$$F \cap B(x, \eta(r)) \subset \bar{V}_\sigma \subset B(x,r)$$
and
$$d(\bar{V}_\sigma, F \backslash \bar{V}_\sigma) \ge \eta(r).$$
Hence F is quasi uniformly disconnected.

Applying Lemma 1, we obtain Theorem 2.

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