

A second order periodic boundary value problem with a parameter and vanishing Green's functions

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Abstract. We consider the following second order periodic boundary value problem with a parameter $\lambda \in (0, \infty)$, $i = 1, 2, \dots, n$,

$$\begin{cases} x_i'' + a_i(t)x_i = \lambda g^i(t)f^i(x), & 0 \leq t \leq T, \\ x_i(0) = x_i(T), \quad x_i'(0) = x_i'(T). \end{cases}$$

By using fixed point theorems in a cone, some existence and nonexistence results for nonnegative solutions are established under different combinations of superlinearity and sublinearity of functions f^i at zero and infinity for an appropriately chosen parameter λ in the case where the associated nonnegative Green's functions may have zeros. The results are illustrated by an example.

1. Introduction

In this paper, we consider the following periodic boundary value problem of second order non-autonomous dynamical systems with a parameter $\lambda \in (0, \infty)$, $i = 1, 2, \dots, n$,

$$\begin{cases} x_i'' + a_i(t)x_i = \lambda g^i(t)f^i(x), & 0 \leq t \leq T, \\ x_i(0) = x_i(T), \quad x_i'(0) = x_i'(T), \end{cases} \quad (1.1)$$

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where $a_i \in C[0, T]$, $g^i : [0, T] \rightarrow (0, \infty)$ is continuous, $f^i \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0, \infty)$ and $f^i(x) > 0$ for $x \neq 0$. We assume that the above basic conditions on a_i, f^i, g^i are always satisfied throughout the paper.

The studies of existence and multiplicity of nonnegative solutions for periodic boundary value problems have attracted lots of mathematician these years (see [1], [13], [14], [17], [10], [12], [18], [11], [22], [24] and references therein), where the major assumption is that the associated Green's functions are of one sign. Recently, GRAEF et al. [7] extended the studies to the case where the associated Green's function needs only to be nonnegative, and established an existence result of nonnegative solutions of (1.1) for $\lambda = 1$ and $n = 1$. Then CABADA and CID [2] present some more results on the existence and nonexistence of nonnegative solutions of (1.1) for $n = 1$. Moreover, we note that several results on the existence of one or two positive solutions of periodic boundary value problems have been obtained in the framework of integral equations with nonnegative kernel in [21]. Here we give further study on this line in this work.

The main purpose of this paper is to establish some existence and nonexistence results of nonnegative solutions of (1.1) in terms of different parameters λ by using fixed point theorems in a cone under the assumption that the associated Green's functions are nonnegative. Our results extends the corresponding results in [7], [2] (see Remark 3.1). For more studies of the boundary value problem with a parameter we refer the readers to [3], [5], [19], [8], [20], [16], [23].

The paper is organized as follows. In Section 2, some preliminary results and notation are presented. In Section 3, the statements of the main results are given in Subsection 3.1. Then the proofs of the main results are presented in Subsection 3.2. Finally, an example is given to illustrate the main results in Subsection 3.3.

2. Preliminaries

Throughout the paper, let \mathbb{R}^n be endowed with norm $|x| = \sum_{i=1}^n |x_i|$ for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $C[0, T]$ be endowed with norm $\|u\| = \max_{0 \leq t \leq T} |u(t)|$ for $u \in C[0, T]$. Let $\mathbb{X} = C([0, T], \mathbb{R}^n)$. Then \mathbb{X} can be regarded as $(C[0, T])^n$. So \mathbb{X} is endowed with norm $\|x\| = \sum_{i=1}^n \|x_i\|$ for $x = (x_1, x_2, \dots, x_n) \in \mathbb{X}$. Even though the notation $\|\cdot\|$ is used for norms in different spaces, no confusion should arise. Let

$$\mathbb{P} = \{x \in \mathbb{X} : x_i(t) \geq 0, t \in [0, T], i = 1, 2, \dots, n\}.$$

Then \mathbb{P} is a normal cone in \mathbb{X} . For $x_1, x_2 \in \mathbb{X}$, we write $x_1 \leq x_2$ if $x_2 - x_1 \in \mathbb{P}$. For a set E , denote by \bar{E} and ∂E the closure and boundary of E , respectively. We also denote $\Omega_r = \{x \in \mathbb{X} : \|x\| < r\}$ for $r > 0$.

A function $x = (x_1, x_2, \dots, x_n)$ with $x_i \in C^2[0, T], i = 1, 2, \dots, n$ is said to be a nontrivial solution of (1.1) if x satisfies (1.1) and $x(t) \not\equiv 0$. Moreover, x is said to be a nonnegative solution of (1.1) if in addition $x \in \mathbb{P}$.

For each $i = 1, 2, \dots, n$, define $\hat{f}^i(\alpha) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $\hat{f}^i(\alpha) = \max\{f^i(u) : u \in \mathbb{R}_+^n, 0 \leq |u| \leq \alpha\}$. Clearly, $\hat{f}^i \in C(\mathbb{R}_+, \mathbb{R}_+)$ and is nondecreasing on \mathbb{R}_+ . For convenience, we introduce the notation

$$\lim_{|x| \rightarrow 0} \frac{f^i(x)}{|x|} = f_0^i, \quad \lim_{|x| \rightarrow \infty} \frac{f^i(x)}{|x|} = f_\infty^i, \quad \lim_{\alpha \rightarrow 0} \frac{\hat{f}^i(\alpha)}{\alpha} = \hat{f}_0^i, \quad \lim_{\alpha \rightarrow \infty} \frac{\hat{f}^i(\alpha)}{\alpha} = \hat{f}_\infty^i.$$

Then we have the following lemma.

Lemma 2.1 ([20]). $f_0^i = \hat{f}_0^i$ and $f_\infty^i = \hat{f}_\infty^i, i = 1, 2, \dots, n$.

We always assume that $G_i(t, s), i = 1, 2, \dots, n$ are the nonnegative Green's functions associated with (1.1), which may have zeros, and

$$\begin{cases} \beta = \min_{1 \leq i \leq n, 0 \leq s \leq T} \int_0^T G_i(t, s) dt > 0, \\ M = \max_{1 \leq i \leq n} M_i, M_i = \max_{0 \leq s, t \leq T} G_i(t, s), i = 1, 2, \dots, n. \end{cases} \tag{2.1}$$

Define a cone

$$\mathbb{K} = \left\{ x \in \mathbb{P} : \int_0^T |x(t)| dt \geq \frac{\beta}{M} \|x\| \right\}.$$

For $\lambda > 0$, let $T_\lambda = (T_\lambda^1, T_\lambda^2, \dots, T_\lambda^n) : \mathbb{X} \rightarrow \mathbb{X}$ be given by

$$T_\lambda^i x(t) = \lambda \int_0^T G_i(t, s) g^i(s) f^i(x(s)) ds, \quad 0 \leq t \leq T, i = 1, 2, \dots, n.$$

It is clear that x is a nonnegative solution of (1.1) if and only if x is a fixed point of T_λ .

Lemma 2.2. $T_\lambda(\mathbb{K}) \subset \mathbb{K}$ and T_λ is completely continuous.

PROOF. The complete continuity of T_λ can be proved easily by the standard method, and we omit the details. Let $x \in \mathbb{K}$. By (2.1),

$$\begin{aligned} \int_0^T |T_\lambda x(t)| dt &= \lambda \sum_{i=1}^n \int_0^T g^i(s) f^i(x(s)) \int_0^T G_i(t, s) dt ds \\ &\geq \lambda \beta \sum_{i=1}^n \int_0^T g^i(s) f^i(x(s)) ds \end{aligned}$$

and

$$\begin{aligned} \|T_\lambda x\| &= \sum_{i=1}^n \|T_\lambda^i x\| = \lambda \sum_{i=1}^n \max_{0 \leq t \leq T} \int_0^T G_i(t, s) g^i(s) f^i(x(s)) ds \\ &\leq \lambda M \sum_{i=1}^n \int_0^T g^i(s) f^i(x(s)) ds. \end{aligned}$$

Then

$$\int_0^T |T_\lambda x(t)| dt \geq \frac{\beta}{M} \|T_\lambda x\|.$$

This implies that $T_\lambda x \in \mathbb{K}$, and $T_\lambda(\mathbb{K}) \subset \mathbb{K}$. □

The following fixed-point theorem of cone expansion/compression type is crucial in the proofs of our results.

Lemma 2.3 ([9], [15]). *Let P be a cone in a Banach space X . Assume that Q_1, Q_2 are bounded open subsets of X with $0 \in Q_1, \bar{Q}_1 \subset Q_2$, and let $A : P \cap (\bar{Q}_2 \setminus Q_1) \rightarrow P$ be completely continuous. Then A has a fixed point in $P \cap (\bar{Q}_2 \setminus Q_1)$ if one of the following statements is true:*

- (i) $Ax \not\leq x$ for $x \in P \cap \partial Q_1$ and $Ax \not\leq x$ for $x \in P \cap \partial Q_2$.
- (ii) $Ax \leq x$ for $x \in P \cap \partial Q_1$ and $Ax \not\leq x$ for $x \in P \cap \partial Q_2$.

3. Existence and nonexistence of nonnegative solutions

3.1. Statements of the main results. The following assumptions on f will be used later.

- (H₁) $f_0^i = 0, i = 1, 2, \dots, n$.
- (H₂) $f_\infty^i = 0, i = 1, 2, \dots, n$.
- (H₃) $f_0^i = \infty$ for some $i \in \{1, 2, \dots, n\}$.
- (H₄) $f_\infty^i = \infty$ and f^i is convex for some $i \in \{1, 2, \dots, n\}$.
- (H₅) $f_0^i < \infty, f_\infty^i < \infty, i = 1, 2, \dots, n$.

Now we are in a position to state our main results, which will be proved in the next Subsection.

Theorem 3.1. *If (H₁) and (H₄) hold or (H₂) and (H₃) hold, (1.1) has a nonnegative solution for $\lambda > 0$.*

Theorem 3.2. *The following statements hold:*

- (i) For any $a \in (0, \infty)$, there exists $\lambda_a > 0$ such that (1.1) has a nonnegative solution x with $\|x\| < a$ for $\lambda \in (0, \lambda_a)$ if (H_3) holds.
- (ii) For any $b \in (0, \infty)$, there exists $\lambda_b > 0$ such that (1.1) has a nonnegative solution x with $\|x\| > b$ for $\lambda \in (0, \lambda_b)$ if (H_4) holds.

By Theorem 3.2, we can get the following corollary immediately.

Corollary 3.1. *If (H_3) and (H_4) hold, there exists $\lambda_0 > 0$ such that (1.1) has two nonnegative solutions for $\lambda \in (0, \lambda_0)$.*

Theorem 3.3. *If (H_5) holds, there exists $\lambda_0 > 0$ such that (1.1) has no nonnegative solution for $\lambda \in (0, \lambda_0)$.*

Remark 3.1. (i) If $\lambda = 1$ and $n = 1$, a result similar to Theorem 3.1 was presented in [7]. Moreover, if $n = 1$, the second part of Theorem 3.1 was given in [2, Theorem 3.7 (2)].

(ii) Theorem 3.2 and Corollary 3.1 extend Theorem 3.7 (1), (4) from $n = 1$ to arbitrary n .

(iii) Some nonexistence results were also given in [2, 7]. However, the condition of Theorem 3.3 is different from them.

3.2. Proofs of the main results. Let us start with some lemmas.

Lemma 3.1. (i) *If (H_1) holds, given $\lambda_0 > 0$, there exists $r_0 > 0$ such that for $\lambda \in (0, \lambda_0)$ and $r \in (0, r_0]$,*

$$T_\lambda x \not\leq x, \quad x \in \partial\Omega_r \cap \mathbb{K}. \tag{3.1}$$

(ii) *If (H_2) holds, given $\lambda_0 > 0$, there exists $r_0 > 0$ such that (3.1) holds for $\lambda \in (0, \lambda_0)$ and $r \geq r_0$.*

(iii) *Given $r_0 > 0$, there exists $\lambda_0 > 0$ such that (3.1) holds for $\lambda \in (0, \lambda_0)$ and $r = r_0$.*

PROOF. If (3.1) is not true for some $\lambda > 0$ and $r > 0$, i.e. there exists $y \in \partial\Omega_r \cap \mathbb{K}$ such that $T_\lambda y(t) \geq y(t)$ for $t \in [0, T]$, then

$$\begin{aligned} r = \|y\| &\leq \|T_\lambda y\| = \lambda \sum_{i=1}^n \max_{0 \leq t \leq T} \int_0^T G_i(t, s) g^i(s) f^i(y(s)) ds \\ &\leq \lambda \sum_{i=1}^n M_i \int_0^T g^i(s) f^i(y(s)) ds. \end{aligned} \tag{3.2}$$

(i) Given $\lambda_0 > 0$, let

$$\eta = \left(\lambda_0 \sum_{i=1}^n M_i \int_0^T g^i(s) ds \right)^{-1} > 0.$$

If (H_1) holds, $\hat{f}_0^i = 0, i = 1, 2, \dots, n$ by lemma 2.1, and then there exists $r_0 > 0$ such that for $r \in (0, r_0]$,

$$\hat{f}^i(r) \leq \eta r, \quad i = 1, 2, \dots, n. \quad (3.3)$$

Suppose that (3.1) is not true for some $\lambda \in (0, \lambda_0)$ and $r \in (0, r_0]$, then there exists $y \in \partial\Omega_r \cap \mathbb{K}$ such that (3.2) holds, and thus

$$\begin{aligned} r &\leq \lambda \sum_{i=1}^n M_i \int_0^T g^i(s) f^i(y(s)) ds \leq \lambda \sum_{i=1}^n M_i \int_0^T g^i(s) \hat{f}^i(r) ds \\ &< \lambda_0 \eta r \sum_{i=1}^n M_i \int_0^T g^i(s) ds = r. \end{aligned} \quad (3.4)$$

This contradiction implies that (i) is true.

(ii) Given $\lambda > 0$. If (H_2) holds, $\hat{f}_\infty^i = 0, i = 1, 2, \dots, n$ by lemma 2.1, and then there exists $r_0 > 0$ such that (3.3) holds for $r \geq r_0$. If (3.1) is not true for some $\lambda \in (0, \lambda_0)$ and $r \geq r_0$, the deduction of contradiction (3.4) will also be valid, and this implies that (ii) holds.

(iii) Given $r_0 > 0$, let

$$\lambda_0 = r_0 \left(\sum_{i=1}^n M_i \hat{f}^i(r_0) \int_0^T g^i(s) ds \right)^{-1} > 0.$$

Suppose that (3.1) is not true for $r = r_0$ and $\lambda \in (0, \lambda_0)$. Then there exists $y \in \partial\Omega_{r_0} \cap \mathbb{K}$ such that (3.2) holds, and

$$r_0 \leq \lambda \sum_{i=1}^n M_i \int_0^T g^i(s) f^i(y(s)) ds < \lambda_0 \sum_{i=1}^n M_i \hat{f}^i(r_0) \int_0^T g^i(s) ds = r_0.$$

This contradiction implies that (iii) holds. \square

Lemma 3.2. (i) If (H_3) holds, given $\lambda_0 > 0$, there exists $r_0 > 0$ such that for $\lambda > \lambda_0$ and $r \in (0, r_0]$,

$$T_\lambda x \not\leq x \quad \text{for } x \in \partial\Omega_r \cap \mathbb{K}. \quad (3.5)$$

(ii) If (H_4) holds, given $\lambda_0 > 0$, there exists $r_0 > 0$ such that (3.5) holds for $\lambda > \lambda_0$ and $r \geq r_0$.

PROOF. Let

$$\underline{g} = \min_{1 \leq i \leq n, 0 \leq t \leq T} g^i(t) > 0.$$

If (3.5) is not true for some $\lambda > 0, r > 0$, i.e., there exists $y \in \partial\Omega_r \cap \mathbb{K}$ such that $T_\lambda y(t) \leq y(t)$ for $t \in [0, T]$, then

$$\begin{aligned} r = \|y\| &\geq \|T_\lambda y\| \geq \frac{1}{T} \int_0^T |T_\lambda y(t)| dt \\ &= \frac{\lambda}{T} \sum_{i=1}^n \int_0^T \int_0^T G_i(t, s) g^i(s) f^i(y(s)) dt ds \geq \frac{\lambda \underline{g} \beta}{T} \sum_{i=1}^n \int_0^T f^i(y(s)) ds. \end{aligned} \quad (3.6)$$

(i) Given $\lambda_0 > 0$, let $\eta = MT(\lambda_0 \underline{g} \beta^2)^{-1} > 0$. If (H_3) holds, there exists $r_0 > 0$ such that for $r \in (0, r_0]$,

$$f^j(x) \geq \eta|x|, \quad |x| \leq r \quad \text{for some } j \in \{1, 2, \dots, n\}.$$

Suppose that (3.5) is not true for some $\lambda > \lambda_0$ and $r \in (0, r_0]$. Then there exists $y \in \partial\Omega_r \cap \mathbb{K}$ such that (3.6) holds, and thus

$$\begin{aligned} r &\geq \frac{\lambda \underline{g} \beta}{T} \sum_{i=1}^n \int_0^T f^i(y(s)) ds \geq \frac{\lambda \underline{g} \beta}{T} \int_0^T f^j(y(s)) ds \\ &\geq \frac{\eta \lambda \underline{g} \beta}{T} \int_0^T |y(s)| ds > \frac{\eta \lambda_0 \underline{g} \beta^2}{MT} \|y\| = \|y\| = r. \end{aligned}$$

This contradiction implies that (i) holds.

(ii) Given $\lambda_0 > 0$, let η be as in (i). If (H_4) holds, there exists $r_0 > 0$ such that for $r \geq r_0$,

$$f^j(x) \geq \eta|x|, \quad |x| \geq \frac{\beta r}{M} \quad \text{for some } j \in \{1, 2, \dots, n\}.$$

Noticing that f^j is convex, then for $z \in \Omega_r \cap \mathbb{K}$,

$$\int_0^T |z(s)| ds \geq \frac{\beta}{M} \|z\| = \frac{\beta r}{M},$$

and thus, in view of Jensen's Inequality (see e.g. [21, Lemma 2.8]), we have

$$\int_0^T f^j(z(s)) ds \geq T f^j \left(\frac{1}{T} \int_0^T z(s) ds \right) \geq T \eta \left| \frac{1}{T} \int_0^T z(s) ds \right| = \eta \int_0^T |z(s)| ds \geq \frac{\eta \beta r}{M}.$$

Suppose that (3.5) is not true for some $\lambda > \lambda_0$ and $r \geq r_0$, then there exists $y \in \partial\Omega_r \cap \mathbb{K}$ such that (3.6) holds, and hence

$$r \geq \frac{\lambda g\beta}{T} \sum_{i=1}^n \int_0^T f^i(y(s))ds \geq \frac{\lambda g\beta}{T} \int_0^T f^j(y(s))ds > \frac{\eta\lambda_0 g\beta^2}{MT} r = r.$$

This contradiction implies that (ii) holds. □

PROOF OF THEOREM 3.1. Fix $\lambda > 0$. If (H₁) is true, it follows from Lemma 3.1 that, given $\lambda_0 > \lambda$, there exists $r_0 > 0$ such that (3.1) holds for $r = r_0$ and the above $\lambda \in (0, \lambda_0)$. Meanwhile, if (H₄) holds, it follows from Lemma 3.2 that, given $\lambda'_0 \in (0, \lambda)$, there exists $r'_0 > 0$ such that (3.5) holds for $r \geq r'_0$ and the above $\lambda > \lambda'_0$. Let $r_1 > \max\{r_0, r'_0\}$. Then (3.5) holds for $r = r_1$. Now by Lemma 2.2 and 2.3, T_λ has a fixed point in $\mathbb{K} \cap (\bar{\Omega}_{r_1} \setminus \Omega_{r_0})$. That is (1.1) has a nonnegative solution for $\lambda > 0$ if (H₁) and (H₄) hold. Similarly, we can prove that (1.1) has a nonnegative solution for $\lambda > 0$ if (H₂) and (H₃) hold. □

PROOF OF THEOREM 3.2. We prove only the case when (H₃) holds. The case when (H₄) holds can be proved similarly, and we omit the details. For any $a > 0$, let $r_1 \in (0, a)$. By Lemma 3.1, there is $\lambda_a > 0$ such that (3.1) holds for $\lambda \in (0, \lambda_a)$ and $r = r_1$. Fix any $\lambda \in (0, \lambda_a)$, and let $\lambda_0 \in (0, \lambda)$. Assume that (H₃) is true. Then by Lemma 3.2, there is $r_0 > 0$ such that (3.5) holds for $r \in (0, r_0]$ and the above $\lambda > \lambda_0$. Let $r_2 \in (0, \min\{r_1, r_0\})$. Then for the above λ , (3.5) holds with $r = r_2$. Now it follows from Lemma 2.2 and 2.3 that T_λ has a fixed point in $x \in \mathbb{K} \cap (\bar{\Omega}_{r_1} \setminus \Omega_{r_2})$ for $\lambda \in (0, \lambda_a)$. That is (1.1) has a nonnegative solution $x \in \mathbb{K} \cap (\bar{\Omega}_{r_1} \setminus \Omega_{r_2})$ for $\lambda \in (0, \lambda_a)$. This completes the proof. □

PROOF OF THEOREM 3.3. It follows from assumption (H₅) that there exists a constant η such that

$$f^i(x) \leq \eta|x| \quad \text{for } x \in \mathbb{R}_+^n, \quad i = 1, 2, \dots, n.$$

Let

$$\lambda_0 = \left(\eta \sum_{i=1}^n M_i \int_0^T g^i(s)ds \right)^{-1}.$$

Suppose that (1.1) has a nonnegative solution x for some $\lambda \in (0, \lambda_0)$. Then

$$\begin{aligned} \|x\| &= \|T_\lambda x\| = \lambda \sum_{i=1}^n \max_{0 \leq t \leq T} \int_0^T G_i(t, s) g^i(s) f^i(x(s)) ds \\ &\leq \lambda \eta \sum_{i=1}^n M_i \int_0^T g^i(s) |x(s)| ds < \lambda_0 \eta \|x\| \sum_{i=1}^n M_i \int_0^T g^i(s) ds = \|x\|. \end{aligned}$$

This is a contradiction. So (1.1) has no nonnegative solution for $\lambda \in (0, \lambda_0)$. □

3.3. Example. We now present an example to illustrate our results. We note that similar examples were discussed in [4], [6], [12], [7], [19]. Consider the following periodic boundary value problem: for $\lambda \in (0, \infty), i = 1, 2, \dots, n$,

$$\begin{cases} x_i'' + \frac{\pi^2}{T^2}x_i = \lambda g^i(t)(a_i|x|^{\alpha/2} + |x|^\alpha), & 0 \leq t \leq T, \\ x_i(0) = x_i(T), \quad x_i'(0) = x_i'(T), \end{cases} \tag{3.7}$$

where $g^i(t), i = 1, 2, \dots, n$ are positive continuous functions on $[0, T]$ and $\alpha > 0, a_i \geq 0$. It is easy to get the Green's functions associated with (3.7): for $i = 1, 2, \dots, n$,

$$G_i(t, s) = \begin{cases} \frac{T}{2\pi} \sin \frac{\pi}{T}(t - s), & 0 \leq s \leq t \leq T, \\ \frac{T}{2\pi} \sin \frac{\pi}{T}(s - t), & 0 \leq t \leq s \leq T. \end{cases}$$

Let

$$\bar{G}_i(x) = \frac{T}{2\pi} \sin \frac{\pi x}{T}, \quad x \in [0, T].$$

Clearly, \bar{G}_i is increasing on $[0, T/2]$ and decreasing on $[T/2, T]$, and $G_i(t, s) = \bar{G}_i(|t - s|)$ for $t, s \in [0, T]$. Moreover,

$$0 = \bar{G}_i(0) \leq G_i(t, s) \leq \bar{G}_i\left(\frac{T}{2}\right) = \frac{T}{2\pi},$$

$$\beta = \min_{1 \leq i \leq n, 0 \leq s \leq T} \int_0^T G_i(t, s) dt = \frac{T^2}{\pi^2}.$$

For the convenience of writing, we denote

(A₁) $a_i > 0$ for some $i \in \{1, 2, \dots, n\}$.

(A₂) $a_i = 0$ for all $i = 1, 2, \dots, n$.

Then we can get easily that

- (H₁) holds if $\alpha \in (2, \infty)$ and (A₁) holds or if $\alpha \in (1, \infty)$ and (A₂) holds.
- (H₂) holds if $\alpha \in (0, 1)$.
- (H₃) holds if $\alpha \in (0, 2)$ and (A₁) holds or if $\alpha \in (0, 1)$ and (A₂) holds.
- (H₄) holds if $\alpha \in (1, \infty)$.
- (H₅) holds if $\alpha = 1$ and (A₂) holds.

Now we can apply Theorem 3.1–3.3 to obtain the following.

Corollary 3.2. (i) (3.7) has a nonnegative solution for all $\lambda > 0$ if one of the following statements is true:

- (a) $\alpha \in (2, \infty)$ and (A_1) holds.
 - (b) $\alpha \in (1, \infty)$ and (A_2) holds.
 - (c) $\alpha \in (0, 1)$.
- (ii) The following statements hold:
- (a) For any $a \in (0, \infty)$, there exists $\lambda_a > 0$ such that (3.7) has a nonnegative solution x with $\|x\| < a$ for $\lambda \in (0, \lambda_a)$ if $\alpha \in (0, 2)$ and (A_1) holds or if $\alpha \in (0, 1)$ and (A_2) holds.
 - (b) For any $b \in (0, \infty)$, there exists $\lambda_b > 0$ such that (3.7) has a nonnegative solution x with $\|x\| > b$ for $\lambda \in (0, \lambda_b)$ if $\alpha \in (1, \infty)$.
- (iii) There exists $\lambda_0 > 0$ such that (3.7) has two nonnegative solutions for $\lambda \in (0, \lambda_0)$ if $\alpha \in (1, 2)$ and (A_1) holds.
- (iv) There exists $\lambda_0 > 0$ such that (3.7) has no nonnegative solution for $\lambda \in (0, \lambda_0)$ if $\alpha = 1$ and (A_2) holds.

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