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# Introduction to generalized topological groups

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**Abstract.** In this paper, we introduce the concept of generalized topological groups as a generalization of topological groups and give their properties. Moreover, we prove that every generalized topological group is a homogeneity space.

## 1. Introduction

Topological groups have the algebraic structure of groups and the topological structure of topological spaces, which are linked by the requirement that the multiplication and the inverse mappings are both generalized continuous. The concept of topological groups is due to Leja (see [16]).

The theory of generalized topological spaces, which was founded by Császár in recent years, is one of the most important development of general topology (see [1], [2], [3], [4], [6], [7]).

The purpose of this paper is to consider generalized topological spaces combining groups. We introduce the concept of generalized topological groups and study their properties.

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## 2. Overview of groups and GTS's

**2.1. On groups.** Let $(X, \cdot)$ be a group. Let $X \times X$ be the cartesian product. We denote two mappings $\rho : X \times X \to X$ and $\lambda : X \to X$ by

$$\rho(x, y) = xy \qquad \text{for any } x, y \in X,$$

$$\lambda(x) = x^{-1} \qquad \text{for any } x \in X.$$

Let $A, B \subset X$ and $x, y \in X$. We denote the identity of $X$ by $e$ and denote

$$xA = \{xa : a \in A\}, \qquad Ax = \{ax : a \in A\}, \qquad xAy = \{xay : a \in A\},$$

$$AB = \{ab : a \in A \text{ and } b \in B\}, \qquad A^{-1} = \{a^{-1} : a \in A\}.$$

Obviously, $AB = \bigcup_{x\in A} xB = \bigcup_{x\in B} Ax$.

Let $H \subset X$. $H$ is called a subgroup of $X$, which is denoted by $H \leq X$, if $(H, \cdot)$ is a group. $H$ is called a normal subgroup of $X$, which is denoted by $H \lhd X$, if $H \leq X$ and $Ha = aH$ for any $x \in X$.

Let $H \leq X$ and $a \in X$. $Ha$ (resp. $aH$) is called a right (resp. left) coset of $H$ in $X$ and $a$ is the representative element of $Ha$ (resp. $aH$). Define a relation $R_H$ on $X$ by

$$xR_H y \Leftrightarrow xy^{-1} \in H \quad \text{for any} \quad x, y \in X.$$

Then $R_H$ is an equivalent relation on $X$ and $[x]_{R_H} = Hx$ for any $x \in X$. $\{Hx : x \in X\}$ is called a quotient set of $X$ with respect to $R_H$. We denote it by $X/H$.

**2.2. On GTS's.** Let $X$ be a set and let $\tau$ be a family of subsets of $X$. $\tau$ is called a generalized topology on $X$, if $\emptyset \in \tau$ and $G_i \in \tau$ for $i \in I$ implies $\bigcup_{i\in I} G_i \in \tau$. The pair $(X, \tau)$ is called a generalized topological space (briefly GTS). The elements of $\tau$ are called $g$-open subsets of $X$ and the complements are called $g$-closed subsets of $X$.

Let $X$ be a GTS and let $H \subset X$. The family of all $g$-open subsets of $X$ and the family of all $g$-closed subsets of $X$ are denoted by $\tau(X)$ and $\mathcal{F}(X)$, respectively. For $x \in X$ and $H \subset X$, we denote $\tau(X, x) = \{U \in \tau(X) : x \in U\}$, $\mathcal{F}(X, x) = \{F \in \mathcal{F}(X) : x \in F\}$ and $\tau_H = \{U \cap H : U \in \tau\}$. Obviously, $\tau_H$ is a generalized topology of $H$ with respect to $X$. Then the pair $(H, \tau(H))$ is called a generalized topological subspace of $X$.

Let $(X, \tau)$ be a generalized topological spaces. For $A \subset X$, the closure and the interior of A are defined as the following:

$$\text{cl}(A) = \cap\{F : F \in \mathcal{F}(X) \text{ and } A \subset F\},$$

$$\text{int}(A) = \cup\{V : V \in \tau(X) \text{ and } V \subset A\}.$$

*Definition 2.1* ([3]). Let $X$ and $Y$ be two GTS's and let $f : X \to Y$ be a mapping. Then

(1) $f$ is called generalized continuous, if $f^{-1}(V) \in \tau(X)$ for any $V \in \tau(Y)$.
(2) $f$ is called generalized continuous at $x \in X$, if for any $V \in \tau(Y, f(x))$, there exists $U \in \tau(X, x)$ such that $f(U) \subset V$.
(3) $f$ is called open (resp. closed), if $f(A) \in \tau(Y)$ (resp. $f(A) \in \mathcal{F}(Y)$) for any $A \in \tau(X)$ (resp. $A \in \mathcal{F}(X)$).
(4) $f$ is called homeomorphism, if $f$ is bijection and $f$, $f^{-1}$ are both generalized continuous.

Let $f : X \to Y$ be generalized continuous and let $H \subset X$. It is easy to prove that $f|_H : H \to Y$ is generalized continuous.

**Proposition 2.2** ([3]). *Let $X, Y$ be two GTS's. Then a mapping $f : X \to Y$ is generalized continuous if and only if $f$ is generalized continuous at $x$ for any $x \in X$.*

**Proposition 2.3.** *Every homeomorphism mapping is open and closed.*

Proof. Let $X$, $Y$ be two GTS's and let $f : X \to Y$ be homeomorphism. Since $f^{-1}$ is generalized continuous, $f(A) = (f^{-1})^{-1}(A) \in \tau(Y)$ for any $A \in \tau(X)$. Thus $f$ is open.

Similarly, we can prove that $f$ is closed. □

*Definition 2.4* ([3]). Let $X$ be a GTS and let $B \subset X$ and $x \in X$. $B$ is called a neighborhood of $x$, if there exists $U \in \tau(X)$ such that $x \in U \subset B$.

*Definition 2.5* ([3]). Let $X$ be a GTS, let $x \in X$ and let $\mathcal{B} \subset \tau$ with $x \in B$ for any $B \in \mathcal{B}$. $\mathcal{B}$ is called a open neighborhood basic of $x$ in $X$, if for any $U \in \tau(X, x)$, there exists $B \in \mathcal{B}$ such that $B \subset U$.

Let $\mathcal{B} \subset 2^X$ satisfy $\emptyset \in \mathcal{B}$. Then all unions of some elements of $\mathcal{B}$ constitute a GT, we denote it by $\tau(\mathcal{B})$. $\mathcal{B}$ is said to be a base for $\tau(\mathcal{B})$ (see [5]).

Let $(X, \tau)$ be a GTS, the union of all elements of $\tau$ is denoted by $M_\tau$.

*Definition 2.6* ([17]). Let $\{(X_i, \tau_i) : i \in \Gamma\}$ be a family of $GTS$'s and let $X = \prod_{i \in \Gamma} X_i$ be the Cartesian product. Let us consider all sets form $\prod_{i \in \Gamma} B_i$, where $B_i \in \tau_i$, and with the exception of a finite number of indices $i$, $B_i = M_{\tau_i}$. We denote $\mathcal{B}$ be the collection of all these sets. $\tau = \tau(\mathcal{B})$ is called having $\mathcal{B}$ as a base of the product of $\{\tau_i : i \in \Gamma\}$. Then $(X, \tau)$ is called the product space of $\{(X_i, \tau_i) : i \in \Gamma\}$.

Let $f$ be a surjection from a GTS $X$ to a set $Y$. It is easy to show that $\tau(Y) = \{B \subset Y : f^{-1}(B) \in \tau(X)\}$ is a generalized topology on $Y$.

*Definition 2.7* ([10]). Let $f$ be a surjection from a GTS $X$ to a set $Y$. Then $\tau(Y)$ is called the quotient topology on $Y$ with respect to $f$ and $(Y, \tau(Y))$ is called the quotient space of $(X, \tau(X))$ with respect to $f$.

*Definition 2.8* ([10]). Let $X$ and $Y$ be two GTS's, and let $f : X \to Y$ be a surjection. $f$ is called a quotient mapping, if $\tau(Y)$ is the quotient topology on $Y$ with respect to $f$.

*Definition 2.9* ([4]). Let $X$ be a GTS and let $A, B \subset X$. $A$ and $B$ are called separated in $X$, if $A \cap \mathrm{cl}(B) = B \cap \mathrm{cl}(A) = \emptyset$.

*Definition 2.10* ([4]). Let $X$ be a GTS and let $Y \subset X$.

(1) $X$ is called a connected space, if there exist two separated sets $A, B \subset X$ such that $X = A \cup B$, then $A = \emptyset$ or $B = \emptyset$.
(2) $Y$ is called a connected subset of $X$, if the subspace $Y$ is a connected space.

**Theorem 2.11** ([4]). *Let $X$ be a GTS. The following are equivalent:*

(1) *$X$ is connected.*
(2) *There exist $A, B \in \tau(X) - \{\emptyset\}$ such that $A \cap B = \emptyset$ and $A \cup B = X$.*

**Lemma 2.12** ([4]). *Let $X$ be a GTS and let $A, B$ be separated in $X$. If $H$ is connected and $H \subset A \cup B$, then either $H \subset A$ or $H \subset B$.*

**Theorem 2.13** ([4]). *Let $f$ be a generalized continuous mapping from a connected space $X$ to a GTG $Y$. Then $f(X)$ is a connected subset of $Y$.*

## 3. The concept of GTG's

*Definition 3.1* ([16]). Let $(X, \cdot)$ be a group and let $(X, \tau)$ be a topological space. Let $X \times X$ be the cartesian product. The pair $(X, \cdot, \tau)$ is called a topological group, if $\rho, \lambda$ are both continuous.

*Definition 3.2.* Let $(X, \cdot)$ be a group and let $(X, \tau)$ be a GTS. Let $X \times X$ be the cartesian product. The pair $(X, \cdot, \tau)$ is called a generalized topological group (briefly GTG), if $\rho, \lambda$ are both generalized continuous.

*Example 3.3.* Let $X = (-\infty, +\infty)$ and let "$\cdot$" be the ordinary addition. The identity of $(X, \cdot)$ is 0 and $x^{-1} = -x$ for any $x \in (-\infty, +\infty)$. Then $(X, \cdot)$ is a group.

Let $\mathcal{B} = \{[a, b] : -\infty < a < b < +\infty\}$ and let $\tau(X) = \{A : A = \cup\mathcal{B}'$ for some $\mathcal{B}' \subset \mathcal{B}\} \cup \{\emptyset\}$, where $[a, b]$ is the general closed interval. Then $(X, \tau(X))$ is a GTS.

Let $[a, b] \in \tau(X)$. Then $\lambda^{-1}([a, b]) = [-b, -a] \in \tau(X)$. Thus $\lambda$ is generalized continuous.

Let $[a, b] \in \tau(X)$. For any $(x_0, y_0) \in \rho^{-1}([a, b])$, we have $x_0+y_0 = \rho((x_0, y_0)) \in [a, b]$. Put

$$\triangle = \min\{x_0 + y_0 - a, b - (x_0 + y_0)\}.$$

Pick $x_1 = x_0 - \frac{1}{4}\triangle$, $x_2 = x_0 + \frac{1}{4}\triangle$, $x_3 = y_0 - \frac{1}{4}\triangle$, $x_4 = y_0 + \frac{1}{4}\triangle$. Then $x_1 \leq x_0 \leq x_2$, $x_3 \leq y_0 \leq x_4$ and $a \leq x_1+x_3 \leq x_0+y_0 \leq x_2+x_4 \leq b$. So $(x_0, y_0) \in [x_1, x_2] \times [x_3, x_4]$. For any $w \in [x_1, x_2] \times [x_3, x_4]$, there exist $w_1 \in [x_1, x_2]$ and $w_2 \in [x_3, x_4]$ such that $w = (w_1, w_2)$. Then $a \leq x_1 + x_3 \leq w_1 + w_2 \leq x_2 + x_4 \leq b$. So $\rho(w) \in [a, b]$. This implies that $w \in \rho^{-1}([a, b])$. Thus $[x_1, x_2] \times [x_3, x_4] \subset \rho^{-1}([a, b])$. Since $[x_1, x_2], [x_3, x_4] \in \tau(X)$, then $\rho^{-1}([a, b]) \in \tau(X \times X)$. Thus $\rho$ is generalized continuous.

Hence $X$ is a GTG.

But $\tau(X)$ is not keeping intersection, so $X$ is not a topological space. Thus $X$ is not a topological group.

Obviously, every group is a GTG when equipped with the discrete topology.

The following theorem gives the decision conditions of GTS's.

**Theorem 3.4.** *Let $(X, \cdot)$ be a group and let $(X, \tau)$ be a GTS. Then the following are equivalent.*

(1) *$X$ is a GTG;*

(2) *The mapping $\mu : X \times X \to X$ is generalized continuous where $\mu(x, y) = xy^{-1}$;*

(3) *For any $x, y \in X$ and any $W \in \tau(X, xy^{-1})$, there exist $U \in \tau(X, x)$ and $V \in \tau(X, y)$ such that $UV^{-1} \subset W$;*

(4) (i) *For any $x, y \in X$ and any $W \in \tau(X, xy)$, there exist $U \in \tau(X, x)$ and $V \in \tau(X, y)$ such that $UV \subset W$,*

(ii) *For any $W \in \tau(X, x^{-1})$, there exists $U \in \tau(X, x)$ such that $U^{-1} \subset W$.*

Proof. (1) $\Rightarrow$ (4) (i) Let $x, y \in X$ and $W \in \tau(X, xy)$. Since $\rho$ is generalized continuous, there exists $L \in \tau(X \times X, (x, y))$ such that $\rho(L) \subset W$. Since $(x, y) \in L \in \tau(X \times X)$, there exist $U, V \in \tau(X)$ such that $(x, y) \in U \times V \subset L$. This implies that $U \in \tau(X, x)$, $V \in \tau(X, y)$ and $\rho(U \times V) \subset W$. Note that $\rho(U \times V) = UV$. Thus $UV \subset W$.

(ii) Let $W \in \tau(X, x^{-1})$. Since $\lambda$ is generalized continuous, $\lambda^{-1}(W) \in \tau(X)$. Put $U = \lambda^{-1}(W)$. Then $U \in \tau(X, x)$ and $\lambda(U) = W$. Note that $\lambda(U) = U^{-1}$. Thus $U^{-1} \subset W$.

(4) $\Rightarrow$ (3) Let $x, y \in X$ and $W \in \tau(X, xy^{-1})$. By (4)($i$), there exist $U \in \tau(X, x)$ and $V_1 \in \tau(X, y^{-1})$ such that $UV \subset W$. By (4)(ii), there exists $V \in \tau(X, y)$ such that $V^{-1} \subset V_1$. Note that $UV^{-1} \subset UV_1$. Thus $UV^{-1} \subset W$.

(3) $\Rightarrow$ (2) Let $(x, y) \in X \times X$ and $W \in \tau(X, xy^{-1})$. By (3), there exist $U \in \tau(X, x)$ and $V \in \tau(X, y)$ such that $UV^{-1} \subset W$. Put $L = U \times V$. Then $L \in \tau(X \times X, (x, y))$. Note that $\mu(L) = UV^{-1}$. Thus $\mu(L) \subset W$. Hence $\mu$ is generalized continuous.

(2) $\Rightarrow$ (1) Since $\mu$ is generalized continuous, $\mu|_{\{e\}\times X}$ is generalized continuous. Note that $\lambda = \mu|_{\{e\}\times X}$. Thus $\lambda$ is generalized continuous.

Denote $g : X \times X \to X \times X$ by $(x, y) \to (x, y^{-1})$. Then we have $\rho = \mu \circ g$. Since $\mu$ is generalized continuous. To prove that $\rho$ is generalized continuous, it is suffice to show that $g$ is generalized continuous.

Let $W \in \tau(X \times X)$. For any $(x, y) \in g^{-1}(W)$, $g(x, y) = (x, y^{-1}) \in W$. Since $W \in \tau(X \times X)$, there exist $U, V \in \tau(X)$ such that $(x, y^{-1}) \in U \times V \subset W$. This implies that $x \in U$ and $\lambda(y) = y^{-1} \in V$. Since $\lambda$ is generalized continuous, $\lambda^{-1}(V) \in \tau(X)$. Put $U_1 = U$ and $V_1 = \lambda^{-1}(V)$. Then $U_1, V_1 \in \tau(X)$ and $(x, y) \in U_1 \times V_1$.

For any $(s, t) \in U_1 \times V_1$, $s \in U_1$ and $t \in V_1$. $t \in V_1$ implies $t^{-1} \in V$. Then $g(s, t) = (s, t^{-1}) \in U \times V \subset W$. Thus $(s, t) \in g^{-1}(W)$. So $U_1 \times V_1 \subset g^{-1}(W)$. By $(x, y) \in U_1 \times V_1 \subset g^{-1}(W)$, $g^{-1}(W) \in \tau(X \times X)$. Thus $g$ is generalized continuous.

Hence $X$ is a GTG. □

## 4. Some properties of GTG's

In this section, we give some properties of GTG's.

**4.1. GTG's and homogeneous spaces.** In [9], Arhangel'skiĭ considered the homogeneity on topological spaces. Similarly, we can define the homogeneity on GTS's as follows.

*Definition 4.1.* A GTS $X$ is called a homogeneous space, if for any $a, b \in X$, there exists a homeomorphism mapping $f : X \to X$ such that $f(a) = b$.

Let $X$ be a GTG. For $a \in X$, we denote respectively the right multiplication $r_a : X \to X$ and the left multiplication $l_a : X \to X$ by

$$r_a(x) = xa \text{ for any } x \in X,$$

$$l_a(x) = ax \text{ for any } x \in X.$$

**Proposition 4.2.** *Let $X$ be a GTG and let $a \in X$. Then $r_a$ and $l_a$ are both homeomorphism.*

PROOF. (1) For any $x, y \in X$ with $x \neq y$. Suppose $r_a(x) = r_a(y)$. Then $xa = ya$. So $(xa)(a^{-1}) = (ya)(a^{-1})$. This implies that $x = y$, a contradiction. Thus $r_a$ is a injection. For any $y \in X$, pick $x = ya^{-1} \in X$, we have $r_a(x) = y$. Then $r_a$ is a surjection. Hence $r_a$ is bijection.

Define a mapping $\varphi : X \to X \times X$ by $\varphi(x) = (x, a)$.

Obviously, $r_a = \rho \circ \varphi$ and $r_a^{-1} = r_{a^{-1}}$.

For any $x \in X$ and any $g$-open neighborhood $W$ of $\varphi(x)$, there exist $U, V \in \tau(X)$ such that $\varphi(x) \in U \times V \subset W$. Then $x \in U$ and $a \in V$. Now $\varphi(U) \subset W$. Thus $\varphi$ is generalized continuous.

Since $X$ is a GTG, $\rho$ is generalized continuous. By $r_a = \rho\circ\varphi$, $r_a$ is generalized continuous.

Now $r_a^{-1} = r_{a^{-1}}$. Similarly, we can prove that $r_a^{-1}$ is generalized continuous. Therefore $r_a$ is homeomorphism.

(2) Similarly, we can prove that $l_a$ is homeomorphism. □

The following Theorem 4.3 is similar to Proposition 1.2 in [19].

**Theorem 4.3.** *Every GTG is a homogeneous space.*

PROOF. Let $X$ be a GTG and let $a, b \in X$. By Proposition 4.2, $r_{a^{-1}b}$ is homeomorphism. Obviously, $r_{a^{-1}b}(a) = b$. Thus $X$ is a homogeneous space. □

### 4.2. Open or closed subsets of GTG's.

**Lemma 4.4.** *Let $X$ be a group. Let $A, B, C \subset X$ and $x \in X$. Then*

(1) *If $A \subset B$, then $xA \subset xB$ and $Ax \subset Bx$.*
(2) *If $A \subset B$, then $AC \subset BC$ and $CA \subset CB$.*
(3) *$(Ax)^{-1} = x^{-1}A$ and $(xA)^{-1} = Ax^{-1}$.*

PROOF. This is obvious. □

**Proposition 4.5.** *Let $X$ be a GTG. Let $A, B \subset X$ and $x \in X$. Then*

(1) *If* $A \in \tau(X)$ *(resp.* $A \in \mathcal{F}(X)$*), then* $Ax, xA \in \tau(X)$ *(resp.* $Ax, xA \in \mathcal{F}(X)$*).*

(2) *If* $A \in \tau(X)$*, then* $AB, BA \in \tau(X)$*.*

Proof. This is straightforward. $\square$

Let $X$ be a topological group and let $A, B \subset X$. If $A \in \mathcal{F}(X)$ and $B$ is a finite set, then $AB, BA \in \mathcal{F}(X)$. In GTG, this conclusion does not hold (See Example 4.6).

*Example 4.6.* Let $X$ be a GTG in Example 3.3. Pick $A = (1,2) \in \mathcal{F}(X)$ and $B = \{0,1\}$. Then $AB = BA = (1,2) \cup (2,3)$. Since $X - AB = X - BA = (-\infty, 1) \cup \{2\} \cup (3, +\infty) \notin \tau$, Thus $AB, BA \notin \mathcal{F}(X)$.

*Definition 4.7.* Let $(X, \cdot, \tau)$ be a GTG and let $\mathcal{B} \subset \tau$ and $x \in X$. $\mathcal{B}$ is called a open neighborhood basic of $x$ in $(X, \cdot, \tau)$, if $\mathcal{B}$ is a open neighborhood basic of $x$ in $(X, \tau)$.

**Proposition 4.8.** *Let* $X$ *be a GTG and let* $\mathcal{B}$ *be a open neighborhood basic of* $e$ *in* $X$*. Then*

(1) *If* $x \in U \in \mathcal{B}$*, then there exists* $B \in \mathcal{B}$ *such that* $Bx \subset U$*.*

(2) *If* $U \in \mathcal{B}$ *and* $x \in X$*, then there exists* $B \in \mathcal{B}$ *such that* $x^{-1}Bx \subset U$*.*

Proof. (1) Since $U \in \tau(X)$, by Proposition 4.5, we have $Ux^{-1} \in \tau(X)$. $x \in U$ means $e \in Ux^{-1}$. Thus there exists $B \in \mathcal{B}$ such that $B \subset Ux^{-1}$. Hence $Bx \subset U$.

(2) Define $f : X \to X$ by $a \to x^{-1}ax$. Then $f$ is generalized continuous. By $U \in \mathcal{B}$, $f^{-1}(U) \in \tau(X)$. Since $f(e) = e \in U$, $e \in f^{-1}(U)$. Then there exists $B \in \mathcal{B}$ such that $B \in f^{-1}(U)$. This implies that $x^{-1}Bx \subset U$. $\square$

### 4.3. GTG's and subgroups.

**Proposition 4.9.** *Let* $(X, \cdot, \tau)$ *be a GTG and let* $H \leq X$*. Then* $(H, \cdot, \tau_H)$ *is a GTG.*

Proof. Let $f = \rho|_{H \times H} : H \times H \to H$ and $g = \lambda|_H : H \to H$.

We only need to prove that $f$ and $g$ are both generalized continuous.

For any $V \in \tau(H)$, then there exists $V_1 \in \tau(X)$ such that $V = V_1 \cap H$. Since $H \leq X$, $H \times H \subset \rho^{-1}(H)$ and $H \subset \lambda^{-1}(H)$.

Note that $\rho$ and $\lambda$ are both generalized continuous. Then $f^{-1}(V) = (H \times H) \cap \rho^{-1}(V) = (H \times H) \cap \rho^{-1}(V_1) \cap \rho^{-1}(H) = (H \times H) \cap \rho^{-1}(V_1) \in \tau(H \times H)$ and $g^{-1}(V) = H \cap \lambda^{-1}(V) = H \cap \lambda^{-1}(V_1) \cap \lambda^{-1}(H) = H \cap \lambda^{-1}(V_1) \in \tau(H)$.

Hence $f$ and $g$ are generalized continuous. $\square$

By Proposition 3.6, we can give the following definition.

*Definition 4.10.* Let $(X, \cdot, \tau)$ be a GTG and let $H \subset X$.

(1) $H$ is called a subgroup of $(X, \cdot, \tau)$, if $H \leq X$.

(2) $H$ is called a $g$-open (resp. $g$-closed) subgroup of $(X, \cdot, \tau)$, if $H \leq X$ and $H \in \tau$ (resp. $H \in \mathcal{F}(X)$).

**Theorem 4.11.** *Let $(X, \cdot, \tau)$ be a GTG. Then*

(1) *Every $g$-open subgroup of $X$ is $g$-closed in $X$.*

(2) *Every subgroup which contains some neighborhood of $e$ is $g$-open in $X$.*

Proof. (1) Let $H$ be a $g$-open subgroup of $X$.
*Claim.* $X - H = \bigcup_{b \notin H} bH$.

For any $y \in X - H$. Since $y = ye$ and $e \in H$, $y \in yH \subset \bigcup_{b \notin H} bH$. Thus $X - H \subset \bigcup_{b \notin H} bH$. For any $y \in \bigcup_{b \notin H} bH$, there exists $b \notin H$ such that $y \in bH$. Then $y = bx$ for some $x \in H$. Thus $b = yx^{-1}$. Suppose $y \in H$. Since $H$ is a group and $x, y \in H$, $b \in H$. This is a contradiction. Hence $X - H \supset \bigcup_{b \notin H} bH$. Then $X - H = \bigcup_{b \notin H} bH$.

Since $H \in \tau(X)$, by Proposition 4.5, $bH \in \tau(X)$ for any $b \notin H$. Thus $X - H \in \tau(X)$. Hence $H \in \mathcal{F}(X)$.

(2) Let $H$ be a subgroup of $X$ with $H$ contain some neighborhood of $e$. Then there exists $U \in \tau(X, e)$ such that $U \subset H$. By Lemma 4.1, $UH \subset HH = H$. Since $e \in U \subset H$, $H = eH \subset \bigcup_{a \in U} aH = UH$. Then $H \subset UH$. This implies $H = UH$. By Proposition 4.5, $H \in \tau(X)$. □

Let $(X, \cdot, \tau(X))$ be a GTG and $H \leq X$. Let $X/H$ be the quotient set of $X$ with respect to $R_H$ where

$$xR_H y \Leftrightarrow xy^{-1} \in H \quad \text{for any} \quad x, y \in X.$$

Put

$$\xi : X \to X/H \text{ by } x \to Hx.$$

Then, we obtain the quotient space of $X$ with respect to $\xi$. We denote it by $(X/H, \tau(X/H))$ or $X/H$ where $\tau(X/H)$ is the quotient topology of $X/H$ with respect to $\xi$, that is

$$\tau(X/H) = \{B \subset X/H : \xi^{-1}(B) \in \tau(X)\}.$$

In this case, $\xi$ is a quotient mapping.

For convenience, $X/H$ is called the coset space of $X$ with respect to $H$.

**Proposition 4.12.** *Let $X$ be a GTG and let $H \leq X$. Let $X/H$ be the coset space of $X$ with respect to $H$. Then $\xi$ is open.*

PROOF. Let $A \in \tau(X)$. By Proposition 4.5, $\xi(A) = HA \in \tau(X/H)$. Thus $\xi$ is open. □

Let $H \lhd X$ and let $X/H$ be a quotient set of $X$ with respect to $R_H$. Define a binary operation on $X/H$ by

$$Hx * Hy = Hxy, \quad \text{for any } Hx, Hy \in X/H.$$

It is easy to prove that $(X/H, *)$ is a group, which is called a quotient group of $X$ with respect to $H$.

**Theorem 4.13.** *Let $(X, \cdot, \tau(X))$ be a GTG and let $H \lhd X$. Then $(X/H, *, \tau(X/H))$ is a GTG.*

PROOF. Define three mappings $\rho_1 : X/H \times X/H \to X/H$, $\lambda_1 : X/H \to X/H$ and $\xi_1 : X \times X \to X/H \times X/H$ by

$$\begin{aligned} \rho_1(Hx, Hy) &= Hxy && \text{for any } Hx, Hy \in X/H, \\ \lambda_1(Hx) &= (Hx)^{-1} && \text{for any } Hx \in X/H, \\ \xi_1(x, y) &= (Hx, Hy) && \text{for any } x, y \in X. \end{aligned}$$

We only need to prove that $\rho_1$ and $\lambda_1$ are both generalized continuous.

It is easy to prove that $\rho_1 \circ \xi_1 = \xi \circ \rho$ and $\lambda_1 \circ \xi = \xi \circ \lambda$. Since $\rho, \xi, \lambda$ are generalized continuous, we obtain that $\xi \circ \rho$ and $\xi \circ \lambda$ are generalized continuous.

*Claim 1.* $\xi_1(U \times V) \in \tau(X/H \times X/H)$ for any $U, V \in \tau(X)$.

Let $(z_1, z_2) \in \xi_1(U \times V)$. Then there exists $(x, y) \in U \times V$ such that $(z_1, z_2) = \xi_1(x, y)$. This implies that $z_1 = Hx,\ z_2 = Hy$. Since $x \in U$ and $y \in V$, $z_1 \in \xi(U)$ and $z_2 \in \xi(V)$. Thus $(z_1, z_2) \in \xi(U) \times \xi(V)$. Suppose $\xi(U) \times \xi(V) - \xi_1(U \times V) \neq \emptyset$. Pick $(w_1, w_2) \in \xi(U) \times \xi(V) - \xi_1(U \times V)$. Since $w_1 \in \xi(U)$ and $w_2 \in \xi(V)$, there exist $a \in U$ and $b \in V$ such that $w_1 = \xi(a)$ and $w_2 = \xi(b)$. $(w_1, w_2) = (Ha, Hb) = \xi_1(a, b) \in \xi_1(U \times V)$, a contradiction. Thus $\xi(U) \times \xi(V) \subset \xi_1(U \times V)$.

By Proposition 4.12, $\xi$ is open, then $\xi(U), \xi(V) \in \tau(X/H)$. Note that $(z_1, z_2) \in \xi(U) \times \xi(V) \subset \xi_1(U \times V)$. Hence $\xi_1(U \times V) \in \tau(X/H \times X/H)$.

*Claim 2.* $\xi_1$ is open.

Let $W \in \tau(X \times X)$. Then there exist $\{U_\alpha : \alpha \in \Gamma\} \bigcup \{V_\alpha : \alpha \in \Gamma\} \subset \tau(X)$, such that $W = \bigcup_{\alpha \in \Gamma} (U_\alpha \times V_\alpha)$.

Now $\xi_1(W) = \bigcup_{\alpha \in \Gamma} \xi(U_\alpha \times V_\alpha)$. By *Claim 1*, $\xi_1(U_\alpha \times V_\alpha) \in \tau(X/H \times X/H)$ for any $\alpha \in \Gamma$. Thus $\xi_1(W) \in \tau(X/H \times X/H)$. Hence $\xi_1$ is open.

Let $U \in \tau(X/H)$. Now $\rho_1 \circ \xi_1 = \xi \circ \rho$. Since $\xi \circ \rho$ is generalized continuous, $(\xi_1^{-1} \circ \rho_1^{-1})(U) = (\rho_1 \circ \xi_1)^{-1}(U) = (\xi \circ \rho)^{-1}(U) \in \tau(X \times X)$. By *Claim 2*, $\xi_1$

is open. Then $\rho_1^{-1}(U) = \xi_1((\xi_1^{-1} \circ \rho_1^{-1})(U)) \in \tau(X/H \times X/H)$. Thus $\rho_1$ is generalized continuous.

Let $V \in \tau(X/H)$. Now $\lambda_1 \circ \xi = \xi \circ \lambda$. Since $\xi \circ \lambda$ is generalized continuous, $(\xi^{-1} \circ \lambda_1^{-1})(V) = (\lambda^{-1} \circ \xi^{-1})(V) = (\xi \circ \lambda)^{-1} \in \tau(X)$. By *Claim 2*, $\xi$ is open. Then $\lambda_1^{-1}(V) = \xi((\xi^{-1} \circ \lambda_1^{-1})(V)) \in \tau(X/H)$. Thus $\lambda_1$ is generalized continuous.

Hence $X/H$ is a GTG. □

### 4.4. Connectedness on GTG's.

**Theorem 4.14.** *Let $(X, \cdot, \tau(X))$ be a GTG and let $H \leq X$.*

(1) *If $(X, \tau(X))$ is connected, then $(X/H, \tau(X/H))$ is connected.*

(2) *If $(H, \tau_H)$ and $(X/H, \tau(X/H))$ are connected, then $(X, \tau(X))$ is connected.*

Proof. (1) Since $\xi$ is generalized continuous, by Theorem 2.13, $X/H$ is connected.

(2) Suppose that $X$ is not connected. Then there exist two separated subsets $A, B \in \tau(X)$ such that $X = A \cup B$.

*Claim 1.* $A = \bigcup\{Ha : Ha \subset A \text{ and } a \in X\}$.

To prove $A = \bigcup\{Ha : Ha \subset A \text{ and } a \in X\}$, we only need to show that $\{Ha : Ha \subset A \text{ and } a \in X\} = \{Ha : Ha \cap A \neq \emptyset \text{ and } a \in X\}$.

Obviously, $\{Ha : Ha \subset A,\ a \in X\} \subset \{Ha : Ha \cap A \neq \emptyset,\ a \in X\}$.

Conversely. Let $F \in \{Ha : Ha \cap A \neq \emptyset \text{ and } a \in X\}$. Then $F = Ha$ with $Ha \cap A \neq \emptyset$ and $a \in X$. Since $r_a$ is generalized continuous, $r_a|_H$ is also generalized continuous. Note that $H$ is connected. By Theorem 2.13, $Ha = r_a(H) = r_a|_H(H)$ is also connected. Now $Ha \subset X = A \cup B$ and $A$ , $B$ are separated. Then $Ha \subset A$ or $Ha \subset B$.

Suppose $Ha \subset B$. By $Ha \cap A \neq \emptyset$. Pick $x \in Ha \cap A$. Then $x \in B$ and so $x \in A \cap B$. This implies that $A \cap B \neq \emptyset$, a contradiction.

Then $F = Ha \subset A$. This implies that $F \in \{Ha : Ha \subset A \text{ and } a \in X\}$. Thus $\{Ha : Ha \subset A,\ a \in X\} \supset \{Ha : Ha \cap A \neq \emptyset,\ a \in X\}$.

Hence $\{Ha : Ha \subset A \text{ and } a \in X\} = \{Ha : Ha \cap A \neq \emptyset \text{ and } a \in X\}$.

*Claim 2.* $B = \bigcup\{Ha : Ha \subset B \text{ and } a \in X\}$.

This is similar to the proof of *Claim 1.*

*Claim 3.* $\xi(A) \cap \xi(B) = \emptyset$.

Suppose $\xi(A) \cap \xi(B) \neq \emptyset$. Pick $W \in \xi(A) \cap \xi(B)$.

$W \in \xi(A)$ implies $W = \xi(x)$ for some $x \in A$. Since $x \in A$ and $A = \bigcup\{Ha : Ha \subset A \text{ and } a \in X\}$, there exists $a \in X$ with $Ha \subset A$ such that $x \in Ha$. So $x = h_1 a$ for some $h_1 \in H$. Obviously, $Hh_1 = H$. Then $W = Hx = Hh_1 a = Ha$. Thus $W \subset A$. Similarly. $W \in \xi(B)$ implies $W \subset B$.

Now $W \subset A \cap B$. Thus $A \cap B \neq \emptyset$, a contradiction.

Hence $\xi(A) \cap \xi(B) = \emptyset$.

Now $X = A \cup B$. Then $X/H = \xi(X) = \xi(A) \cup \xi(B)$. Since $\xi$ is open, $\xi(A),\ \xi(B) \in \tau(X/H)$.

By $\xi(A) \cap \xi(B) = \emptyset$, $X/H$ is not connected. This is a contradiction.

Hence $X$ is connected. □

*Example 4.15.* Let $X$ be a GTG in Example 3.3. Pick $A = (- \in fty, 0]$ and $B = (0, +\infty)$. Then $\mathrm{cl}(A) = (-\infty, 0]$ and $\mathrm{cl}(B) = (0, +\infty)$. So $A$ and $B$ are separated. Note that $A \cup B = X$ and $A, B \neq \emptyset$. Thus $X$ is not connected.

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