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Blaschke–Minkowski homomorphisms and affine surface area

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Abstract. Schuster introduced the notion of Blaschke–Minkowski homomorphisms and first considered related Shephard type problems. In this paper, we obtain affirmative and negative parts of a generalization of the Winterniz problem for affine surface area with respect to Blaschke–Minkowski homomorphisms.

1. Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}^n_e$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we denote by $\omega_n = V(B)$ its volume.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty,\infty)$, is defined by (see [8], [43])

$$h(K,x) = \max\{x\cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

The projection body ΠK of $K \in \mathcal{K}^n$ is the origin-symmetric convex body whose support function is defined by (see [8], [43])

$$h_{\Pi K}(u) = \frac{1}{2}\int_{S^{n-1}} |u\cdot v| dS(K,v),$$

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for $u, v \in S^{n-1}$. Here $S(K,\cdot)$ denotes the surface area measure of K. The classical projection body is a very important notion for the study of projections in the Brunn–Minkowski theory, and has received considerable attention over the last decades (see [6]–[8], [11], [12], [22], [23], [31], [32], [43]).

In [51] Shephard posed the following problem: Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, is there the implication

$$\Pi K \subseteq \Pi L \Rightarrow V(K) \leq V(L)? \tag{1.1}$$

Petty and Schneider both showed that the answer to this problem is affirmative if the body L belongs to the class of projection bodies (zonoids). In addition, Schneider showed that if K is sufficiently smooth and has positive curvature but is not a zonoid, then there is an L such that (1.1) does not hold.

A convex body $K \in \mathcal{K}^n$ is said to have a curvature function $f(K,\cdot) : S^n \to \mathbb{R}$ (see [33]), if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot). \tag{1.2}$$

Let $\mathcal{F}^n$ denote the set of convex bodies in $\mathcal{K}^n$ that have a positive continuous curvature function. For $F \in \mathcal{F}^n$, the classical affine surface area of K, $\Omega(K)$, is defined by (see [33])

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{\frac{n}{n+1}} dS(u). \tag{1.3}$$

When the volume in (1.1) is replaced by the affine surface area Ω, the question is called the Winterniz problem and was solved by Lutwak [33].

The study of affine surface areas went back to Blaschke in [5] about one hundred years ago, and its L_p counterpart was first introduced by Lutwak in [35]. Important applications of the L_p-affine surface areas can be found in the articles [30], [38], [52], [53], [56], [57]. One of the most important results regarding the L_p-affine surface area is its related L_p-affine isoperimetric inequality (see [35], [58]). The classification of valuations about affine surface area see the more recent contributions [16], [28], [29]. Recently, the L_p-affine surface area was further extended to the Orlicz Brunn–Minkowski theory, which are natural extensions of the L_p-affine surface area (see [16], [25], [29]).

A function Φ defined on the space $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ and taking values in an ablelian semigroup is called a valuation if

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi K + \Phi L, \tag{1.4}$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{K}^n$.

The theory of real valued valuation is at the center of convex geometry. A systematic study was initiated by Blaschke in the 1930s and continued by Hadwiger culminating in his famous classification of continuous and rigid motion invariant valuations on convex bodies. The surveys (see [36], [37]) and the book (see [21]) are excellent sources for the classical theory of valuations. For some of the more recent results, see [1]–[4], [13], [17]–[19], [24], [26], [39]–[41], [49], [50], [54], [59]–[61]. Motivated by recent results of Ludwig [22], [23] characterizing the projection body map as unique Minkowski valuation (that is, valuation with respect to Minkowski addition) intertwining linear translations, Schuster [46] gave the following definition:

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Blaschke–Minkowski homomorphism if it satisfies the following conditions:

(a) Φ is continuous.

(b) Φ is Blaschke Minkowski additive, i.e., for all $K, L \in \mathcal{K}^n$

$$\Phi(K \# L) = \Phi K + \Phi L.$$

(c) Φ intertwines rotations, i.e., for all $K \in \mathcal{K}^n$ and $\vartheta \in SO(n)$

$$\Phi(\vartheta K) = \vartheta \Phi K.$$

Here $\Phi K + \Phi L$ denotes the Minkowski sum (see (2.1)) of the Blaschke–Minkowski homomorphisms ΦK and ΦL and $K \# L$ is the Blaschke sum of the convex bodies K and L (see (2.9)). $SO(n)$ is the group of rotations in n dimensions. Note that every Blaschke–Minkowski homomorphism is a Minkowski valuation.

Together with the definition of Blaschke–Minkwoski homomorphisms, Schuster [47] started an investigation of Shephard type problems for them and obtained the following results:

Theorem 1.A. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke–Minkowski homomorphism. If $K \in \mathcal{K}^n$ and a translate of L is contained in $\Phi\mathcal{K}^n$, then*

$$\Phi K \subseteq \Phi L \Rightarrow V(K) \leq V(L),$$

and $V(K) = V(L)$ if and only if K and L are translates of each other.

Here $\Phi\mathcal{K}^n$ denotes the range of Φ.

Theorem 1.B. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke–Minkowski homomorphism and $\mathcal{K}^n_e \subseteq \mathcal{K}^n(\Phi)$. If $K \in \mathcal{K}^n_e$ is a polynomial convex body and has positive curvature, then if $K \notin \Phi\mathcal{K}^n$, there exists a convex body $L \in \mathcal{K}^n_e$, such that*

$$\Phi K \subseteq \Phi L,$$

but

$$V(K) > V(L).$$

Here $\mathcal{K}^n(\Phi)$ denotes the injectivity set of Φ (see [47]) and a convex body is called polynomial if the spherical harmonics expansion of its support function is finite.

In this article, we investigate a generalization of the Winterniz problem for Blaschke–Minkowski homomorphisms.

Theorem 1.1. *Let $K \in \mathcal{F}^n$. If $L \in \mathcal{W}^n$ and $\Phi K \subseteq \Phi L$, then*

$$\Omega(K) \leq \Omega(L),$$

with equality if and only if K is a translate of L.

Here

$$\mathcal{W}^n = \{Q \in \mathcal{F}^n : \text{there exists } Z \in \Phi\mathcal{K}^n \text{ with } f(Q,\cdot) = h(Z,\cdot)^{-(n+1)}\},$$

where $f(Q,\cdot)$ is the curvature function of Q.

Theorem 1.2. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke–Minkowski homomorphism and $L \in \mathcal{K}^n$. If L is not an origin-symmetric convex body, then there exists $K \in \mathcal{K}^n_e$, such that*

$$\Phi K \subseteq \Phi L,$$

but

$$\Omega(K) > \Omega(L).$$

2. Preliminaries

2.1. Mixed volumes. Here, we recall some basic notions and notations about mixed volumes which can be found in the books [8] and [43].

For $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1, \lambda_2 \geq 0$ (not both zero), the support function of the Minkowski linear combination $\lambda_1 K_1 + \lambda_2 K_2$ is

$$h(\lambda_1 K_1 + \lambda_2 K_2, \cdot) = \lambda_1 h(K_1, \cdot) + \lambda_2 h(K_2, \cdot). \tag{2.1}$$

The volume of a Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$ of convex bodies $K_1, \ldots, K_m$ is an n-homogeneous polynomial given by

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n} V(K_{i_1}, \ldots, K_{i_n})\lambda_{i_1} \ldots \lambda_{i_n}. \tag{2.2}$$

The coefficients $V(K_{i_1}, \ldots, K_{i_n})$ are called mixed volumes of $K_{i_1}, \ldots, K_{i_n}$. These functions are nonnegative, symmetric and translation invariant. Moreover, they are monotone (with respect to set inclusion), multilinear with respect to Minkowski addition and their diagonal form is ordinary volume, i.e., $V(K, \ldots, K) = V(K)$.

Denote by $V_i(K, L)$ the mixed volume $V(K, \ldots, K, L, \ldots, L)$, where K appears $n - i$ times and L appears i times.

For $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, there exists a Borel measure on S^{n-1}, $S(K_1, \ldots, K_{n-1}, \cdot)$, called the mixed surface area measure of $K_1, \ldots, K_{n-1}$ which is symmetric and has the property that, for each $K \in \mathcal{K}^n$,

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}, u). \tag{2.3}$$

The measures $S_i(K, \cdot) = S(K, \ldots, K, B, \ldots, B, \cdot)$, where B appears i times and K appears $n - i - 1$ times, are called the surface area measures of order $n - i - 1$ of K. If $i = 0$, then we write $S(K, \cdot)$ for $S_0(K, \cdot)$, for the surface area measure of K. Thus from the definition of $V_i(K, L)$, and (2.3), we have

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS(K, u). \tag{2.4}$$

2.2. Mixed affine surface area. Lutwak in [34] defined the ith mixed affine surface area as follows: For $K, L \in \mathcal{F}^n$ and real i, the ith mixed affine surface area, $\Omega_i(K, L)$, of K and L is defined by

$$\Omega_i(K, L) = \int_{S^{n-1}} f(K, u)^{\frac{n-i}{n+1}} f(L, u)^{\frac{i}{n+1}} dS(u). \tag{2.5}$$

From definitions (1.3) and (2.5), it obviously follows that

$$\Omega_i(K, K) = \Omega(K). \tag{2.6}$$

Further, Lutwak in [34] proved the following cyclic inequality for the ith mixed affine surface area.

Theorem 2.A. *If $K, L \in \mathcal{F}^n$, $i, j, k \in \mathbb{R}$ and $i < j < k$, then*

$$\Omega_i(K,L)^{k-j}\Omega_k(K,L)^{j-i} \geq \Omega_j(K,L)^{k-i}, \tag{2.7}$$

with equality if and only if K and L are homothetic.

If we take $i = -1$, $j = 0$, $k = n$ in (2.7)*, it follows from* (1.3) *and* (2.5) *that if $K, L \in \mathcal{F}^n$, then*

$$\Omega_{-1}(K,L)^n \geq \Omega(K)^{n+1}\Omega(L)^{-1}, \tag{2.8}$$

with equality if and only if K and L are homothetic.

From (2.8), we easily obtain

Theorem 2.1. *Let $K, L \in \mathcal{F}^n$, then*

$$\Omega_{-1}(K,Q) = \Omega_{-1}(L,Q)$$

for all $Q \in \mathcal{F}^n$ if and only if K and L are homothetic.

The Blaschke combination of convex bodies with non-empty interiors is as follows (see [8], [43]): If $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$ (not both zero), then there exists a convex body $\lambda \odot K \# \mu \odot L$, such that

$$S(\lambda \odot K \# \mu \odot L, \cdot) = \lambda S(K,\cdot) + \mu S(L,\cdot). \tag{2.9}$$

This addition and scalar multiplication are called Blaschke addition and scalar multiplication.

Blaschke addition is the most important operation between sets in convex geometry, and has found many applications in geometry. For example, it was applied by Petty [42] and Schneider [44] in their independent solutions of Shephard's problem for origin-symmetric convex bodies. Blaschke sums also appear in the theory of valuations (see [14], [15], [20], [27], [45], [48]). Moreover, for the characterization related to Blaschke addition, we refer to [9], [10].

Taking $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in the Blaschke combination $\lambda \odot K \# \mu \odot L$, leads to the Blaschke body, ∇K, of $K \in \mathcal{K}^n$ (see [8, 43]):

$$\nabla K = \frac{1}{2} \odot K \# \frac{1}{2} \odot (-K). \tag{2.10}$$

From the definitions of the affine surface area and the Blaschke body, Lutwak in [34] obtained the following result.

Theorem 2.B. *If $K \in \mathcal{F}^n$, then*

$$\Omega(\nabla K) \geq \Omega(K), \tag{2.11}$$

with equality if and only if K is an origin-symmetric convex body.

2.3. Spherical convolution. In the following we state some material on convolutions, which can be found in the reference [48].

As usual, $SO(n)$ and S^{n-1} will be equipped with the invariant probability measures. Let $\mathcal{C}(SO(n))$ and $\mathcal{C}(S^{n-1})$ be the spaces of continuous functions on $SO(n)$ and S^{n-1} with uniform topology and let $\mathcal{M}(SO(n))$ and $\mathcal{M}(S^{n-1})$ denote their dual spaces of signed finite Borel measures with the weak topology. If $\mu, \sigma \in \mathcal{M}(SO(n))$, the convolution $\mu * \sigma$ is defined by

$$\int_{SO(n)} f(\vartheta) d(\mu * \sigma)(\vartheta) = \int_{SO(n)} \int_{SO(n)} f(\eta\tau) d\mu(\eta) d\sigma(\tau), \tag{2.12}$$

for every $f \in \mathcal{C}(SO(n))$ and $\vartheta \in SO(n)$. The sphere S^{n-1} can be identified with the homogeneous space $SO(n)/SO(n-1)$, where $SO(n-1)$ denotes the subgroup of rotations leaving the pole $\hat{e}$ of S^{n-1} fixed. The projection from $SO(n)$ onto S^{n-1} is $\vartheta \mapsto \hat{\vartheta} := \vartheta\hat{e}$. Every $f \in \mathcal{C}(S^{n-1})$ gives rise to a right $SO(n-1)$ invariant function on $SO(n)$ defined by $\check{f}(\vartheta) = f(\hat{\vartheta})$. In fact, $\mathcal{C}(S^{n-1})$ is isomorphic to the subspace of right $SO(n-1)$-invariant function in $\mathcal{C}(SO(n))$ and this correspondence carries over to an identification of the space $\mathcal{M}(S^{n-1})$ with right $SO(n-1)$-invariant measures in $\mathcal{M}(SO(n))$.

For $\mu \in \mathcal{M}(SO(n))$, the convolutions $\mu * f \in \mathcal{C}(SO(n))$ and $f * \mu \in \mathcal{C}(SO(n))$ with a function $f \in \mathcal{C}(SO(n))$ are defined by

$$(f * \mu)(\eta) = \int_{SO(n)} f(\eta\vartheta^{-1}) d\mu(\vartheta), \quad (\mu * f)(\eta) = \int_{SO(n)} \vartheta f(\eta) d\mu(\vartheta). \tag{2.13}$$

For $\vartheta \in SO(n)$, the left translation ϑf of $f \in \mathcal{C}(SO(n))$ is defined by

$$\vartheta f(\eta) = f(\vartheta^{-1}\eta). \tag{2.14}$$

A function $f \in \mathcal{C}(S^{n-1})$ is called zonal, if $\vartheta f = f$ for every $\vartheta \in SO(n-1)$. Zonal functions depend only on the value $u \cdot \hat{e}$. The set of continuous zonal function on S^{n-1} will be denoted by $\mathcal{C}(S^{n-1}, \hat{e})$ and the definition of $\mathcal{M}(S^{n-1}, \hat{e})$ is analogous.

Theorem 2.C ([48]). *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke–Minkowski homomorphism, then there is a generating function $g \in \mathcal{C}(S^{n-1}, \hat{e})$, unique up to addition of a linear function, such that*

$$h(\Phi K, \cdot) = S(K, \cdot) * g. \tag{2.15}$$

3. Proofs of the main results

In this section, we complete the proofs of Theorems 1.1 and 1.2. For the proof of Theorem 1.1, we need the following lemma:

Lemma 3.1 ([47])**.** *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke–Minkowski homomorphism, then for $K, L \in \mathcal{K}^n$*

$$V_1(K, \Phi L) = V_1(L, \Phi K). \tag{3.1}$$

Proof of Theorem 1.1. Since $Q \in \mathcal{W}^n$, there exists $Z \in \Phi\mathcal{K}^n$ such that

$$f(Q, \cdot)^{-\frac{1}{n+1}} = h(Z, \cdot).$$

Since $Z \in \Phi\mathcal{K}^n$, there exists $M \in \mathcal{K}^n$ such that $Z = \Phi M$. Hence, using (1.2), (2.5) and (3.1), we have

$$\begin{aligned}\frac{\Omega_{-1}(L,Q)}{\Omega_{-1}(K,Q)} &= \frac{\int_{S^{n-1}} f(Q,u)^{-\frac{1}{n+1}} dS(L,u)}{\int_{S^{n-1}} f(Q,u)^{-\frac{1}{n+1}} dS(K,u)} = \frac{\int_{S^{n-1}} h(Z,u)dS(L,u)}{\int_{S^{n-1}} h(Z,u)dS(K,u)} = \frac{V_1(L,Z)}{V_1(K,Z)} \\ &= \frac{V_1(L,\Phi M)}{V_1(K,\Phi M)} = \frac{V_1(M,\Phi L)}{V_1(M,\Phi K)} = \frac{\int_{S^{n-1}} h(\Phi L,u)dS(M,u)}{\int_{S^{n-1}} h(\Phi K,u)dS(M,u)}.\end{aligned}$$

Since $\Phi K \subseteq \Phi L$, it follows that

$$\Omega_{-1}(K, Q) \le \Omega_{-1}(L, Q). \tag{3.2}$$

By Theorem 2.1, we know that equality holds in (3.2) if and only if K is a translate of L. Since $L \in \mathcal{W}^n$, taking $Q = L$ in (3.2), and using (2.6) and inequality (2.8), we get

$$\Omega(L) \ge \Omega_{-1}(K, L) \ge \Omega(K)^{\frac{n+1}{n}} \Omega(L)^{-\frac{1}{n}},$$

i.e.,

$$\Omega(K) \le \Omega(L). \tag{3.3}$$

By the equality conditions of (2.8) and (3.2), we see that equality holds in (3.3) if and only if K is a translate of L. □

Lemma 3.2. *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is an even Blaschke–Minkowski homomorphism and $K \in \mathcal{K}^n$, then*

$$\Phi(\nabla K) = \Phi K.$$

Proof. From (2.15), (2.9) and (2.10), it follows that

$$\begin{aligned} h(\Phi(\nabla K), u) &= S(\nabla K, u) * g = S\left(\frac{1}{2} \odot K \# \frac{1}{2} \odot (-K), u\right) * g \\ &= \left(\frac{1}{2} S(K, u) + \frac{1}{2} S(-K, u)\right) * g \\ &= \frac{1}{2} S(K, u) * g + \frac{1}{2} S(-K, u) * g \\ &= \frac{1}{2} h(\Phi K, u) + \frac{1}{2} h(\Phi(-K), u). \end{aligned}$$

Since $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is even, i.e., $\Phi(-K) = \Phi K$, it follows that

$$\Phi(\nabla K) = \Phi K. \qquad \square$$

Proof of Theorem 1.2. Since L is not origin-symmetric, by Theorem 2.B, we know that

$$\Omega(\nabla L) > \Omega(L).$$

Choose $\varepsilon > 0$, such that $\Omega((1-\varepsilon)\nabla L) > \Omega(L)$, and let $K = (1-\varepsilon)\nabla L$. Then

$$\Omega(K) > \Omega(L),$$

but from Lemma 3.2, and the fact that $\Phi((1-\varepsilon)K) = (1-\varepsilon)^{n-1}\Phi K$, we obtain

$$\Phi K = \Phi((1-\varepsilon)\nabla L) = (1-\varepsilon)^{n-1}\Phi(\nabla L) = (1-\varepsilon)^{n-1}\Phi L \subseteq \Phi L. \qquad \square$$

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