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Number of representations of integers by binary forms

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Abstract. We give improved upper bounds for the number of solutions of the Thue equation $F(x,y)=h$ where F is an irreducible binary form of degree ≥ 3.

1. Introduction

Throughout this paper, let $F(x,y) = a_0x^r + a_1x^{r-1}y + \cdots + a_ry^r$ be an irreducible binary form of degree $r \geq 3$, with integer coefficients. We assume without loss of generality that the content of F, i.e. $\gcd(a_0,\ldots,a_r)$ is 1. Let h be a nonzero integer. In a seminal work in 1909, Thue proved that the equation

$$F(x,y)=h \tag{1}$$

has only finitely many solutions in integers x and y. For this purpose, he developed a method based on Diophantine approximation of algebraic numbers by rationals. Since then, these equations are known as Thue equations. Thue's method does not give any bound for the size of solutions, thus it is ineffective. Nevertheless, it can be used to give bounds for the number of solutions. Let $N_F(r,h)$ denote the number of primitive solutions of (1), i.e., solutions (x,y) with $\gcd(x,y)=1$. In 1929, Siegel proposed that $N_F(r,h)$ can be bounded by a function depending only on r and h, otherwise independent of F, i.e., there exists a positive number $Z(r,h)$ depending on r and h such that

$$N_F(r,h) \leq Z(r,h)$$

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for any form F of degree r. In 1983, EVERTSE [5] showed that $Z(r,h)$ can be taken as

$$7^{15\left(\binom{r}{3}+1\right)^2} + 6 \times 7^{2\binom{r}{3}(\omega(h)+1)}$$

where $\omega(h)$ denotes the number of distinct prime factors of h. A closely related equation is

$$|F(x,y)| = h. \tag{2}$$

If (x,y) is a solution of (2), then $(-x,-y)$ is also a solution of (2). Let $N_F^{(1)}(r,h)$ denote the number of primitive solutions of (2) with (x,y) and (-x,-y) identified as one solution. Clearly, $N_F(r,h) + N_F(r,-h) = 2N_F^{(1)}(r,h)$. Suppose

$$N_F^{(1)}(r,h) \leq Z^{(1)}(r,h). \tag{3}$$

Then $N_F(r,h)+N_{-F}(r,h) \leq 2Z^{(1)}(r,h)$. Thus we may take $Z(r,h) = 2Z^{(1)}(r,h)$. In 1987, BOMBIERI and SCHMIDT [2] proved that there is an absolute constant c_1 such that

$$N_F^{(1)}(r,h) \leq c_1 r^{1+\omega(h)},$$

for any form F of degree r. Further they showed that $c_1 = 215$ if r is large. In 1991, STEWART [12] showed that

$$N_F(r,h) = 2800\left(1 + \frac{1}{8\epsilon r}\right) r^{1+\omega(g)} \tag{4}$$

for all $r \geq 3$ with ϵ any positive real number and g is a divisor of h satisfying some conditions (see (7) below or Theorem 1 of [12]). This is a refinement of the result of Bombieri and Schmidt. In the above results, the method is based on counting large and small solutions for certain forms equivalent to F and having large discriminant. In [13], ZHANG and YUAN obtained

$$N_F^{(1)}(r,h) \leq 21r^{1+\omega(h)}$$

if $D(F) \geq 19^{r(r-1)}$ and $r \geq 24$. They gave similar bounds for $4 \leq r \leq 23$. On the other hand, using linear forms in logarithms and geometry of numbers, AKHTARI [1] has shown that

$$N_F^{(1)}(r,h) \leq (11r-2)r^{\omega(h)}$$

if the discriminant of F is larger than some effectively computable constant which depends only on r. In the results of [1], [2], and [13], the exponent of r is $1+\omega(h)$.

In fact, following [2], in these papers, it was enough to find an upper bound for $N_F^{(1)}(r,1)$ and then

$$N_F^{(1)}(r,h) \le r^{\omega(h)} N_F^{(1)}(r,1).$$

Upper bounds for $N_F^{(1)}(r,h)$ were also considered by Győry in [6] and [7]. One may easily see from the result of Győry in [6, Corollary 3] that

$$N_F^{(1)}(r,h) \le 25r + (r+2)\frac{\theta+8}{4\theta}$$

if

$$|D(F)| > r^r(3.5^r h^2)^{\frac{2(r-1)}{1-\theta}}$$

for any θ with $0<\theta<1$. In [7, Theorem 1.G(ii)], by fixing $\theta = 1/3$, he showed that

$$N_F^{(1)}(r,h) \le 32r + 11$$

if

$$|D(F)| \ge (3^r h)^{6(r-1)}.$$

We improve the results mentioned above from the papers [1], [2], [6], [7] and [13] as follows.

Theorem 1. *We have*

$$N_F^{(1)}(r,h) \le c_0 r^{1+\omega(h)}$$

where

$$c_0 = \begin{cases} 210 & \text{if } r \ge 23 \\ 236 & \text{if } 14 \le r \le 22. \end{cases} \tag{5}$$

For $3 \le r \le 13$, the values of c_0 are given in Table 1. Further

$$N_F^{(1)}(r,h) \le c_0' r^{1+\omega(h)} \quad \text{if } |D(F)| \ge p_0^{r(r-1)}$$

where

$$(c_0', p_0) = \begin{cases} (10, 29) & \text{if } r \ge 24 \\ (10, 53) & \text{if } 18 \le r \le 23 \\ (10.88, 53) & \text{if } 14 \le r \le 17. \end{cases} \tag{6}$$

For $3 \le r \le 13$, the values of (c_0', p_0) are given in Table 1.

Note.

(i) The value of c_0 in (5) corresponds to $c_1 = 215$ in the result of Bombieri and Schmidt mentioned earlier. Thus Theorem 1 is explicit and gives a better estimate for all $r \ge 23$.

(ii) The value of c_0' in (6) is better than that of [1] for $r \geq 11$. We do not use linear forms in logarithms and geometry of numbers as in [1]. For all values of $r \geq 4$, the value of c_0' is better than those obtained in [13].

(iii) We choose p_0 large to make c_0' small. On the other hand, since it is known that $c_0 = c_0'(p_0+1)$, it is calculated by choosing that p_0 for which $c_0'(p_0+1)$ is small.

In 1938, Erdős and Mahler [4] had shown that if F has nonzero discriminant, $h > c_2$ and g a divisor of h with $g > h^{6/7}$ then

$$N_F(r,h) \leq c_3 r^{1+\omega(g)},$$

where c_2 and c_3 are positive numbers depending only on F. Stewart [12] improved this result as follows. For any prime p and integers $r \geq 2, k$ and $D \neq 0$, $g \neq 0$, define

$$T(r,k,p,D) = \min\left\{\frac{r-1}{r}k, \min_{0\leq j\leq r-2}\left(\frac{\operatorname{ord}_p D}{(j+1)(j+2)} + \frac{j}{j+2}k\right)\right\}.$$

Let

$$G(g,r,D(F)) = \prod_{p|g} p^{T(r,\operatorname{ord}_p g,p,D(F))}.$$

Then (4) holds provided

$$g^{1+\epsilon}|D(F)|^{1/r(r-1)} \geq G(g,r,D(F))|h|^{\frac{2}{r}+\epsilon}, \epsilon > 0. \tag{7}$$

Remark 1. Since $r \geq 3$, the power of $|h|$ in (7) is less than 6/7, thus sharpening the result of Erdős and Mahler.

Remark 2. Suppose $g = |h|$. From the definition of T, we get

$$T(r,\operatorname{ord}_p g,p,D(F)) \leq \frac{\operatorname{ord}_p D(F)}{r(r-1)} + \frac{r-2}{r}\operatorname{ord}_p g.$$

Hence

$$G(g,r,D(F)) \leq g^{\frac{r-2}{r}}|D(F)|^{r(r-1)} \leq |h|^{\frac{r-2}{r}}|D(F)|^{r(r-1)}.$$

So (7) holds with g replaced by h.

Remark 3. It is well known that $\omega(h)$ has normal order $\log\log h$. Suppose $\psi(X,Y)$ denotes the Dickman function which counts the number of integers $\leq X$ having all its prime factors $\leq Y$. These are $Y-$smooth numbers which are very well studied. See [8] for a survey of smooth numbers. The following estimate is due to RANKIN, see [3].

$$\psi(X,Y) \le X \exp\left\{-\frac{\log_3 Y}{\log Y}\log X + \log_2 Y + O\left(\frac{\log_2 Y}{\log_3 Y}\right)\right\}$$

Taking $X = h$, we find that the number of integers not exceeding h and having very small prime factors are *few* in number. Hence for a positive proportion of h, we may take g to be a prime satisfying (7). Then $\omega(g) = 1$ and we get

$$Z(r,h) = 2800\left(1 + \frac{1}{8\epsilon r}\right)r^2.$$

There are values of h for which $\omega(h)$ is as large as $\frac{c\log h}{\log\log h}$ with c an absolute constant while $\omega(g) = 1$. Hence the above estimate is much better. We improve the result of Stewart in the following theorem.

Theorem 2. *Suppose g is a divisor of h such that*

$$|D(F)| \ge \left(\max\left(1, \frac{(G(g,r,D(F))^r}{g^{r-2}}\right)\right)^{r-1}\left(\frac{h}{g}\right)^{\mu} \tag{8}$$

with $\mu = \mu_1(r-1)$, say. Let c_0, c_0', p_0 be as in Theorem 1. Then

(i) $N_F(r,h) \le 2c_0 r^{1+\omega(g)}$ *if* $\mu_1 = 2.83$.

(ii) $N_F(r,h) \le 2c_0' r^{1+\omega(g)}$ *if* $|D(F)| \ge p_0^{r(r-1)}$ *and*

$$\mu_1 = \begin{cases} 3.066 & \text{if } r \ge 24 \\ 3.62 & \text{if } 14 \le r \le 23 \end{cases}$$

and as in Table 1 for $3 \le r \le 13$.

Remark 4. Assume $G(g,r,D(F)) \ge g^{1-2/r}$. On comparing the condition for $|D(F)|$ in (8) with that of (7) due to Stewart, we find that (8) is better whenever $2 + \epsilon r \ge \mu_1$. Thus (8) is better for $\epsilon \ge .28$ if (i) holds and $\epsilon \ge .045$ for $r \ge 24$; $\epsilon \ge .116$ for $14 \le r \le 23$ if (ii) holds. Similar remark holds for $4 \le r \le 13$ by using Table 1.

Our method is based on the Thue–Siegel principle as enunciated in [2] and Diophantine approximation methods. We divide the primitive solutions (x,y) according as $0 \le y < Y_0$, $Y_0 \le y < M(F)^q$ and $y \ge M(F)^q$ where Y_0 and q are chosen judiciously depending on r. In fact, for the calculation of c_0 we find that

$Y_0 = 3$ gives a better value and for computing c_0', $Y_0 = 2$ for $r \geq 11$ and $Y_0 = 1$ for $4 \leq r \leq 10$ yield better bounds. This is a simple analogue of *small, medium* and *large* solutions considered by MUELLER and SCHMIDT [11]. The parameter q is taken as 2 in all earlier works. Here we find that it is more economical to take q smaller than 2 for large values of r. For instance for $r \geq 24$, q is taken as 1.54 for computing c_0'(see the proof of Theorem 1). These choices result in the improved bounds given in Theorems 1 and 2.

2. Lemma on discriminant

Suppose $\gamma_1, \ldots, \gamma_r$ denote the roots of the equation $F(x,1) = 0$. Denote by

$$D(F) = a_0^{2r-2} \prod_{i<j} (\gamma_i - \gamma_j)^2$$

and

$$M(F) = |a_0| \prod_{i=1}^{r} \max(1, |\gamma_i|),$$

the discriminant and Mahler height of F, respectively. We begin with an elementary result which describes the change in the discriminant of a form when an element of $GL(2,\mathbb{Z})$ acts on it.

Lemma 3. *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z})$ and let $F_A(x,y)$ denote the form $F(ax+by, cx+dy)$. Then $D(F_A) = (\det A)^{r(r-1)} D(F)$.*

PROOF. The coefficient of x^r in $F_A(x,y)$ is $F_A(1,0) = F(a,c)$. Therefore

$$D(F_A) = F(a,c)^{2r-2} \prod_{i<j} (\beta_i - \beta_j)^2,$$

where $\beta_1, \ldots, \beta_r$ denote the roots of the equation $F_A(x,1) = 0$. Let $1 \leq i \leq r$. Since $F_A(\beta_i, 1) = 0$, we have

$$F(a\beta_i + b, c\beta_i + d) = 0.$$

Hence

$$\frac{a\beta_i + b}{c\beta_i + d} = \gamma_j \quad \text{for some } j \text{ with } 1 \leq j \leq r$$

which gives

$$\beta_i = \frac{\gamma_j d - b}{a - \gamma_j c}.$$

By changing the indices, if necessary, we may assume that

$$\beta_i = \frac{\gamma_i d - b}{a - \gamma_i c}, \quad 1 \le i \le r.$$

For $i \neq j$, we have

$$\beta_i - \beta_j = \frac{\gamma_i d - b}{a - \gamma_i c} - \frac{\gamma_j d - b}{a - \gamma_j c} = \frac{(\det A)(\gamma_i - \gamma_j)}{(a - \gamma_i c)(a - \gamma_j c)}.$$

Observe that

$$(F(a, c))^{r-1} = \prod_{i<j} (a - \gamma_i c)(a - \gamma_j c).$$

Therefore

$$\begin{aligned} D(F_A) &= F(a, c)^{2r-2} \prod_{i<j} \left(\frac{(\det A)(\gamma_i - \gamma_j)}{(a - \gamma_i c)(a - \gamma_j c)} \right)^2 \\ &= (\det A)^{r(r-1)} a_0^{2r-2} \prod_{i<j} (\gamma_i - \gamma_j)^2 = (\det A)^{r(r-1)} D(F). \end{aligned}$$

□

3. Equation (2) when h has a large divisor g

Stewart [12] expanded the $p-$ adic technique of Bombieri and Schmidt [2] to reduce the problem of solving (2) to a set of equalities where the forms have large discriminant and h is reduced to h/g where g is a divisor of h satisfying some conditions. The following lemma is an adaptation of ([12], Theorem 1).

Lemma 4. *Let g be a divisor of h such that*

$$\frac{g^{(r-2)(r-1)} |D(F)|}{G(g, r, D(F))^{r(r-1)}} \ge \left(\frac{h}{g} \right)^{\mu}. \tag{9}$$

Then, there is a set W of at most $r^{\omega(g)}$ binary forms with the property that distinct primitive solutions (x, y) of (2) correspond to distinct triples $(\widetilde{F}, x', y')$ where $\widetilde{F}$ is in W and (x', y') is a pair of co-prime integers for which

$$|\widetilde{F}(x', y')| = \frac{h}{g}.$$

Further, if $\widetilde{F}$ is in W, then

$$C(\widetilde{F}) = 1 \quad \text{and} \quad |D(\widetilde{F})| \ge \left(\frac{h}{g} \right)^{\mu}.$$

PROOF. We follow the arguments of the proof of Theorem 1 in [12]. Suppose (x, y) is a primitive solution of (2). Let p be a prime divisor of h and let k denote the highest power of p dividing h. Then

$$F(x, y) \equiv 0 \pmod{p^k}.$$

Let $p \nmid y$. Then

$$F(xy^{-1}, 1) \equiv 0 \pmod{p^k}.$$

Let Ω_p be the completion of the algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. We denote the p-adic value in $\mathbb{Q}_p$ and its extension to Ω_p by $|\ |_p$. Consider the ring R_p of elements in Ω_p whose p-adic value is ≤ 1. Let s be the number of zeros of $F(z, 1)$ in R_p.

By Theorem 2 of [12], there is an integer $t = t(k)$, $0 \leq t \leq s$ and integers $b_1, \dots, b_t, u_1, \dots, u_t$ with $0 \leq u_i \leq T = T(r, k, p, D(F))$ such that

$$xy^{-1} \equiv b_i \pmod{p^{k-u_i}} \quad \text{for some } i$$

i.e., there is an integer A such that

$$x = p^{k-u_i} A + b_i y.$$

For $1 \leq i \leq t$, put

$$F_i(X, Y) = F(p^{k-u_i} X + b_i Y, Y).$$

By Theorem 2 of [12], p^k divides $C(F_i)$. Since

$$|F_i(A, y)| = |F(p^{k-u_i} A + b_i y, y)| = |F(x, y)| = h$$

and k is the highest power of p dividing h, k is also the highest power of p dividing $C(F_i)$. Let $q \neq p$ be a prime dividing $C(F_i)$. Let $P = p^{k-u_i}$ and $Q = b_i$. Then $F_i(X, Y) = F(PX + QY, Y)$. So

$$F_i(X, Y) = a_0 P^r X^r + (a_0 r P^{r-1} Q + a_1 P^{r-1}) X^{r-1} Y + \ldots + a_r Y^r.$$

Since q divides each of the coefficients and is co-prime to P, we obtain that q divides $C(F)$, which is 1. Thus $C(F_i) = p^k$. Put $\widetilde{F}_i(X, Y) = p^{-k} F_i(X, Y)$. Then $C(\widetilde{F}_i) = 1$. Also

$$\begin{aligned}
\widetilde{F}_i(X, Y) &= F_i\left(\begin{pmatrix} p^{-k/r} & 0 \\ 0 & p^{-k/r} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}\right) \\
&= F\left(\begin{pmatrix} p^{k-u_i} & b_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-k/r} & 0 \\ 0 & p^{-k/r} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}\right).
\end{aligned}$$

Therefore by Lemma 3,

$$D(\widetilde{F}_i) = p^{-2k(r-1)} p^{(k-u_i)r(r-1)} D(F).$$

Also

$$\widetilde{F}_i(A, y) = p^{-k} F_i(x, y) = hp^{-k} \quad \text{for} \quad 1 \leq i \leq t.$$

Let now $p|y$. Then x is invertible modulo p^k since $\gcd(x, y) = 1$. In this case,

$$F(1, yx^{-1}) \equiv 0 \pmod{p^k}.$$

By Theorem 2 of [12] there exist integers $w = w(k)$, $b_{t+1}, \ldots, b_w, u_{t+1}, \ldots, u_w$ with $0 \leq u_i \leq T = T(r, k, p, D(F))$ such that

$$yx^{-1} \equiv b_i \pmod{p^{k-u_i}} \quad \text{for some } i \text{ with } t+1 \leq i \leq w.$$

Let s_1 be the number of zeros α of $F(1, z)$ with $|\alpha|_p < 1$. Then $w - t \leq s_1$. Every non zero root of $F(1, z)$ is the inverse of a non zero root of $F(z, 1)$. Hence s_1 is the number of non zero roots γ of $F(z, 1)$ with $|\gamma|_p > 1$. Therefore

$$w \leq t + s_1 \leq s + s_1 = r.$$

We argue as before, with the roles of x and y interchanged to obtain

$$y = p^{k-u_i} A' + b_i x,$$

and a form

$$\widetilde{F_i'}(X, Y) = p^{-k} F(X, b_i X + p^{k-u_i} Y)$$

such that

$$D(\widetilde{F_i'}) = p^{-2k(r-1)} p^{(k-u_i)r(r-1)} D(F)$$

and

$$|\widetilde{F_i'}(x, A')| = p^{-k} h \quad \text{for} \quad t+1 \leq i \leq w.$$

By repeating this process for each prime factor of g, we get a set W of at most $r^{\omega(g)}$ binary forms with the property that distinct primitive solutions (x, y) of (2) correspond to distinct triples $(\widetilde{F}, x', y')$ where $\widetilde{F}$ is in W and (x', y') is a pair of co-prime integers for which

$$|\widetilde{F}(x', y')| = \frac{h}{g}.$$

Further,

$$C(\widetilde{F}) = 1.$$

Suppose $g = p_1^{k_1} \dots p_l^{k_l}$. Then

$$\begin{aligned}\left|D(\widetilde{F})\right| &= \frac{(p_1^{k_1-u_1} \dots p_l^{k_l-u_l})^{r(r-1)}}{(p_1^{k_1} \dots p_l^{k_l})^{2r-2}} |D(F)| \\ &= \frac{g^{r(r-1)}}{g^{2r-2}} (p_1^{-u_1} \dots p_l^{-u_l})^{r(r-1)} |D(F)| \geq \frac{g^{(r-1)(r-2)}|D(F)|}{\left(\prod_{p|g} p^{T(r,\mathrm{ord}_p\, g, p, D(F))}\right)^{r(r-1)}}\end{aligned}$$

since $u_i \leq T(r, \mathrm{ord}_p\, g, p, D(F))$ for $1 \leq i \leq l$. Thus

$$\left|D(\widetilde{F})\right| \geq \frac{g^{(r-1)(r-2)}|D(F)|}{G(g,r,D(F))^{r(r-1)}} \geq \left(\frac{h}{g}\right)^{\mu}$$

by (9). □

4. Forms with discriminant larger than a power of a prime

By Lemma 4 and the discussions in the Introduction it is enough to consider forms F satisfying

$$|F(x,y)| = n \text{ with } C(F) = 1 \quad \text{and } |D(F)| \geq n^{\mu} \tag{10}$$

where μ is as given in (9). Let $N_F^{(2)}(r,n)$ denote the number of primitive solutions of (10). We give an upper bound for $N_F^{(2)}(r,n)$ in terms of the number of solutions of forms having even larger discriminant. Let p be a prime number and G an irreducible form of degree r satisfying

$$|G(x,y)| = n \text{ with } C(G) = 1 \quad \text{and } |D(G)| \geq p^{r(r-1)} n^{\mu}. \tag{11}$$

Let $A \in SL(2,\mathbb{Z})$. Then G_A has $C(G_A) = 1$ and $|D(G_A)| \geq p^{r(r-1)} n^{\mu}$. Also

$$|G_A(x,y)| = n$$

has the same number of solutions as $|G(x,y)| = n$. Hence it is enough to consider (11) with G having smallest Mahler height among all forms $SL(2,\mathbb{Z})-$equivalent to it. Let $N^{(1)}(r,n;p)$ denote the maximum number of solutions of (11) for all forms G.

Lemma 5. *We have*

$$N_F^{(2)}(r,n) \leq (p+1) N^{(1)}(r,n;p).$$

Further for any form G with $D(G)$ satisfying the condition in (11) *we have*

$$M(G) \geq p^{r/2} n^{\mu/(2r-2)} r^{-r/(2r-2)}. \tag{12}$$

Proof. Let (x,y) be a primitive solution of (10). Suppose

$$A_0 = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A_j = \begin{pmatrix} 0 & -1 \\ p & j \end{pmatrix} \quad \text{for } 1 \le j \le p.$$

Then $\binom{x}{y} = A_j \binom{x'}{y'}$ for some $j \in \{0,1,\dots,p\}$ and some integers x', y'. For, if x is divisible by p, we can take $j=0$, $x' = x/p$ and $y' = y$. If x and p are co-prime, there exist integers a and b such that $ax + bp = 1$. Now,

$$\begin{aligned} y &= x(ay) + p(by) = x(pq - j) + p(by) \text{ for some integers } q \text{ and } j \text{ with } 1 \le j \le p \\ &= x(-j) + p(xq + by). \end{aligned}$$

Taking $x' = xq + by$ and $y' = -x$, we get $\gcd(x', y') = 1$ and $\binom{x}{y} = A_j \binom{x'}{y'}$. Since x and y satisfy

$$|F(x,y)| = n,$$

x' and y' satisfy

$$|F_{A_j}(x', y')| = n.$$

Thus, for every primitive solution of $|F(x,y)| = n$, there exists a primitive solution of $|F_{A_j}(x',y')| = n$ for some j with $0 \le j \le p$. So, if n_j denotes the number of solutions of $|F_{A_j}(x,y)| = n$ with $0 \le j \le p$, we get

$$N_F^{(1)}(r,n) \le n_0 + n_1 + \ldots + n_p.$$

Also note that

$$|D(F_{A_j})| \ge p^{r(r-1)}.$$

Hence

$$N_F^{(1)}(r,n) \le (p+1) N^{(1)}(r,n;p).$$

It is a well-known result of Mahler [10] that

$$|D(F)| \le r^r \; M(F)^{2r-2}.$$

Therefore the Mahler height of such forms satisfies

$$M(F) \ge p^{r/2} n^{\mu/(2r-2)} r^{-r/(2r-2)}. \qquad \square$$

5. Lemma of Lewis and Mahler

The following lemma is a refinement, due to Stewart [12, Lemma 3], of an estimate of Lewis and Mahler [9].

Lemma 6. *Let $G(x,y)$ be an irreducible form of degree r. For any (x,y) with $y \neq 0$, we have*

$$\min_{\alpha} \left| \alpha - \frac{x}{y} \right| \leq \frac{2^{r-1} r^{(r-1)/2} (M(G))^{r-2} |G(x,y)|}{|D(G)|^{1/2} |y|^r},$$

where the minimum on the left is over the roots of $G(x,1)$.

As an immediate consequence of the above lemma, we get the following corollary.

Corollary 1. *Let G be an irreducible form of degree r satisfying* (11). *Suppose $\mu \geq 2$ and $p \geq 3$. Then*

$$\left| \alpha_0 - \frac{x}{y} \right| := \min_{\alpha} \left| \alpha - \frac{x}{y} \right| \leq \frac{M(G)^{r-2}}{2|y|^r}. \tag{13}$$

□

6. Thue–Siegel principle

For the purpose of stating Thue–Siegel principle as given in ([2], p. 74), we introduce some notations. Let t, τ be positive numbers such that

$$t < \sqrt{2/r}, \quad \sqrt{2 - rt^2} < \tau < t.$$

Put

$$\lambda = \frac{2}{t - \tau}$$

and

$$A_1 = \frac{t^2}{2 - rt^2} \left(\log M(G) + \frac{r}{2} \right).$$

Suppose that $\lambda < r$. We say that a rational number x/y is a *very good approximation* to an algebraic number α of degree r if

$$\left| \alpha - \frac{x}{y} \right| < (4e^{A_1} H(x,y))^{-\lambda}$$

where $H(x,y) = \max(|x|, |y|)$.

Lemma 7. *If α is of degree r and x/y, x'/y' are two very good approximations to α, then*

$$\log(4e^{A_1}) + \log H(x', y') \leq \delta^{-1}\{\log(4e^{A_1}) + \log H(x, y)\}$$

where

$$\delta = \frac{rt^2 + \tau^2 - 2}{r - 1}.$$

For application we choose

$$t = \sqrt{2/(r + a^2)}, \quad \tau = bt$$

with $0 < a < b < 1$. Then

$$\lambda = \frac{2}{(1-b)t} = \frac{\sqrt{2(r + a^2)}}{1 - b} > \frac{\sqrt{2r}}{1 - b}, \tag{14}$$

$$A_1 = \frac{1}{a^2}\left(\log M(G) + \frac{r}{2}\right) \quad \text{and} \quad \delta = \frac{2(b^2 - a^2)}{(r-1)(r + a^2)}. \tag{15}$$

7. Large solutions

In this section we estimate the number of primitive solutions (x, y) of (11) when y is large.

Lemma 8. *Let G be a form satisfying* (11). *Suppose $y \geq M(G)^q$ with $q > 1$. Let B be a number satisfying*

$$B = \frac{1}{2}\left(\frac{r^{\frac{r}{2r-2}}}{p^{\frac{r}{2}}}\right)^3 + 1. \tag{16}$$

Then

$$|x| \leq BM(G)|y|. \tag{17}$$

Proof. By (13),

$$\left|\alpha_0 - \frac{x}{y}\right| \leq \frac{M(G)^{r-2}}{2|y|^r} = \frac{1}{2M(G)^{(q-1)r+2}} \leq \frac{1}{2M(G)^2}.$$

This implies that

$$\left|\frac{x}{y}\right| \leq |\alpha_0| + \frac{1}{2M(G)^2} \leq M(G) + \frac{1}{2M(G)^2}$$

by the definition of $M(G)$. From (12) and (16) we get

$$B-1 \geq \frac{1}{2}\left(\frac{r^{\frac{r}{2r-2}}}{p^{\frac{r}{2}}}\right)^3 \geq \frac{1}{2M(G)^3}.$$

Thus

$$\left|\frac{x}{y}\right| \leq M(G)+(B-1)M(G)=BM(G).$$

This proves the lemma. □

Let α_i be a root of $G(x,1)=0$. In the next lemma we count all those large primitive solutions (x,y) of (11) which are closest to α_i.

Lemma 9. *Let G be a form satisfying* (11). *For $1 \leq i \leq r$, set*

$$I_i=\left\{(x,y):\left|\alpha_i-\frac{x}{y}\right|=\min_{\alpha}\left|\alpha-\frac{x}{y}\right| \text{ and } y \geq M(G)^q\right\}.$$

Let a, b, λ,δ and B be as given in sections 6 and 7 with $r>\lambda$. Let

$$\nu=\frac{\log(4B)+(r/2a^2)}{\frac{r\log p}{2}-\frac{r\log r}{2r-2}} \quad \text{and} \quad \eta=\frac{(\nu+2+\frac{1}{a^2})\lambda-2}{r-\lambda}. \tag{18}$$

Then

$$|I_i| \leq 2+\left[\frac{\log\eta-\log(q-1)}{\log(r-1)}\right]+\left[\frac{1}{\log(r-1)}\log\left(\frac{(\nu+2+\frac{1}{a^2})r-2}{\delta((\nu+2+\frac{1}{a^2})\lambda-2)}\right)\right].$$

Proof. Enumerate the primitive solutions in I_i as $(x_1,y_1),(x_2,y_2),\dots$ with $y_1 \leq y_2 \leq \cdots$. Put $y_j=M(G)^{1+\delta_j}$. Then $1+\delta_j \geq q>1$. Hence $\delta_j>0$ for $j \geq 1$. Further

$$\frac{1}{y_jy_{j+1}} \leq \left|\frac{x_{j+1}}{y_{j+1}}-\frac{x_j}{y_j}\right| \leq \left|\frac{x_{j+1}}{y_{j+1}}-\alpha_i\right|+\left|\alpha_i-\frac{x_j}{y_j}\right| \leq \frac{M(G)^{r-2}}{y_j^r}$$

by (13). Thus,

$$y_j^{r-1} \leq M(G)^{r-2}y_{j+1}$$

i.e.,

$$M(G)^{(1+\delta_j)(r-1)} \leq M(G)^{r-2}M(G)^{1+\delta_{j+1}}.$$

Hence

$$\delta_{j+1} \geq (r-1)\delta_j.$$

It follows by induction that

$$\delta_j \geq (r-1)^{j-1}\delta_1 \quad \text{for } j \geq 1.$$

This also shows that

$$\delta_j \geq (r-1)^{j-1}(q-1) \quad \text{for } j \geq 1. \tag{19}$$

Similarly,

$$\delta_{k+l} \geq (r-1)^l \delta_k. \tag{20}$$

By the choice of ν and (12) we have

$$M(G)^\nu \geq 4Be^{r/2a^2}$$

and by (15) and (17),

$$(4e^{A_1}H(x_j, y_j))^\lambda \leq (4e^{A_1}BM(G)|y_j|)^\lambda = M(G)^{\left(2+\nu+\delta_j+\frac{1}{a^2}\right)\lambda}. \tag{21}$$

By Corollary 1 with $\alpha_0 = \alpha_i$, we have

$$\left|\alpha_i - \frac{x_j}{y_j}\right| \leq M(G)^{-2-r\delta_j}.$$

Thus by (21), x_j/y_j is a very good approximation to α_i if

$$r\delta_j + 2 \geq \left(\nu + 2 + \delta_j + \frac{1}{a^2}\right)\lambda. \tag{22}$$

Let

$$J := 1 + \left[\frac{\log\eta - \log(q-1)}{\log(r-1)}\right].$$

Then by (19), we have $\delta_j \geq \eta$ for $j > J$. Thus by (22) and the definition of η, we find that x_j/y_j is a very good approximation to α_i for $j \geq J+1$. $\square$

Claim. The number of very good approximations to α_i is at most

$$1 + \left[\frac{1}{\log(r-1)}\log\left(\frac{(\nu+2+\frac{1}{a^2})r - 2}{\delta((\nu+2+\frac{1}{a^2})\lambda - 2)}\right)\right]. \tag{23}$$

We prove the claim. As seen above, x_{J+1}/y_{J+1} and x_{J+l}/y_{J+l} with $l \geq 1$ are very good approximations to α_i. Then by the Thue–Siegel principle,

$$\log(4e^{A_1}) + \log H(x_{J+l}, y_{J+l}) \leq \delta^{-1}\{\log(4e^{A_1}) + \log H(x_{J+1}, y_{J+1})\}.$$

This implies that

$$\log y_{J+l} \le \delta^{-1}\{\log(4e^{A_1}) + \log(BM(G)y_{J+1})\}.$$

Since $4Be^{A_1} \le M(G)^{\nu+\frac{1}{a^2}}$,

$$\log y_{J+l} \le \delta^{-1}\left\{ \log\left(\frac{M(G)^{\nu+\frac{1}{a^2}}}{B}\right) + \log(BM(G)y_{J+1})\right\}.$$

By the definition of δ_j's we get

$$(1+\delta_{J+l})\log M(G) \le \delta^{-1}\left(\nu + \frac{1}{a^2} + 2 + \delta_{J+1}\right)\log M(G).$$

Thus by (20),

$$(r-1)^{l-1} \le \frac{\delta_{J+l}}{\delta_{J+1}} \le \delta^{-1}\left(1+\frac{\nu+\frac{1}{a^2}+2}{\delta_{J+1}}\right) \le \delta^{-1}\left(1+\frac{\nu+\frac{1}{a^2}+2}{\eta}\right)$$
$$= \frac{(\nu+2+\frac{1}{a^2})r-2}{\delta\big((\nu+2+\frac{1}{a^2})\lambda-2\big)}.$$

Taking logarithm of both sides, we obtain (23) since the number of very good approximations is l. Thus

$$|I_i| \le J + l$$

which gives the assertion of the lemma. □

8. Small solutions

In this section we estimate the number of primitive solutions of (11) with $Y_0 \le y < M(G)^q$ where Y_0 is a positive integer. Let $\mathbf{x} = (x, y)$ and set

$$L_i(\mathbf{x}) = x - \alpha_i y \quad \text{for } 1 \le i \le r.$$

For $\mathbf{x} = (x, y)$ and $\mathbf{x}_0 = (x_0, y_0)$, let

$$D(\mathbf{x}, \mathbf{x}_0) = xy_0 - x_0 y.$$

We use the following estimate from ([2], Lemma 3 and (4.2)).

Lemma 10. *Let* $\mathbf{x} = (x, y)$ *be a solution of* (11). *Then there exists a number* $\beta_i = \beta_i(\mathbf{x})$ *and an integer* $m = m(\mathbf{x})$ *such that*

$$\frac{1}{|L_i(\mathbf{x})|} \geq \left(|m - \beta_i| - \frac{1}{2}\right)|y| - 1 \quad \text{for } 1 \leq i \leq r. \tag{24}$$

Here, $\beta_1, \ldots, \beta_r$ *are such that the form*

$$J(v, w) = n(v - \beta_1 w) \ldots (v - \beta_r w)$$

is equivalent to G.

Put

$$\chi_i = \left\{\mathbf{x} : |G(x, y)| = n, Y_0 \leq y < M(G)^q \text{ and } |L_i(\mathbf{x})| \leq \frac{1}{2y}\right\} \text{ for } 1 \leq i \leq r.$$

Lemma 11. *Suppose* $\mathbf{x} \neq \widetilde{\mathbf{x}}$ *with* $y \leq \widetilde{y}$ *belong to* χ_i. *Then*

$$\frac{\widetilde{y}}{y} \geq \frac{2Y_0}{5Y_0 + 2} \max(1, |m - \beta_i|) \tag{25}$$

where $\beta_i = \beta_i(\mathbf{x})$ *and* $m = m(\mathbf{x})$.

Proof. Since

$$D(\mathbf{x}, \widetilde{\mathbf{x}}) = x\widetilde{y} - \widetilde{x}y = \begin{vmatrix} x & y \\ \widetilde{x} & \widetilde{y} \end{vmatrix} = \begin{vmatrix} x - \alpha_i y & y \\ \widetilde{x} - \alpha_i \widetilde{y} & \widetilde{y} \end{vmatrix},$$

we have

$$1 \leq |D(\mathbf{x}, \widetilde{\mathbf{x}})| \leq y|L_i(\widetilde{\mathbf{x}})| + \widetilde{y}|L_i(\mathbf{x})| \leq \frac{y}{2\widetilde{y}} + \widetilde{y}|L_i(\mathbf{x})| \leq \frac{1}{2} + \widetilde{y}|L_i(\mathbf{x})|.$$

By (24),

$$\widetilde{y} \geq \frac{1}{2}\left(\left(|m - \beta_i| - \frac{1}{2}\right) y - 1\right).$$

Therefore,

$$\frac{\widetilde{y}}{y} \geq \frac{1}{2}\left(|m - \beta_i| - \frac{1}{2} - \frac{1}{y}\right) \geq \frac{1}{2}\left(|m - \beta_i| - \frac{Y_0 + 2}{2Y_0}\right).$$

This together with $\widetilde{y} \geq y$ shows that

$$\frac{\widetilde{y}}{y} \geq \max\left\{1, \frac{1}{2}\left(|m - \beta_i| - \frac{Y_0 + 2}{2Y_0}\right)\right\}.$$

It is easy to see that the right hand side exceeds

$$\frac{2Y_0}{5Y_0 + 2} \max\{1, |m - \beta_i|\}$$

which implies the assertion. $\square$

The following lemma is an immediate consequence of (24).

Lemma 12. *Suppose* $\mathbf{x}$ *is a solution of* (11) *with* $y \geq Y_0$ *and* $|L_i(\mathbf{x})| > 1/(2y)$. *Then*

$$|m - \beta_i| \leq \frac{5}{2} + \frac{1}{Y_0}.$$

where $\beta_i = \beta_i(\mathbf{x})$ *and* $m = m(\mathbf{x})$.

PROOF. This follows immediately from (24). □

Lemma 13. *Let*

$$\mu = \frac{(r-1)\log p}{\log\left(\frac{5Y_0+2}{2Y_0}\right)}.$$

The number of primitive solutions of (11) *with* $Y_0 \leq y \leq M(G)^q$ *does not exceed*

$$r + \frac{qr}{1 - \frac{\log\left(\frac{5Y_0+2}{2Y_0}\right)}{\frac{1}{2}\log p - \frac{1}{2r-2}\log r}}$$

provided the denominator in the expression above is positive.

PROOF. Fix $1 \leq i \leq r$. For each set χ_i which is not empty, let $\mathbf{x}_1^{(i)}, \ldots, \mathbf{x}_\sigma^{(i)}$ denote the elements of χ_i ordered so that $y_1 \leq \ldots \leq y_\sigma$. Put $\mathbf{x}(i) = \mathbf{x}_\sigma^{(i)}$. Let χ be the set of solutions of (11) with $Y_0 \leq y$, except $\mathbf{x}(1), \ldots, \mathbf{x}(r)$. By (25),

$$\prod_{\mathbf{x}\in\chi\cap\chi_i}\left(\frac{2Y_0}{5Y_0+2}\max(1, |m(\mathbf{x}) - \beta_i(\mathbf{x})|)\right) \leq \prod_{i=1}^{\sigma-1}\frac{y_{i+1}}{y_i} \leq \frac{y_\sigma}{y_1} \leq M(G)^q.$$

For $\mathbf{x} \in \chi \setminus \chi_i$, using Lemma 12, we get

$$\frac{2Y_0}{5Y_0+2}\max(1, |m(\mathbf{x}) - \beta_i(\mathbf{x})| \leq 1.$$

Therefore

$$\prod_{\mathbf{x}\in\chi}\left(\frac{2Y_0}{5Y_0+2}\max(1, |m(\mathbf{x}) - \beta_i(\mathbf{x})|)\right) \leq M(G)^q.$$

Note that

$$\prod_{1\leq i\leq r}\max(1, |m(\mathbf{x}) - \beta_i(\mathbf{x})|) = \frac{M(\hat{J})}{n}$$

where $\hat{J} = n(v - (\beta_1 - m)w)\ldots(v - (\beta_r - m)w)$ and $\hat{J}$ is equivalent to J and hence to G. Thus we get

$$\prod_{1\leq i\leq r}\max(1, |m(\mathbf{x}) - \beta_i(\mathbf{x})|) \geq \frac{M(G)}{n}.$$

Hence

$$\left(\frac{2Y_0}{5Y_0+2}\right)^{r|\chi|}(M(G)/n)^{|\chi|}\leq \prod_{\mathbf{x}\in\chi}\prod_{1\leq i\leq r}\frac{2Y_0}{5Y_0+2}\max(1,|m(\mathbf{x})-\beta_i(\mathbf{x})|)\leq M(G)^{qr}.$$

Thus

$$|\chi|\leq \frac{qr\log M(G)}{\log M(G)-\log n-r\log\left(\frac{5Y_0+2}{2Y_0}\right)}.$$

Using (12), we get

$$|\chi|\leq \frac{qr}{1-\frac{\log n+r\log\left(\frac{5Y_0+2}{2Y_0}\right)}{\frac{\mu\log n}{2r-2}+\frac{r}{2}\log p-\frac{r}{2r-2}\log r}}.$$

Since $\frac{a+b}{c+d}\leq \max(a/c,b/d)$,

$$|\chi|\leq \frac{qr}{\min\left(1-\frac{2r-2}{\mu},1-\frac{\log\left(\frac{5Y_0+2}{2Y_0}\right)}{\frac{1}{2}\log p-\frac{1}{2r-2}\log r}\right)}.$$

Hence by the definition of μ we get

$$|\chi|\leq \frac{qr}{1-\frac{\log\left(\frac{5Y_0+2}{2Y_0}\right)}{\frac{1}{2}\log p-\frac{1}{2r-2}\log r}}.$$

So the number of solutions with $Y_0\leq y<M(G)^q$ is at most $|\chi|+r$ which gives the assertion of the lemma. □

9. Parametric estimate for $N^{(1)}(r,n,p)$

Let G satisfy (11). Number of solutions of (11) with $y\leq Y_0-1$, including $(1,0)$ is at most

$$(Y_0-1)r+1.$$

We now combine Lemmas 9 and 13 to get

$$N^{(1)}(r,n,p)\leq r(S+L) \tag{26}$$

where

$$S=Y_0-1+\frac{1}{r}+\frac{q}{1-\frac{\log\left(\frac{5Y_0+2}{2Y_0}\right)}{\frac{1}{2}\log p-\frac{1}{2r-2}\log r}}$$

and

$$L=2+\left[\frac{\log\eta-\log(q-1)}{\log(r-1)}\right]+\left[\frac{1}{\log(r-1)}\log\left(\frac{(\nu+2+\frac{1}{a^2})r-2}{\delta((\nu+2+\frac{1}{a^2})\lambda-2)}\right)\right],$$

with λ,δ given by (14) and (15); ν,η given by (18).

10. Proof of Theorems 1 and 2

Proof of Theorem 1. As noted in the Introduction, we have

$$N_F^{(1)}(r,h) \le r^{\omega(h)} N_F^{(1)}(r,1).$$

Thus it is enough to find an upper bound for $N_F^{(1)}(r,1)$. Note that $N_F^{(1)}(r,1) = N_F^{(2)}(r,1)$. Hence by (11) and Lemma 4,

$$N_F^{(1)}(r,1) \le \begin{cases} N^{(1)}(r,1,p) & \text{whenever } \ |D(F)| \ge p^{r(r-1)} \\ (p+1)N^{(1)}(r,1,p) & \text{otherwise.} \end{cases} \tag{27}$$

For any given r, we choose a, b, p and Y_0 so that $r > \lambda$ and the right hand side of (26) is as small as possible. Let $r \ge 24$. Take $a = .5$, $b = .54$, $p = 29$ and $Y_0 = 2$. Then $\eta \le 12.21$. Taking $q = 1.54 > 1 + \eta/(r-1)$ we find that $L \le 4$ and $S \le 5.8608$. Thus the right hand side of (26) is at most $9.8608r$ which gives the assertion of Theorem 1, by (27).

Let $14 \le r \le 23$. In these cases take $a = .4$, $b = .48$, $p = 53$ and $Y_0 = 2$. Further take $q = 2.04$ for $18 \le r \le 23$ and $q = 2$ for $14 \le r \le 17$. Then the right hand side of (26) is at most $10r$ if $18 \le r \le 23$ and $10.88r$ if $14 \le r \le 17$ proving the assertion of Theorem 1.

Let $3 \le r \le 13$. In these cases take $q = 2$. Also take $Y_0 = 2$ for $11 \le r \le 13$ & 3 and $Y_0 = 1$ for $4 \le r \le 10$. Further take $a = .4$, $b = .48$ if $9 \le r \le 13$; $a = .3$, $b = .36$ if $r = 6, 7, 8$; $a = .2$, $b = .24$ if $r = 4, 5$ and $a = .1$, $b = .15$ if $r = 3$. The choice of p and the resulting c_0' is given in Table 1. Note that the values of a, b, p are as in [13] but the values of c_0' obtained are always better than those given in [13]. By choosing different values for a, b, p it is possible to get slightly improved bounds, but the improvement is not significant.

To obtain c_0, we choose $Y_0 = 3$; $p = 17$ for $r \ge 4$ and $p = 19$ for $r = 3$. Further let

$$(a,b,q) = \begin{cases} (.4, .48, 1.04) & \text{if } \ r \ge 12 \\ (.4, .48, 1.1) & \text{if } \ r = 10, 11 \\ (.3, .36, 1.1) & \text{if } \ 6 \le \ r \le 9 \\ (.2, .24, 1.1) & \text{if } \ r = 4, 5 \\ (.1, .15, 1.1) & \text{if } \ r = 3. \end{cases}$$

Then

$$N_F(r,h) \le c_0 r^{1+\omega(h)}$$

where $c_0 = 210, 236$ if $r \ge 23, 14 \le r \le 22$, respectively and for $3 \le r \le 13$, c_0 is listed in Table 1. This completes the proof of Theorem 1. □

Note. Let c_0' and p_0 be as in the statement of Theorem 1. In the above proof, we showed that

$$N_F^{(1)}(r,h) \le c_0' r \quad \text{if} \ \ |D(F)| \ge p_0^{r(r-1)} h^{\mu}$$

Further, as can be seen through Sections 5–10, these estimates hold not only for Thue equations but also for the Thue inequality

$$|F(x,y)| \le h. \tag{28}$$

Since $p_0 \le 101$ and $\mu \le 4.21(r-1)$, our lower bound for $|D(G)|$ and upper bound for the number of primitive solutions of (28) are both better than those in [7, Theorem 1.G (ii)].

Proof of Theorem 2. As noted in the Introduction, $N_F(r,h) \le 2N_F^{(1)}(r,h)$ and by Lemmas 4 and 5 we get

$$\begin{aligned} N_F^{(1)}(r,h) &\le r^{\omega(g)} N_F^{(2)}(r,h/g) \\ &\le \begin{cases} r^{\omega(g)}(p+1)N^{(1)}(r,h/g;p) & \text{if} \ \ |D(F)| \ge (h/g)^{\mu} \\ r^{\omega(g)} N^{(1)}(r,h/g;p) & \text{if} \ \ |D(F)| \ge p^{r(r-1)}(h/g)^{\mu}. \end{cases} \end{aligned}$$

Recall from Lemma 13 that $\mu = \frac{(r-1)\log p}{\log(\frac{5Y_0+2}{2Y_0})}$. We make the same choices for a, b, p_0, Y_0 and q as in the proof of Theorem 1 for each r. It follows that

$$N_F(r,h) \le \begin{cases} 2c_0 r^{1+\omega(g)} & \text{if} \ \ |D(F)| \ge (h/g)^{\mu_1(r-1)} \\ 2c_0' r^{1+\omega(g)} & \text{if} \ \ |D(F)| \ge p_0^{r(r-1)}(h/g)^{\mu_1'(r-1)} \end{cases}$$

with c_0, c_0' given by Theorem 1. Further while calculating c_0, take $\mu_1 = 2.83$ for $r \ge 3$ and when c_0' is calculated take $\mu_1' = 3.066, 3.62$ for $r \ge 24$, $14 \le r \le 23$, respectively. Also for $3 \le r \le 13$, we record μ_1' in Table 1. This completes the proof of Theorem 2. $\square$

r	p_0	μ'_1	c'_0	c_0
13	53	3.62	10.896	237
12	59	3.72	10.74	239
11	61	3.75	10.717	247
10	67	3.36	11.572	268
9	71	3.41	12.492	271
8	73	3.43	12.493	294
7	79	3.49	12.398	300
6	83	3.53	13.39	327
5	89	3.59	14.38	376
4	97	3.66	17.369	456
3	101	3.684	25.546	696

Table 1

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