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Co-commutators with generalized derivations in prime and semiprime rings

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Abstract. Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, F and G two nonzero generalized derivations of R, I an ideal of R and $f(x_1,\dots,x_n)$ be a multilinear polynomial over C which is not central valued on R. If

$$F^2(f(x_1,\dots,x_n))f(x_1,\dots,x_n) - f(x_1,\dots,x_n)G^2(f(x_1,\dots,x_n)) = 0$$

for all $x_1,\dots,x_n \in I$, then one of the following holds:
(1) $F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
(2) $F(x) = xa$ and $G(x) = bx$ for all $x \in R$ with $a^2 = b^2$;
(3) $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
(4) $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2$ and $f(x_1,\dots,x_n)^2$ is central valued on R;
(5) $F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 = b^2 \in C$.

We finally extend the result to a semiprime ring R in case $F^2(x)x - xG^2(x) = 0$ for all $x \in R$.

Throughout this paper R always denotes an associative prime ring with center $Z(R)$, extended centroid C and U its Utumi quotient ring. The Lie commutator of x and y is denoted by $[x,y]$ and defined by $[x,y] = xy - yx$ for $x,y \in R$. An

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additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Evidently, any derivation is a generalized derivation. Thus, the generalized derivation covers both the concepts of derivation and left multiplier mapping. The left multiplier mapping means an additive mapping $F : R \to R$ such that $F(xy) = F(x)y$ holds for all $x, y \in R$. We denote by s_4 the standard identity in four variables.

A well known result of Posner [22] states that if d is a derivation of R such that $d(x)x - xd(x) \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. Several authors generalized Posner's theorem. For instance, Brešar proved in [3] that if d and δ are two derivations of R such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later Lee and Wong [19] consider the situation $d(x)x - x\delta(x) \in Z(R)$ for all x in some noncentral Lie ideal L of R and obtained the result that either $d = \delta = 0$ or R satisfies s_4. In [15], Lee and Shiue consider the situation $d(x)x - x\delta(x) \in C$ for all $x \in \{f(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in R\}$, where $f(x_1, \ldots, x_n)$ is any polynomial over C and obtained that either $d = \delta = 0$, or $\delta = -d$ and $f(x_1, \ldots, x_n)^2$ is central valued on RC, except when $\mathrm{char}(R) = 2$ and $\dim_C RC = 4$.

In [8], the first author and Sharma studied the case when R is a prime ring of $\mathrm{char}(R) \neq 2$, d a derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C and I a right ideal of R such that $d^2(x)x - xd^2(x) = 0$ for all $x \in \{f(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in I\}$, then either $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is satisfied by I, or there exists $b \in U$ such that $d(x) = [b, x]$ for all $x \in R$, with $b^2 = 0$ and $bI = (0)$.

Recently, in [1] Argac and the second author of this paper, studied the case $F(x)x - xG(x) = 0$ for all $x \in \{f(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in I\}$, where F and G two generalized derivations of R and I is an ideal of R and then determined the structure of the maps.

In the present paper, we shall study the situation $F^2(x)x - xG^2(x) = 0$ for all $x \in \{f(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in I\}$, where F and G are two generalized derivations of R and I is an ideal of R.

Lemma 1 ([1, Lemma 1]). *Let R be a noncommutative prime ring, $a, b \in U$, $p(x_1, \ldots, x_n)$ be any polynomial over C, which is not an identity for R. If $ap(r) - p(r)b = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

(1) *$a = b \in C$,*

(2) *$a = b$ and $p(x_1, \ldots, x_n)$ is central valued on R,*

(3) $\mathrm{char}(R) = 2$ *and* R *satisfies* s_4

Lemma 2 ([1, Lemma 3])**.** *Let* R *be a noncommutative prime ring with Utumi quotient ring* U *and extended centroid* C*, and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C*, which is not central valued on* R*. Suppose that there exist* $a, b, c, q \in U$ *such that* $(af(r) + f(r)b)f(r) - f(r)(cf(r) + f(r)q) = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$*. Then one of the following holds:*

(1) $a, q \in C$ *and* $q - a = b - c = \alpha \in C$*;*

(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exists* $\alpha \in C$ *such that* $q - a = b - c = \alpha$*;*

(3) $\mathrm{char}(R) = 2$ *and* R *satisfies* s_4*.*

In particular, from above Lemma, we have the followings:

Lemma 3. *Let* R *be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring* U *and extended centroid* C*, and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C*, which is not central valued on* R*. Suppose that there exist* $c, q \in U$ *such that* $f(r)(cf(r) + f(r)q) = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$*. Then* $q = -c \in C$*.*

Lemma 4. *Let* R *be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring* U *and extended centroid* C*, and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C*, which is not central valued on* R*. Suppose that there exist* $c \in U$ *such that* $f(r)cf(r) = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$*. Then* $c = 0$*.*

Lemma 5. *Let* R *be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring* U *and extended centroid* C*, and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C*, which is not central valued on* R*. Suppose that there exist* $a, b \in U$ *such that* $(af(r) + f(r)b)f(r) = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$*. Then* $-a = b \in C$*.*

Lemma 6 (Lemma 1 in [6])**.** *Let* C *be an infinite field and* $m \geq 2$*. If* $A_1, \ldots, A_k$ *are not scalar matrices in* $M_m(C)$ *then there exists some invertible matrix* $P \in M_m(C)$ *such that any matrices* $PA_1P^{-1}, \ldots, PA_kP^{-1}$ *have all non-zero entries.*

Proposition 1. *Let* $R = M_m(C)$ *be the ring of all* $m \times m$ *matrices over the infinite field* C*,* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C *and* $a, a', b, b', c, q, q' \in R$*. If*

$$a'f(r)^2 + 2af(r)bf(r) + f(r)b'f(r) - 2f(r)cf(r)q - f(r)^2q' = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$, *then either* a *or* b *and either* c *or* q *are central.*

Proof. By our assumption R satisfies the generalized identity

$$a' f(x_1, \dots, x_n)^2 + 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n) + f(x_1, \dots, x_n)b'f(x_1, \dots, x_n) \\ -2f(x_1, \dots, x_n)cf(x_1, \dots, x_n)q - f(x_1, \dots, x_n)^2 q'. \tag{1}$$

We assume first that $a \notin Z(R)$ and $b \notin Z(R)$. Now we shall show that this case leads a contradiction.

Since $a \notin Z(R)$ and $b \notin Z(R)$, by Lemma 6 there exists a C-automorphism ϕ of $M_m(C)$ such that $a_1 = \phi(a)$, $b_1 = \phi(b)$ have all non-zero entries. Clearly a_1, b_1, $c_1 = \phi(c)$, $a'_1 = \phi(a')$, $b'_1 = \phi(b')$, $q_1 = \phi(q)$ and $q'_1 = \phi(q')$ must satisfy the condition (). Without loss of generality we may replace a, a', b, b', c, q, q' with a_1, a'_1, b_1, b'_1, c_1, q_1, q'_1 respectively.

Here e_{kl} denotes the usual matrix unit with 1 in (k, l)-entry and zero elsewhere. Since $f(x_1, \dots, x_n)$ is not central, by [17] (see also [20]), there exist $u_1, \dots, u_n \in M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(u_1, \dots, u_n) = \gamma e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(r_1, \dots, r_n) : r_1, \dots, r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \dots, r_n \in M_m(C)$ such that $f(r_1, \dots, r_n) = e_{ij}$. Hence by () we have

$$2ae_{ij}be_{ij} + e_{ij}b'e_{ij} - 2e_{ij}ce_{ij}q = 0 \tag{2}$$

and then left multiplying by e_{ij}, it follows $2e_{ij}ae_{ij}be_{ij} = 0$, which is a contradiction, since a and b have all non-zero entries. Thus we conclude that either a or b are central.

Similarly we can prove that c or q are central. □

Proposition 2. *Let* $R = M_m(C)$ *be the ring of all matrices over the field* C *with* $\mathrm{char}(R) \neq 2$ *and* $f(x_1, \dots, x_n)$ *a non-central multilinear polynomial over* C *and* $a, a', b, b', c, q, q' \in R$. *If*

$$a' f(r)^2 + 2af(r)bf(r) + f(r)b'f(r) - 2f(r)cf(r)q - f(r)^2 q' = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$, *then either* a *or* b *and either* c *or* q *are central.*

Proof. If one assumes that C is infinite, then the conclusions follow by Proposition 1.

Now let C be finite and K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial

$f(x_1,\ldots,x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned}P(x_1,\ldots,x_n) &= a'f(x_1,\ldots,x_n)^2 + 2af(x_1,\ldots,x_n)bf(x_1,\ldots,x_n)\\ &\quad + f(x_1,\ldots,x_n)b'f(x_1,\ldots,x_n)\\ &\quad - 2f(x_1,\ldots,x_n)cf(x_1,\ldots,x_n)q - f(x_1,\ldots,x_n)^2q' \qquad (3)\end{aligned}$$

which is a generalized polynomial identity for R.

Moreover, it is a multi-homogeneous of multi-degree $(2,\ldots,2)$ in the indeterminates $x_1,\ldots,x_n$.

Hence the complete linearization of $P(x_1,\ldots,x_n)$ is a multilinear generalized polynomial $\Theta(x_1,\ldots,x_n,y_1,\ldots,y_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1,\ldots,x_n,x_1,\ldots,x_n) = 2^nP(x_1,\ldots,x_n).$$

Clearly the multilinear polynomial $\Theta(x_1,\ldots,x_n,y_1,\ldots,y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $\mathrm{char}(C)\neq 2$ we obtain $P(r_1,\ldots,r_n)=0$ for all $r_1,\ldots,r_n\in\overline{R}$ and then conclusion follows from Proposition 1. □

Lemma 7. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that for some $a,b,c,q\in U$, $F(x)=ax+xb$ and $G(x)=cx+xq$ for all $x\in R$ such that*

$$F^2(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)G^2(f(x_1,\ldots,x_n)) = 0$$

for all $x_1,\ldots,x_n\in R$. Then one of the following holds:

1. *$F(x)=x(a+b)$ and $G(x)=x(c+q)$ for all $x\in R$ with $(a+b)^2=(c+q)^2\in C$;*
2. *$F(x)=x(a+b)$ and $G(x)=(c+q)x$ for all $x\in R$ with $(a+b)^2=(c+q)^2$;*
3. *$F(x)=(a+b)x$ and $G(x)=x(c+q)$ for all $x\in R$ with $(a+b)^2=(c+q)^2\in C$;*
4. *$F(x)=(a+b)x$ and $G(x)=x(c+q)$ for all $x\in R$ with $(a+b)^2=(c+q)^2$ and $f(x_1,\ldots,x_n)^2$ is central valued on R;*
5. *$F(x)=(a+b)x$ and $G(x)=(c+q)x$ for all $x\in R$, with $(a+b)^2=(c+q)^2\in C$.*

Proof. By hypothesis, we have

$$\begin{aligned}h(x_1,\ldots,x_n) &= a^2f(x_1,\ldots,x_n)^2 + 2af(x_1,\ldots,x_n)bf(x_1,\ldots,x_n)\\ &\quad + f(x_1,\ldots,x_n)(b^2-c^2)f(x_1,\ldots,x_n) - 2f(x_1,\ldots,x_n)cf(x_1,\ldots,x_n)q\\ &\quad - f(x_1,\ldots,x_n)^2q^2 = 0 \qquad (4)\end{aligned}$$

for all $x_1, \ldots, x_n \in R$. Since R and U satisfy same generalized polynomial identity (see [4]), U satisfies $h(x_1, \ldots, x_n) = 0$. Suppose that $h(x_1, \ldots, x_n)$ is a trivial GPI for U. Let $T = U *_C C\{x_1, x_2, \ldots, x_n\}$, the free product of U and $C\{x_1, \ldots, x_n\}$, the free C-algebra in noncommuting indeterminates $x_1, x_2, \ldots, x_n$. Then, $h(x_1, \ldots, x_n)$ is zero element in $T = U *_C C\{x_1, \ldots, x_n\}$. This implies that $\{a^2, a, 1\}$ is a linearly dependent over C. Let $\alpha a^2 + \beta a + \gamma = 0$. If $\alpha = 0$, then $\beta \neq 0$, and hence $a \in C$. If $\alpha \neq 0$, then $a^2 = \lambda a + \mu$ for some $\lambda, \mu \in C$. In this case our identity reduces to

$$\begin{aligned}(\lambda a + \mu) f(x_1, \ldots, x_n)^2 &+ 2af(x_1, \ldots, x_n) bf(x_1, \ldots, x_n) \\ &+ f(x_1, \ldots, x_n)(b^2 - c^2) f(x_1, \ldots, x_n) - 2f(x_1, \ldots, x_n) cf(x_1, \ldots, x_n) q \\ &- f(x_1, \ldots, x_n)^2 q^2 = 0. \end{aligned} \tag{5}$$

If $a \notin C$, then

$$\lambda af(x_1, \ldots, x_n)^2 + 2af(x_1, \ldots, x_n) bf(x_1, \ldots, x_n) = 0 \tag{6}$$

that is

$$af(x_1, \ldots, x_n)(\lambda + 2b) f(x_1, \ldots, x_n) = 0. \tag{7}$$

This implies $\lambda + 2b = 0$. Since $\mathrm{char}(R) \neq 2$, this implies $b \in C$. Thus we conclude that either $a \in C$ or $b \in C$.

Similarly we can prove that either $c \in C$ or $q \in C$.

Next suppose that $h(x_1, \ldots, x_n)$ is a non-trivial GPI for U. In case C is infinite, we have $h(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$. By Martindale's theorem [21], R is then a primitive ring with nonzero socle $\mathrm{soc}(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [11, p. 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Lemma 6, we get that a or b and c or q are in C. If V is infinite dimensional over C, then for any $e^2 = e \in \mathrm{soc}(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. We want to show that in this case also a or b and c or q are in C. To prove this, let at least one of a and b and at least one of c and q are not in C. Then at least one of a and b and at least one of c and q does not centralize the nonzero ideal $\mathrm{soc}(R)$. Hence there exist $h_1, h_2, h_3, h_4 \in \mathrm{soc}(R)$ such that either $[a, h_1] \neq 0$ or $[b, h_2] \neq 0$

and $[c, h_3] \neq 0$ or $[q, h_4] \neq 0$. By Litoff's theorem [10], there exists idempotent $e \in \mathrm{soc}(R)$ such that $ah_1, h_1a, bh_2, h_2b, ch_3, h_3c, qh_4, h_4q, h_1, h_2, h_3, h_4 \in eRe$. We have $eRe \cong M_k(C)$ with $k = \dim_C Ve$. Since R satisfies generalized identity

$$\begin{aligned} &e\{a^2 f(ex_1e, \ldots, ex_ne)^2 + 2af(ex_1e, \ldots, ex_ne)bf(ex_1e, \ldots, ex_ne) \\ &\quad + f(ex_1e, \ldots, ex_ne)(b^2 - c^2)f(ex_1e, \ldots, ex_ne) \\ &\quad - 2f(ex_1e, \ldots, ex_ne)cf(ex_1e, \ldots, ex_ne)q - f(ex_1e, \ldots, ex_ne)^2q^2\}e = 0, \end{aligned} \tag{8}$$

the subring eRe satisfies

$$\begin{aligned} &ea^2ef(x_1, \ldots, x_n)^2 + 2eaef(x_1, \ldots, x_n)ebef(x_1, \ldots, x_n) \\ &\quad + f(x_1, \ldots, x_n)e(b^2 - c^2)ef(x_1, \ldots, x_n) \\ &\quad - 2f(x_1, \ldots, x_n)ecef(x_1, \ldots, x_n)eqe - f(x_1, \ldots, x_n)^2eq^2e = 0. \end{aligned} \tag{9}$$

Then by the above finite dimensional case, either eae or ebe and either ece or eqe are central elements of eRe. Thus $ah_1 = (eae)h_1 = h_1eae = h_1a$ or $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$ and $ch_3 = (ece)h_3 = h_3(ece) = h_3c$ or $qh_4 = (eqe)h_4 = h_4eqe = h_4q$, a contradiction.

Thus up to now, we have proved that either a or b and c or q are in C. Thus we have the following four cases:

Case-I: $a, c \in C$.
In this case, we have

$$f(x_1, \ldots, x_n)(a^2+b^2+2ab)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2(c^2+2cq+q^2) = 0 \tag{10}$$

that is

$$f(x_1, \ldots, x_n)\{(a^2+b^2+2ab)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)(c^2+2cq+q^2)\} = 0 \tag{11}$$

for all $x_1, \ldots, x_n \in R$. Then by Lemma 3, we have $a^2+b^2+2ab = c^2+2cq+q^2 \in C$, that is $(a+b)^2 = (c+q)^2 \in C$. Hence we obtain that $F(x) = x(a+b)$ and $G(x) = x(c+q)$ with $(a+b)^2 = (c+q)^2 \in C$, which is our conclusion (1).

Case-II: $a, q \in C$.
In this case, we have,

$$f(x_1, \ldots, x_n)(a^2 + b^2 + 2ab - q^2 - 2cq - c^2)f(x_1, \ldots, x_n) = 0 \tag{12}$$

for all $x_1, \ldots, x_n \in R$. Then by Lemma 4, $a^2 + b^2 + 2ab - q^2 - 2cq - c^2 = 0$, that is $(a+b)^2 = (c+q)^2$. Thus we have $F(x) = x(a+b)$ and $G(x) = (c+q)x$ with $(a+b)^2 = (c+q)^2$.

Case-III: $b, c \in C$.
In this case, we have

$$(a^2 + 2ab + b^2)f(x_1, \dots, x_n)^2 - f(x_1, \dots, x_n)^2(c^2 + 2cq + q^2) = 0 \quad (13)$$

for all $x_1, \dots, x_n \in R$. Then by Lemma 1, we have any one of the following two cases:

- $(a + b)^2 = (c + q)^2 \in C$. Then $F(x) = (a + b)x$ and $G(x) = x(c + q)$ for all $x \in R$.
- $(a + b)^2 = (c + q)^2$ and $f(x_1, \dots, x_n)^2$ is central valued on R. Then $F(x) = (a + b)x$ and $G(x) = x(c + q)$ for all $x \in R$.

Case-IV: $b, q \in C$.
In this case, we have

$$(a^2 + 2ab + b^2)f(x_1, \dots, x_n)^2 - f(x_1, \dots, x_n)(c^2 + 2cq + q^2)f(x_1, \dots, x_n) = 0 \quad (14)$$

that is

$$\{(a^2 + 2ab + b^2)f(x_1, \dots, x_n) - f(x_1, \dots, x_n)(c^2 + 2cq + q^2)\}f(x_1, \dots, x_n) = 0 \quad (15)$$

for all $x_1, \dots, x_n \in R$. Then by Lemma 5, we have $a^2 + 2ab + b^2 = c^2 + 2cq + q^2 \in C$, which is $(a + b)^2 = (c + q)^2 \in C$. Thus $F(x) = (a + b)x$ and $G(x) = (c + q)x$ for all $x \in R$, with $(a + b)^2 = (c + q)^2 \in C$. □

Lemma 8. *Let R be a noncommutative prime ring of characteristic different from 2 and $f(x_1, \dots, x_n)$ a multilinear polynomial over C. If for any $i = 1, \dots, n$,*

$$\left[\sum_{i=0}^{n} f(x_1, \dots, t_i, \dots, x_n), f(x_1, \dots, x_n)\right] = 0 \quad (16)$$

for all $t_i, x_1, \dots, x_n \in R$, then the polynomial $f(x_1, \dots, x_n)$ is central-valued on R.

Proof. Let a be a noncentral element of R. Then replacing t_i with $[a, x_i]$, we have that

$$\left[\sum_{i=0}^{n} f(x_1, \dots, [a, x_i], \dots, x_n), f(x_1, \dots, x_n)\right] = 0 \quad (17)$$

which gives,

$$[a, f(x_1, \dots, x_n)]_2 = 0 \quad (18)$$

for all $x_1, \dots, x_n \in R$ implying $f(x_1, \dots, x_n)$ is central-valued on R [16, Theorem]. □

Theorem 1. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, F and G two nonzero generalized derivations of R, I an ideal of R and $f(x_1,\dots,x_n)$ a multilinear polynomial over C which is not central valued on R. If*

$$F^2(f(x_1,\dots,x_n))f(x_1,\dots,x_n) - f(x_1,\dots,x_n)G^2(f(x_1,\dots,x_n)) = 0$$

for all $x_1,\dots,x_n \in I$, then one of the following holds:

(1) $F(x) = xa$ *and* $G(x) = xb$ *for all* $x \in R$ *with* $a^2 = b^2 \in C$*;*

(2) $F(x) = xa$ *and* $G(x) = bx$ *for all* $x \in R$ *with* $a^2 = b^2$*;*

(3) $F(x) = ax$ *and* $G(x) = xb$ *for all* $x \in R$ *with* $a^2 = b^2 \in C$*;*

(4) $F(x) = ax$ *and* $G(x) = xb$ *for all* $x \in R$ *with* $a^2 = b^2$ *and* $f(x_1,\dots,x_n)^2$ *is central valued on* R*;*

(5) $F(x) = ax$ *and* $G(x) = bx$ *for all* $x \in R$*, with* $a^2 = b^2 \in C$*.*

Proof. In [14, Theorem 3], Lee proved that every generalized derivation g on a dense right ideal of R can be uniquely extended to a generalized derivation of U and thus can be assumed to be defined on the whole U with the form $g(x) = ax + d(x)$ for some $a \in U$ and d is a derivation of U. In light of this, we may assume that there exist $a, b \in U$ and derivations d, δ of U such that $F(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$. Since I, R and U satisfy the same generalized polynomial identities (see [4]) as well as the same differential identities (see [17]), without loss of generality, to prove our results, we may assume $F^2(f(x_1,\dots,x_n))f(x_1,\dots,x_n) - f(x_1,\dots,x_n)G^2(f(x_1,\dots,x_n)) = 0$ for all $x_1,\dots,x_n \in U$, where d, δ are two derivations on U.

If F and G both are inner generalized derivations of R, the by Lemma 7 we obtain our conclusions. Thus we assume that not both of F and G are inner. Hence U satisfies

$$\begin{aligned}\{F(a)f(x_1,\dots,x_n) &+ 2ad(f(x_1,\dots,x_n)) + d^2(f(x_1,\dots,x_n))\}f(x_1,\dots,x_n)\\ &- f(x_1,\dots,x_n)\{G(b)f(x_1,\dots,x_n) + 2b\delta(f(x_1,\dots,x_n))\\ &+ \delta^2(f(x_1,\dots,x_n))\} = 0. \qquad (19)\end{aligned}$$

By assumption, d and δ can not be both inner derivations of U. Assume that d and δ are C-dependent modulo inner derivations of U, say $d = \lambda\delta + ad_p$, where $\lambda \in C$, $p \in U$ and $ad_p(x) = [p, x]$ for all $x \in R$. Then δ can not be inner derivation of U. Moreover,

$$\begin{aligned}d^2(x) &= d(\lambda\delta(x) + [p,x]) = d(\lambda)\delta(x) + \lambda d\delta(x) + [d(p),x] + [p,d(x)]\\ &= d(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda[p,\delta(x)] + [d(p),x] + [p,[p,x]]. \qquad (20)\end{aligned}$$

From (19), we obtain that U satisfies

$$\begin{aligned}&\big\{F(a)f(x_1,\dots,x_n)+2\lambda a\delta(f(x_1,\dots,x_n))+2a[p,f(x_1,\dots,x_n)]\\&\quad+d(\lambda)\delta(f(x_1,\dots,x_n))+\lambda^2\delta^2(f(x_1,\dots,x_n))+2\lambda[p,\delta(f(x_1,\dots,x_n))]\\&\quad+[d(p),f(x_1,\dots,x_n)]+[p,[p,f(x_1,\dots,x_n)]]\big\}f(x_1,\dots,x_n)\\&\quad-f(x_1,\dots,x_n)\big\{G(b)f(x_1,\dots,x_n)\\&\quad+2b\delta(f(x_1,\dots,x_n))+\delta^2(f(x_1,\dots,x_n))\big\}=0.\qquad(21)\end{aligned}$$

This gives

$$\begin{aligned}&\Big\{F(a)f(x_1,\dots,x_n)+(2\lambda a+d(\lambda))\Big(f^{\delta}(x_1,\dots,x_n)+\sum_i f(x_1,\dots,\delta(x_i),\dots,x_n)\Big)\\&+2a[p,f(x_1,\dots,x_n)]+\lambda^2\Big(f^{\delta^2}(x_1,\dots,x_n)+2\sum_i f^{\delta}(x_1,\dots,\delta(x_i),\dots,x_n)\\&+\sum_i f(x_1,\dots,\delta^2(x_i),\dots,x_n)+\sum_{i\neq j} f(x_1,\dots,\delta(x_i),\dots,\delta(x_j),\dots,x_n)\Big)\\&+2\lambda\Big[p,f^{\delta}(x_1,\dots,x_n)+\sum_i f(x_1,\dots,\delta(x_i),\dots,x_n)\Big]+[d(p),f(x_1,\dots,x_n)]\\&+[p,[p,f(x_1,\dots,x_n)]]\Big\}f(x_1,\dots,x_n)-f(x_1,\dots,x_n)\Big\{G(b)f(x_1,\dots,x_n)\\&+2b\Big(f^{\delta}(x_1,\dots,x_n)+\sum_i f(x_1,\dots,\delta(x_i),\dots,x_n)\Big)\\&+f^{\delta^2}(x_1,\dots,x_n)+2\sum_i f^{\delta}(x_1,\dots,\delta(x_i),\dots,x_n)+\sum_i f(x_1,\dots,\delta^2(x_i),\dots,x_n)\\&+\sum_{i\neq j} f(x_1,\dots,\delta(x_i),\dots,\delta(x_j),\dots,x_n)\Big\}=0.\qquad(22)\end{aligned}$$

Then by Kharchenko's theorem [12], we have that U satisfies

$$\begin{aligned}&\Big\{F(a)f(x_1,\dots,x_n)+(2\lambda a+d(\lambda))\Big(f^{\delta}(x_1,\dots,x_n)+\sum_i f(x_1,\dots,y_i,\dots,x_n)\Big)\\&\quad+2a[p,f(x_1,\dots,x_n)]+\lambda^2(f^{\delta^2}(x_1,\dots,x_n)+2\sum_i f^{\delta}(x_1,\dots,y_i,\dots,x_n)\\&\quad+\sum_i f(x_1,\dots,t_i,\dots,x_n)+\sum_{i\neq j} f(x_1,\dots,y_i,\dots,y_j,\dots,x_n))\\&\quad+2\lambda\Big[p,f^{\delta}(x_1,\dots,x_n)+\sum_i f(x_1,\dots,y_i,\dots,x_n)\Big]+[d(p),f(x_1,\dots,x_n)]\end{aligned}$$

$$+ [p, [p, f(x_1, \dots, x_n)]]\Big\} f(x_1, \dots, x_n) - f(x_1, \dots, x_n)\Big\{G(b) f(x_1, \dots, x_n)$$
$$+ 2b(f^{\delta}(x_1, \dots, x_n) + \sum_i f(x_1, \dots, y_i, \dots, x_n))$$
$$+ f^{\delta^2}(f(x_1, \dots, x_n)) + 2\sum_i f^{\delta}(x_1, \dots, y_i, \dots, x_n) + \sum_i f(x_1, \dots, t_i, \dots, x_n)$$
$$+ \sum_{i \neq j} f(x_1, \dots, y_i, \dots, y_j, \dots, x_n)\Big\} = 0. \tag{23}$$

In particular, U satisfies the blended component

$$\Big\{\lambda^2 \sum_i f(x_1, \dots, t_i, \dots, x_n)\Big\} f(x_1, \dots, x_n)$$
$$- f(x_1, \dots, x_n)\Big\{\sum_i f(x_1, \dots, t_i, \dots, x_n)\Big\} = 0. \tag{24}$$

In particular, when $t_1 = x_1$ and $t_2 = \cdots = t_n = 0$, we have from above that

$$(\lambda^2 - 1) f(x_1, \dots, x_n)^2 = 0. \tag{25}$$

This implies that $\lambda^2 = 1$. Then (24) becomes

$$\Big\{\sum_i f(x_1, \dots, t_i, \dots, x_n)\Big\} f(x_1, \dots, x_n)$$
$$- f(x_1, \dots, x_n)\Big\{\sum_i f(x_1, \dots, t_i, \dots, x_n)\Big\} = 0. \tag{26}$$

In this case by Lemma 8, $f(x_1, \dots, x_n)$ is central valued on R, a contradiction.

The situation when $\delta = \lambda d + ad_q$ is similar.

Next assume that d and δ are C-independent modulo inner derivations of U. Let $f^d(x_1, \dots, x_n)$ and $f^{d^2}(x_1, \dots, x_n)$ be the polynomials obtained from $f(x_1, \dots, x_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$ and $d^2(\alpha_\sigma)$ respectively. Then we have

$$d(f(x_1, \dots, x_n)) = f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n)$$

and

$$d^2(f(x_1, \dots, x_n)) = f^{d^2}(x_1, \dots, x_n) + 2\sum_i f^d(x_1, \dots, d(x_i), \dots, x_n)$$
$$+ \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n). \tag{27}$$

Then we have from (19) that U satisfies

$$\Big\{F(a)f(x_1,\dots,x_n)+2af^d(x_1,\dots,x_n)+2a\sum_i f(x_1,\dots,d(x_i),\dots,x_n)$$
$$+f^{d^2}(x_1,\dots,x_n)+2\sum_i f^d(x_1,\dots,d(x_i),\dots,x_n)+\sum_i f(x_1,\dots,d^2(x_i),\dots,x_n)$$
$$+\sum_{i\neq j} f(x_1,\dots,d(x_i),\dots,d(x_j),\dots,x_n)\Big\}f(x_1,\dots,x_n)$$
$$-f(x_1,\dots,x_n)\Big\{G(b)f(x_1,\dots,x_n)+2bf^\delta(x_1,\dots,x_n)$$
$$+2b\sum_i(f(x_1,\dots,\delta(x_i),\dots,x_n))+f^{\delta^2}(x_1,\dots,x_n)$$
$$+2\sum_i f^\delta(x_1,\dots,\delta(x_i),\dots,x_n)+\sum_i f(x_1,\dots,\delta^2(x_i),\dots,x_n)$$
$$+\sum_{i\neq j} f(x_1,\dots,\delta(x_i),\dots,\delta(x_j),\dots,x_n)\Big\}=0. \tag{28}$$

Since neither d nor δ is inner, by Kharchenko's theorem [12], we have from above that U satisfies

$$\Big\{F(a)f(x_1,\dots,x_n)+2af^d(x_1,\dots,x_n)+2a\sum_i f(x_1,\dots,y_i,\dots,x_n)$$
$$+f^{d^2}(x_1,\dots,x_n)+2\sum_i f^d(x_1,\dots,y_i,\dots,x_n)$$
$$+\sum_i f(x_1,\dots,t_i,\dots,x_n)+\sum_{i\neq j} f(x_1,\dots,y_i,\dots,y_j,\dots,x_n)\Big\}f(x_1,\dots,x_n)$$
$$-f(x_1,\dots,x_n)\Big\{G(b)f(x_1,\dots,x_n)+2bf^\delta(x_1,\dots,x_n)$$
$$+2b\sum_i(f(x_1,\dots,y_i,\dots,x_n))+f^{\delta^2}(x_1,\dots,x_n)+2\sum_i f^\delta(x_1,\dots,y_i,\dots,x_n)$$
$$+\sum_i f(x_1,\dots,t_i,\dots,x_n)+\sum_{i\neq j} f(x_1,\dots,y_i,\dots,y_j,\dots,x_n)\Big\}=0. \tag{29}$$

In particular U satisfies the blended component

$$\Big\{\sum_i f(x_1,\dots,t_i,\dots,x_n)\Big\}f(x_1,\dots,x_n)$$
$$-f(x_1,\dots,x_n)\Big\{\sum_i f(x_1,\dots,t_i,\dots,x_n)\Big\}=0. \tag{30}$$

Then by Lemma 8, $f(x_1,\dots,x_n)$ is central valued on R, a contradiction. □

As a reduction of previous Theorem we also have:

Theorem 2. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, F and G two nonzero generalized derivations of R, $a, b \in U$ and f, g derivations of R such that $F(x) = ax + f(x)$, $G(x) = bx + g(x)$, for all $x \in R$. If $F^2(x)x - xG^2(x) = 0$ for all $x \in R$, then $a^2 = b^2$ and either R is commutative or one of the following holds:*

(1) $F(x) = xa$ and $G(x) = xb$ *for all* $x \in R$ *with* $a^2 \in C$;

(2) $F(x) = xa$ and $G(x) = bx$ *for all* $x \in R$;

(3) $F(x) = ax$ and $G(x) = xb$ *for all* $x \in R$ *with* $a^2 \in C$;

(4) $F(x) = ax$ and $G(x) = bx$ *for all* $x \in R$, *with* $a^2 \in C$.

Proof. By applying Theorem 1, we have that either R is commutative or one of the following holds:

- $F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
- $F(x) = xa$ and $G(x) = bx$ for all $x \in R$ with $a^2 = b^2$;
- $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
- $F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 = b^2 \in C$.

Therefore, in order to prove the present Theorem, we now assume that R is commutative and then show that $a^2 = b^2$.

By our assumption, R satisfies the following differential identity

$$a^2x + 2af(x) + f(a)x + f^2(x) - b^2x - 2bg(x) - g(b)x - g^2(x) = 0. \qquad (31)$$

Notice that, if both f and g are inner derivations of R, then relation (31) implies that $(a^2 - b^2)R = (0)$ and so we are done. Hence we assume that at least one of f and g is not an inner derivation of R.

First, we assume that f and g are linearly C-independent modulo inner derivations, that is, f and g are linearly C-independent modulo 0. Thus, by Kharchenko's Theorem [12], we have from (31) that R satisfies

$$a^2x_1 + 2ax_2 + f(a)x_1 + x_3 - b^2x_1 - 2bx_4 - g(b)x_1 - x_5 = 0. \qquad (32)$$

In particular, we have $2ax_2 = 2bx_4$. Since $\mathrm{char}(R) \neq 2$, it follows $a = b$ and so $a^2 = b^2$.

Next we assume that f and g are linearly C-dependent modulo 0, that is, there exists $0 \neq \lambda \in C$ such that $f(x) = \lambda g(x)$, for all $x \in R$. Then by (31), R

satisfies

$$a^2x + 2\lambda ag(x) + \lambda g(a)x + \lambda g(\lambda)g(x) + \lambda^2 g^2(x) - b^2x - 2bg(x) - g(b)x - g^2(x) = 0. \quad (33)$$

In light of previous comments, we may also assume $g \neq 0$, i.e. g is not inner. Again by Kharchenko's Theorem [12], R satisfies

$$a^2x_1 + 2\lambda ax_2 + \lambda g(a)x_1 + \lambda g(\lambda)x_2 + \lambda^2 x_3 - b^2x_1 - 2bx_2 - g(b)x_1 - x_3 = 0. \quad (34)$$

For $x_1 = x_2 = 0$, we have $\lambda^2 x_3 = x_3$ for all $x_3 \in R$, that is, $\lambda^2 x = x$ for all $x \in R$. In particular $\lambda^2 g(x) = g(x)$ for all $x \in R$. This implies that

$$g(x) = g(\lambda^2 x) = g(\lambda^2)x + \lambda^2 g(x) = g(\lambda^2)x + g(x),$$

which implies

$$0 = g(\lambda^2)x = 2\lambda g(\lambda)x \quad \forall x \in R.$$

This gives $\lambda g(\lambda) = 0$. Thus (34) reduces to

$$a^2x_1 + 2\lambda ax_2 + \lambda g(a)x_1 - b^2x_1 - 2bx_2 - g(b)x_1 = 0. \quad (35)$$

In particular, R satisfies $2\lambda ax_2 - 2bx_2 = 0$. Therefore $b = \lambda a$, so that $b^2 = (\lambda a)^2 = \lambda^2 a^2 = a^2$, and the proof is complete. □

Remark 1. By [14, Theorem 3], every generalized derivation F on R can be uniquely extended to a generalized derivation of U and there exist $a \in U$ and a derivation $f : U \to U$ such that $F(x) = ax + f(x)$, for all $x \in U$. Starting from this, we may trivially write $F(x) = ax + xa - xa + f(x) = xa + [a, x] + f(x) = xa + f'(x)$, where $f' : U \to U$ is a derivation defined as $f'(x) = f(x) - [x, a]$, for all $x \in U$. Thus, for any generalized derivation F there exist two derivations f and f' of U (associated with F), such that both $F(x) = ax + f(x)$ and $F(x) = xa + f'(x)$.

In light of this, we may state Theorem 2 as follows: *Let f, f' be the associated derivations with F and g, g' the associated ones with G. Denote $F(x) = ax + f(x) = xa + f'(x)$ and $G(x) = bx + g(x) = xb + g'(x)$, for all $x \in R$.If $F^2(x)x - xG^2(x) = 0$ for all $x \in R$, then $a^2 = b^2$ and either R is commutative or one of the following holds:*

(1) *$f' = 0$, $g' = 0$, $F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 \in C$;*

(2) *$f' = 0$, $g = 0$, $F(x) = xa$ and $G(x) = bx$ for all $x \in R$;*

(3) *$f = 0$, $g' = 0$, $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 \in C$;*

(4) $f = 0$, $g = 0$, $F(x) = ax$ *and* $G(x) = bx$ *for all* $x \in R$, *with* $a^2 \in C$.

We finally examine the case of 2-torsion free semiprime rings. The first result we obtained is the following:

Lemma 9. *Let R be a semiprime ring of characteristic different from 2 with Utumi quotient ring U, extended centroid C and center $Z(R) \neq (0)$. Let F and G be two nonzero generalized derivations of R respectively defined as $F(x) = ax + f(x)$, $G(x) = bx + g(x)$, for f and g derivations of R and $a, b \in U$. If $F^2(x)x - xG^2(x) = 0$, for all $x \in R$, then $a^2 = b^2$.*

Proof. Since R is semiprime, by Proposition 2.5.1 in [2], the derivations f and g can be uniquely extended on U. Since U and R satisfy the same differential identities (see [17]), then $F^2(x)x - xG^2(x) = 0$, for all $x \in U$. Let B be the complete boolean algebra of idempotents in C and M be any maximal ideal of B.

Since U is a B-algebra orthogonal complete (see for insatnce (2) of Fact 1 in [5], page 42), MU is a prime ideal of U, which is both f-invariant and g-invariant. Denote $\overline{U} = U/MU$ and $\overline{f}$ the derivation induced by f on $\overline{U}$ and $\overline{g}$ the derivation induced by g on $\overline{U}$. In particular, $\overline{U}$ is a prime ring and so, by Theorem 2, $\overline{a^2} - \overline{b^2} = 0$ in $\overline{U}$. This implies that, for any maximal ideal M of B, $a^2 - b^2 \in MU$, therefore $a^2 - b^2 \in \bigcap_M MU = 0$. □

Theorem 3. *Let R be a semiprime ring of characteristic different from 2 with Utumi quotient ring U, extended centroid C and center $Z(R) \neq (0)$. Let F and G be two nonzero generalized derivations of R respectively defined as $F(x) = ax + f(x)$, $G(x) = bx + g(x)$, for f and g derivations of R and $a, b \in U$. If $F^2(x)x - xG^2(x) = 0$, for all $x \in R$, then $a^2 = b^2$ and either R contains a non-zero central ideal or $f(Z(R)) = f'(Z(R)) = g(Z(R)) = g'(Z(R)) = (0)$, $f(a) = f'(a) = g(b) = g'(b) = 0$, $ff'(R) = f'f(R) = gg'(R) = g'g(R) = (0)$ and $Z(R)(G^2(x) - a^2x) = Z(R)(F^2(x) - xa^2) = (0)$ for all $x \in R$, or one of the following holds:*

(1) $F(x) = xa$ and $G(x) = xb$ *for all* $x \in R$ *with* $a^2 \in C$;

(2) $F(x) = xa$ and $G(x) = bx$ *for all* $x \in R$;

(3) $F(x) = ax$ and $G(x) = xb$ *for all* $x \in R$ *with* $a^2 \in C$;

(4) $F(x) = ax$ and $G(x) = bx$ *for all* $x \in R$, *with* $a^2 \in C$.

Proof. By Lemma 9, $a^2 = b^2$. In the sequel we assume that R does not contain any non-zero central ideal.

In light of Remark 1 we may write $F(x) = ax + f(x) = xa + f'(x)$ and $G(x) = bx + g(x) = xb + g'(x)$, for all $x \in R$.

Firstly we consider the case when both F and G are centralizers. Hence either $f(R)=(0)$ or $f'(R)=(0)$ and either $g(R)=(0)$ or $g'(R)=(0)$. In other words either $F(x)=ax$ or $F(x)=xa$ and either $G(x)=bx$ or $G(x)=xb$, for any $x\in R$.

Let P be a prime ideal of R, set $\overline{R}=R/P$ and write $\overline{r}=r+P\in\overline{R}$, for all $r\in R$. We notice that in any case $F(P)\subseteq P$ and $G(P)\subseteq P$. Thus $\overline{F}:\overline{R}\to\overline{R}$ and $\overline{G}:\overline{R}\to\overline{R}$ are generalized derivations of $\overline{R}=R/P$, for any ideal P. By the prime case it follows that one of the following cases must occur:

(1) $F(x)=ax$ and $G(x)=bx$, for all $x\in R$. Thus $0=a^2x^2-xb^2x=[a^2x,x]$, for all $x\in R$ and by Lemma 3.2 in [7] and since R is not commutative, we have that $a^2\in Z(R)$;

(2) $F(x)=xa$ and $G(x)=bx$, for all $x\in R$. In this case we are done, since $a^2=b^2$.

(3) $F(x)=ax$ and $G(x)=xb$, for all $x\in R$. Thus $0=a^2x^2-x^2b^2=[a^2,x^2]$ and by Main Theorem in [13] and since R does not contain any non-zero central ideal, it follows $a^2\in Z(R)$;

(4) $F(x)=xa$ and $G(x)=xb$, for all $x\in R$. In this case $0=xa^2x-x^2b^2=[xa^2,x]$, for all $x\in R$. Once again, by Lemma 3.2 in [7] we have that $a^2\in Z(R)$.

By the previous argument, we now assume that either F or G is not a centralizer, without loss of generality we consider the case $g(R)\neq(0)$. We will prove that $f(Z(R))=f'(Z(R))=g(Z(R))=g'(Z(R))=(0)$, $f(a)=f'(a)=g(b)=g'(b)=0$, $ff'(R)=f'f(R)=gg'(R)=g'g(R)=(0)$ and $Z(R)(G^2(x)-a^2x)=Z(R)(F^2(x)-xa^2)=(0)$ for all $x\in R$.

As above, let P be a prime ideal of R, set $\overline{R}=R/P$ and write $\overline{r}=r+P\in\overline{R}$ for all $r\in R$. We start from

$$\overline{F^2(\overline{r})\overline{r}-\overline{r}G^2(\overline{r})}=\overline{0},\quad \forall\overline{r}=r+x\in\overline{R},\quad x\in P \tag{36}$$

and by computations we get

$$\begin{aligned}&\big(a^2r+af(r)+2af(x)+f(ar)+f^2(r)+f^2(x)\big)r\\ -r&\big(b^2r+bg(r)+2bg(x)+g(br)+g^2(r)+g^2(x)\big)\in P,\quad\forall x\in P,\ r\in R.\end{aligned}\tag{37}$$

Replace x with xy in (37), for any $y\in P$, then it follows

$$\begin{aligned}&\big(a^2r+af(r)+f(ar)+f^2(r)+2f(x)f(y)\big)r\\ -r&\big(b^2r+bg(r)+g(br)+g^2(r)+2g(x)g(y)\big)\in P,\quad\forall x,y\in P,\ r\in R.\end{aligned}\tag{38}$$

By comparing (37) with (38) we get

$$\big(f^2(x)+2af(x)-2f(x)f(y)\big)r-r\big(g^2(x)+2bg(x)-2g(x)g(y)\big)\in P,\quad \forall x,y\in P,\ r\in R. \tag{39}$$

We divide our argument into three cases:

Case 1. $f(P)\subseteq P$, $g(P)\nsubseteq P$.

In this case $\overline{g(P)}$ is a non-zero ideal of $\overline{R}$. Moreover by (39) we have that

$$r\big(g^2(x)+2bg(x)-2g(x)g(y)\big)\in P,\quad \forall x,y\in P,\ r\in R. \tag{40}$$

Replace y with ys in (40), for any $s\in R$, then it follows

$$r\big(g^2(x)+2bg(x)-2g(x)g(y)s\big)\in P,\quad \forall x,y\in P,\ r,s\in R. \tag{41}$$

On the other hand, replacing y with sy in (40) we also have

$$r\big(g^2(x)+2bg(x)-2g(x)sg(y)\big)\in P,\quad \forall x,y\in P,\ r,s\in R. \tag{42}$$

Hence by (41) and (42) we get $rg(x)[s,g(y)]\in P$, that is $\overline{Rg(P)[R,g(P)]}=(\overline{0})$. Since $\overline{g(P)}$ is a non-zero ideal of the prime ring $\overline{R}$, we conclude that $\overline{g(P)}\subseteq Z(\overline{R})$, that is $\overline{R}$ is commutative.
An analogous argument shows that if $g(P)\subseteq P$ and $f(P)\nsubseteq P$, then $\overline{R}$ is commutative (we omit it for brevity).

Case 2. $f(P)\nsubseteq P$ and $g(P)\nsubseteq P$.

Notice that both $\overline{f(P)}$ and $\overline{g(P)}$ are non-zero ideals of $\overline{R}$. Also in this case we start from (39) and replace here y with ys, for any $s\in R$. Thus

$$\big(f^2(x)+2af(x)-2f(x)f(y)s\big)r-r\big(g^2(x)+2bg(x)-2g(x)g(y)s\big)\in P,\quad \forall x,y\in P,\ r,s\in R. \tag{43}$$

On the other hand, replacing y with sy in (39) we have

$$\big(f^2(x)+2af(x)-2f(x)sf(y)\big)r-r\big(g^2(x)+2bg(x)-2g(x)sg(y)\big)\in P,\quad \forall x,y\in P,\ r,s\in R \tag{44}$$

and comparing (43) with (44) it follows

$$f(x)[f(y),s]r-rg(x)[g(y),s]\in P,\quad \forall x,y\in P,\ r,s\in R \tag{45}$$

that is

$$\overline{f(x)[f(y),s]r - rg(x)[g(y),s]} = \overline{0}, \quad \forall \overline{r}, \overline{s} \in \overline{R},\ x, y, \in P. \tag{46}$$

Denote $\overline{u} = \overline{f(x)[f(y),s]}$ and $\overline{v} = \overline{g(x)[g(y),s]}$, so that $\overline{ur} - \overline{rv} = \overline{0}$, for any $\overline{r} \in \overline{R}$. In this case, it is well known that $\overline{u} = \overline{v} \in Z(\overline{R})$, which means

$$\overline{f(x)[f(y),s]} \in Z(\overline{R}), \quad \forall \overline{s} \in \overline{R}, \quad x, y, \in P. \tag{47}$$

For $s = rt$ in (47) and since $\overline{f(P)R} \subseteq \overline{f(P)}$, it follows that

$$\overline{f(x)[f(y),r]t} \in Z(\overline{R}), \quad \forall \overline{r}, \overline{t} \in \overline{R}, \quad x, y, \in P. \tag{48}$$

By the primeness of $\overline{R}$, either $\overline{t} \in Z(\overline{R})$ for any $\overline{t} \in \overline{R}$, or $\overline{f(x)[f(y),r]} = \overline{0}$, for any $\overline{r} \in \overline{R}$ and $x, y, \in P$. Therefore we may assume that

$$\overline{f(x)[f(y),t]} = \overline{0}, \quad \forall \overline{t} \in \overline{R}, \quad x, y, \in P. \tag{49}$$

By using Theorem 1 in [18], one has that either $\overline{f(x)} = \overline{0}$ or $\overline{f(y)} \in Z(\overline{R})$. In any case we have $\overline{f(P)} \subseteq Z(\overline{R})$. Since $\overline{f(P)}$ is a non-zero ideal of the prime ring $\overline{R}$, it follows that $\overline{R}$ is commutative.

Case 3. $f(P) \subseteq P$ and $g(P) \subseteq P$.

In this case we remark that f, f', g, g' induce canonical derivations $\overline{f}$, $\overline{f'}$, $\overline{g}$ and $\overline{g'}$ on $\overline{R}$. It follows from the prime case and Remark 1 that either $\overline{R}$ is commutative, or the following holds simultaneously:

(1) either $f(R) \subseteq P$ or $f'(R) \subseteq P$;

(2) either $g(R) \subseteq P$ or $g'(R) \subseteq P$.

The argument contained in Cases 1, 2, 3 implies that:

$$[f(R)f'(R), R] \subseteq \bigcap_i P_i = (0), \quad [f(R), f'(R)] \subseteq \bigcap_i P_i = (0),$$
$$[f(f'(R)), R] \subseteq \bigcap_i P_i = (0). \tag{50}$$

and also

$$[g(R)g'(R), R] \subseteq \bigcap_i P_i = (0), \quad [g(R), g'(R)] \subseteq \bigcap_i P_i = (0),$$
$$[g(g'(R)), R] \subseteq \bigcap_i P_i = (0). \tag{51}$$

In the next step we prove that $f(R)f'(R) = (0)$. To do this, by contradiction we consider the case $f(R)f'(R) \neq (0)$.

For all $x, y, t \in R$ and since $f(R)f'(R) \subseteq Z(R)$ we have both $f(x)f'(yt) \in Z(R)$ and $f(xy)f'(t) \in Z(R)$ that is

$$f(x)f'(y)t + f(x)yf'(t) \in Z(R)$$

and

$$f(x)yf'(t) + xf(y)f'(t) \in Z(R).$$

Comparing these last two relations we get

$$f(x)f'(y)t - xf(y)f'(t) \in Z(R), \quad \forall x, y, t \in R. \tag{52}$$

In case $f(Z(R)) = (0)$, we have for all $y, t \in R$,

$$0 = f\Big(f(y)f'(t)\Big) = f^2(y)f'(t) + f(y)(ff')(t), \tag{53}$$

which implies $f(y)(ff')(t) \in Z(R)$, since $f^2(y)f'(t) \in Z(R)$. Moreover, by (50) we also have that $(ff')(R) \subseteq Z(R)$. Suppose that there exists $t \in R$ such that $0 \neq \alpha = (ff')(t) \in Z(R)$, then $\alpha f(y) \in Z(R)$ for all $y \in R$. Moreover, $f(\alpha) = 0$ implies $f(\alpha y) = \alpha f(y) \in Z(R)$, that is $f(\alpha R) \subseteq Z(R)$. Therefore R must contain a non-zero central ideal, which is a contradiction, unless when $(0) = f(\alpha R) = \alpha f(R)$. In this last case, since $\alpha \in f(R)$, it follows the contradiction $\alpha^2 = 0$. Thus the previous argument shows that $(ff')(R) = (0)$, and so by (53) we get

$$f^2(y)f'(t) = 0, \quad \forall y, t \in R. \tag{54}$$

Replace y with $yf'(t)y$ in (54) and then get $0 = f(y)f'(t)f(y)f'(t) = \beta^2$, for $\beta = f(y)f'(t) \neq 0$, which is again a contradiction.

Assume now $f(Z(R)) \neq (0)$ and let $0 \neq z \in Z(R)$ be such that $f(z) = \beta \neq 0$. Replacing x with zx in (52) and using again (52), we have $\beta x f'(y)t \in Z(R)$ for all $x, y, t \in R$, that is $R(\beta f'(R))R \subseteq Z(R)$. Therefore, R contains a non-zero central ideal which is a contradiction, unless when $\beta f'(R) = (0)$. On the other hand $f'(z) = [a, z] + f(z) = \beta$, thus $0 = \beta f'(z) = \beta^2$, a contradiction again.

In light of previous contradictions, it is proved that, if R does not contain any non-zero central ideal, then $f(R)f'(R) = (0)$ (similarly one can prove $f'(R)f(R) = (0)$).

Thus $f(R)Rf'(R) = (0)$ and $(0) = f(a)Rf'(a) = f(a)Rf(a)$ implying $f(a) = 0$ and also $f'(a) = 0$. Similarly, for any $z \in Z(R)$, $0 = f(z)Rf'(z) = f(z)Rf(z)$ implying $f(Z(R)) = (0)$ and also $f'(Z(R)) = (0)$. Moreover, for any $x, y \in R$, $0 = f(f(x)f'(y)) = f^2(x)f'(y) + f(x)ff'(y) = f(x)ff'(y)$. Replacing x with

$f'(y)$, we have that $(ff'(y))^2 = 0$ for all $y \in R$. Since by (50), $ff'(R) \subseteq Z(R)$, it follows that $ff'(R) = (0)$. Analogously $f'f(R) = (0)$.

In a similar way $g(b) = g'(b) = 0$, $g(Z(R)) = g'(Z(R)) = (0)$, $g(R)g'(R) = g'(R)g(R) = (0)$, $gg'(R) = (0)$ and $g'g(R) = (0)$.

By using all the previous conditions, we have

$$F^2(x) = \big(xa + f'(x)\big)a + f'\big(xa + f'(x)\big) = xa^2 + 2f'(x)a + f'^2(x)$$

and

$$G^2(x) = b\big(bx + g(x)\big) + g\big(bx + g(x)\big) = a^2x + 2bg(x) + g^2(x)$$

so that, for all $x \in R$,

$$0 = F^2(x)x - xG^2(x) = 2f'(x)ax + f'^2(x)x - 2xbg(x) - xg^2(x). \tag{55}$$

Replace x with $x + y$ for any $y \in Z(R)$ in (55), we have

$$y\big(F^2(x) - G^2(x) + [a^2, x]\big) = 0. \tag{56}$$

Right multiplying (56) by x and since $F^2(x)x = xG^2(x)$, we have

$$Z(R)[G^2(x) - a^2x, x] = 0, \quad \forall x \in R. \tag{57}$$

Denote $\Delta(x) = G^2(x) - a^2x = 2bg(x) + g^2(x)$. By computations we get

$$\begin{aligned}\Delta(xy) &= G^2(xy) - a^2xy = G^2(x)y + 2G(x)g(y) + xg^2(y) - a^2xy\\ &= \Delta(x)y + 2G(x)g(y) + xg^2(y).\end{aligned} \tag{58}$$

Moreover, by using $G(x) = xb + g'(x)$ and $g'(R)g(R) = (0)$ in (58), it follows

$$\begin{aligned}\Delta(xy) &= \Delta(x)y + 2(xb + g'(x))g(y) + xg^2(y)\\ &= \Delta(x)y + 2xbg(y) + xg^2(y) = \Delta(x)y + x\Delta(y).\end{aligned} \tag{59}$$

This implies that Δ is a derivation of R satisfying $Z(R)[\Delta(x), x] = (0)$ for all $x \in R$. In particular, $\Delta(zx) = z\Delta(x)$ for all $z \in Z(R)$, then $[\Delta(zx), zx] = z^2[\Delta(x), x] = 0$, for all $x \in R$. This means that Δ is commuting on the non-zero ideal zR of R, for any $0 \neq z \in Z(R)$. Since R does not contain any non-zero central ideal, by [13] it follows $\Delta(zR) = (0)$, that is $z\Delta(R) = (0)$ and $Z(R)(G^2(x) - a^2x) = (0)$ for all $x \in R$, as required.

We finally remark that, starting again from (56) and left multiplying by x, we obtain $Z(R)[F^2(x) - xa^2, x] = (0)$ for all $x \in R$. The same above argument shows that $Z(R)(F^2(x) - xa^2) = (0)$ for all $x \in R$. □

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