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Summability process by singular operators

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Abstract. The aim of this paper is to obtain some approximation theorems for a sequence of singular operators that do not have to be positive in general. In the approximation, we mainly use a general matrix summability process introduced by Bell [Order summability and almost convergence, *Proc. Amer. Math. Soc.* **38** (1973) 548–552], which includes many well-known convergence methods, such as, the ordinary convergence, almost convergence, the Cesàro mean, the order summability, and so on. An application presented at the end of the paper shows that our approximation result is more applicable than the classical aspects.

1. Introduction

A sequence of singular operators is written in the form

$$L_j(f;x) = \int_a^b f(y)K_j(x,y)dy, \quad j \in \mathbb{N},\ x \in [a,b], \tag{1.1}$$

where $f : [a,b] \to \mathbb{R}$ and $K_j : [a,b] \times [a,b] \to \mathbb{R}$ are any functions that make sense the above integral. Here, K_n is called kernel and has the property that for functions f of a certain class and in a certain sense, $L_j(f)$ converges to f as $j \to \infty$. By sense of convergence, we mean type of convergence, such as, the classical uniform convergence on $C[a,b]$, the convergence in L_p $(p \geq 1)$, the almost convergence introduced by Lorentz [20], the statistical convergence given

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by Fast [11] (see also [12], [22]), and so on. In order to get such an approximation to f, so far, there has been defined many well-known singular operators. For example, the best known singular operator is Dirichlet's operator (integral), which represents the partial sums of the Fourier series of a function f. Another example is Fejér's operator which represents the arithmetic means of Dirichlet's operator. We recommend [8] to the reader for a detailed information regarding the approximation to a function by singular operators in the spaces L^p and $C[a,b]$. Furthermore, convergence results given in certain function spaces linked to the Mellin transforms for Mellin-type singular operators may be found in [7], [21]. On the other hand, there are many studies on approximating a function by linear operators via some convergence methods, such as, almost convergence, statistical convergence, summability process (see, for instance, [1], [2], [10], [13], [18], [23], [24], [25], [30]). However, all of them need the positivity of the operator. Recall that a linear operator L is called positive provided that $f \geq 0$ implies $L(f) \geq 0$. However, some Korovkin-type approximation theorems for a class of general singular operators which are not necessarily positive have recently been studied in the papers [3], [4]. In this paper, we mainly use the matrix summability process for the approximation by the singular operators L_j given by (1.1). Observe that the singular operator L_j $(j \in \mathbb{N})$ is linear, but does not have to be positive in general. Thus, with this property the results obtained in the present paper are the first one in the framework of summability process.

As usual, $C[a,b]$ denotes the space of all real valued continuous functions defined on $[a,b]$. Then $C[a,b]$ is a Banach space with the usual norm $\|\cdot\|$ defined by $\|f\| := \sup_{x\in[a,b]} |f(x)|$, $f \in C[a,b]$. Also, $L_p := L_p[a,b]$ $(p \geq 1)$ denotes the space of all functions whose pth power is integrable on $[a,b]$. In this case, L_p is a Banach space with the norm $\|\cdot\|_p$ defined by $\|f\|_p := (\int_a^b |f(x)|^p dx)^{1/p}$.

Now we recall the concept of $\mathcal{A}$-summability method (process) of a sequence, which was first introduced by Bell [5], [6]. Let $\mathcal{A} = \{A^{(n)}\} = \{[a_{kj}^{(n)}]\}$ $(j,k,n \in \mathbb{N})$ is a sequence of infinite matrices of real numbers. For a sequence of real numbers, $x := \{x_j\}$, the double sequence $\mathcal{A}x := \{(\mathcal{A}x)_k^{(n)}\}$, where $(\mathcal{A}x)_k^{(n)} = \{\sum_{j=1}^{\infty} a_{kj}^{(n)} x_j\}$ $(k,n \in \mathbb{N})$, is called the $\mathcal{A}$-transform of x whenever the series converges for all k and n. A sequence x is said to be $\mathcal{A}$-summable (or, $\mathcal{A}$-convergent) to some number L if $\lim_{k\to\infty} (\mathcal{A}x)_k^{(n)} = L$, uniformly in $n \in \mathbb{N}$. We say that $\mathcal{A}$ is regular if $\lim_{j\to\infty} x_j = L$ implies $\lim_{k\to\infty} (\mathcal{A}x)_k^{(n)} = L$. Bell [5] proved that $\mathcal{A}$ is regular if and only if (a) for each $j \in \mathbb{N}$, $\lim_{k\to\infty} a_{kj}^{(n)} = 0$, uniformly in n; (b) $\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} = 1$, uniformly in n; (c) for each $k,n \in \mathbb{N}$, $\sum_{j=1}^{\infty} |a_{kj}^{(n)}| < \infty$, and there exist integers N and B such that $\sum_{j=1}^{\infty} |a_{kj}^{(n)}| \leq B$ for $k \geq N$ and all

$n \in \mathbb{N}$. This result generalizes the the classical Silverman–Toeplitz theorem. $\mathcal{A}$-summability method (process) contains many well-known (regular) convergence types:

- If we take $\mathcal{A} = \{A\} = \{[a_{kj}]\}$ for all $n \in \mathbb{N}$, then we get A-convergence which is the ordinary matrix summability method [14].
- If $\mathcal{A} = \{I\}$, where I is the identity matrix, for all $n \in \mathbb{N}$, then we get the ordinary convergence.
- If $\mathcal{A} = \{C_1\}$, where C_1 is the Cesàro matrix of order one, for all $n \in \mathbb{N}$, then we get the arithmetic mean (Cesàro mean) convergence.
- This summability method includes the order summability introduced by Jurkat and Peyerimhoff [15], [16].
- If we define $\mathcal{A} = \{[a_{kj}^{(n)}]\}$ by

$$a_{kj}^{(n)} := \begin{cases} \frac{1}{k}, & \text{if } n \le j \le n+k-1 \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

then $\mathcal{A}$-summability process reduces to the almost convergence introduced by Lorentz [20]. Recall that a sequence $x = \{x_j\}$ is almost convergent to L if

$$\lim_{k\to\infty} \frac{x_n + x_{n+1} + \cdots + x_{n+k-1}}{k} = L, \text{ uniformly in } n \in \mathbb{N},$$

which is denoted by $F - \lim x = L$. Many interesting properties of almost convergent sequences may be found in the papers and cited therin: [9], [17], [27], [28], [29].

2. $\mathcal{A}$-summability process by singular operators

We first get the following approximation theorem.

Now consider the sequences of singular operators $\{L_j\}$ defined by (1.1). Let (a_k) be a sequence of positive numbers. We say that $\{L_j(f)\}$ is uniformly $\mathcal{A}$-summable to a function $f \in C[a,b]$ with the rate of $o(a_k)$ if

$$\left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\| = o(a_k) \text{ as } k \to \infty, \quad \text{uniformly in } n,$$

where supremum in the norm $\|\cdot\|$ is taken over $[a,b]$. Similarly, we say that $\{L_j(f)\}$ is $\mathcal{A}$-summable to a function f in L_p with the rate of $o(a_k)$ if

$$\left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\|_p = o(a_k) \text{ as } k \to \infty, \quad \text{uniformly in } n.$$

Then we get the following approximation result with respect to $\mathcal{A}$-summability process.

Theorem 2.1. *Let $\{L_j\}$ be a sequence of singular operators defined as in (1.1), and let $\mathcal{A} = \{[a_{kj}^{(n)}]\}$ $(j,k,n \in \mathbb{N})$ be a regular summability method of infinite matrices with non-negative entries. Assume that (a_k), (b_k) are two sequences satisfying $a_k \geq \alpha > 0$ and $b_k \geq \beta > 0$ for every $k \in \mathbb{N}$. Assume that*

$$\int_a^b |K_j(x,y)|\,dy \leq M \tag{2.1}$$

holds for all $j \in \mathbb{N}$ and all $x \in [a,b]$. If the following conditions

$$L_j(f_0) \text{ is uniformly } \mathcal{A}\text{-summable to } f_0 \text{ with the rate of } o(a_k) \tag{2.2}$$

and, for each $\delta > 0$,

$$\int_{|y-\cdot|\geq\delta;\ y\in[a,b]} |K_j(\cdot,y)|\,dy \text{ is uniformly } \mathcal{A}\text{-summable to } 0 \text{ with the rate of } o(b_k) \tag{2.3}$$

hold, then for every $f \in C[a,b]$, we have

$$L_j(f) \text{ is uniformly } \mathcal{A}\text{-summable to } f \text{ with the rate of } o(c_k),$$

where (c_k) is the sequence defined by $c_k := \max\{a_k, b_k\}$ for $k \in \mathbb{N}$.

Proof. For the proof of this theorem, we mainly apply the idea by Orlicz (see [20], [26]) to the framework of $\mathcal{A}$-summability process. Let $f \in C[a,b]$. By the linearity of the operators L_j, we may write that, for each $k,n \in \mathbb{N}$ and for every $x \in [a,b]$,

$$\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) = \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j\left(f(y)-f(x);x\right) + f(x) \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x),$$

which implies that

$$\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\right| \leq \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_a^b |f(y)-f(x)|\,|K_j(x,y)|\,dy + C\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\right|, \quad (2.4)$$

where $C := \|f\|$. By the uniform continuity of f on $[a,b]$, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $x, y \in [a,b]$ satisfying $|y - x| < \delta$. Hence, it follows from the last inequality that

$$\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\right| \leq \varepsilon \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{|y-x|<\delta} |K_j(x,y)|\,dy + 2C \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{|y-x|\geq\delta} |K_j(x,y)|\,dy + C\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\right|,$$

which yields that

$$\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\right| \leq M\varepsilon \sum_{j=1}^{\infty} a_{kj}^{(n)} + C\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\right| + 2C \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{|y-x|\geq\delta} |K_j(x,y)|\,dy$$

where M is the same as in (2.1). Since $c_k = \max\{a_k, b_k\}$ for every $k \in \mathbb{N}$, we may write that

$$\frac{1}{c_k}\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\right| \leq \frac{M\varepsilon}{\gamma} \sum_{j=1}^{\infty} a_{kj}^{(n)} + \frac{C}{a_k}\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\right| + \frac{2C}{b_k} \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{|y-x|\geq\delta} |K_j(x,y)|\,dy$$

for $k \in \mathbb{N}$ and $x \in [a,b]$, where $\gamma := \min\{\alpha, \beta\}$. Thus, we get, for every $k, n \in \mathbb{N}$, that

$$\frac{1}{c_k}\left\|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f\right\| \leq \frac{M\varepsilon}{\gamma} \sum_{j=1}^{\infty} a_{kj}^{(n)} + \frac{C}{a_k}\left\|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0) - f_0\right\| + \frac{2C}{b_k}\left\|\sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{|y-\cdot|\geq\delta} |K_j(\cdot,y)|\,dy\right\|$$

Now, using (2.2) and (2.3) and also considering the regularity of $\mathcal{A}$, the right hand side of the last equality tends to zero as $k \to \infty$, uniformly in n. Thus, the proof is completed. $\square$

Now, we replace the condition (2.3) by using the concept of modulus of continuity which is one of the most effective tool in the approximation theory. Recall that the modulus of continuity of a function $f \in C[a,b]$, denoted by $\omega(f,\delta)$, is defined to be $\omega(f,\delta) = \sup_{|y-x|\le\delta} |f(y) - f(x)|$, $\delta > 0$. According to this definition, $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$. Then we get the next result.

Theorem 2.2. *Let $\{L_j\}$ be a sequence of singular operators defined as in* (1.1), *and let $\mathcal{A} = \{[a_{kj}^{(n)}]\}$ $(j,k,n \in \mathbb{N})$ be a regular summability method of infinite matrices with non-negative entries. Assume that (a_k) and (b_k) are any positive sequences. Assume further that* (2.1) *and* (2.2) *holds. If the condition*

$$\omega\left(f, \left\|\delta_k^{(n)}(\cdot,p)\right\|\right) = o(b_k) \text{ as } k \to \infty, \qquad \text{uniformly in } n, \tag{2.5}$$

holds for $f \in C[a,b]$, where

$$\delta_k^{(n)}(x,p) := \left(\sum_{j=1}^{\infty} a_{kj}^{(n)} \int_a^b |x-y|^p |K_j(x,y)| dy\right)^{1/p}, \ p \ge 1, \tag{2.6}$$

then, we have

$$L_j(f) \text{ is uniformly } \mathcal{A}\text{-summable to } f \text{ with the rate of } o(c_k),$$

where (c_k) is the same as in Theorem 2.1.

Proof. Let $f \in C[a,b]$. Then, by using the well-known inequality

$$|f(y) - f(x)| \le \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right)$$

for any $\delta > 0$, it follows from (2.4) that

$$\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\right| \le \omega(f,\delta) \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_a^b \left(\frac{|x-y|}{\delta} + 1\right) |K_j(x,y)|\, dy$$
$$+ C\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\right|,$$

where $C := \|f\|$ as in the proof of Theorem 2.1. Hence, from (2.1) we get

$$\begin{aligned}
&\Big|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\Big| \\
&\quad \leq \omega(f,\delta)\Big\{M\sum_{j=1}^{\infty} a_{kj}^{(n)} + \frac{1}{\delta}\sum_{j=1}^{\infty} a_{kj}^{(n)}\int_a^b |x-y|\,|K_j(x,y)|\,dy\Big\} \\
&\qquad + C\Big|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\Big|, \qquad (2.7)
\end{aligned}$$

where M is the same as in (2.1). If $p > 1$, then using Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we observe that

$$\begin{aligned}
\int_a^b |x-y|\,|K_j(x,y)|\,dy &\leq \Big(\int_a^b |x-y|^p\,|K_j(x,y)|\,dy\Big)^{1/p}\Big(\int_a^b |K_j(x,y)|\,dy\Big)^{1/q} \\
&\leq M^{1/q}\Big(\int_a^b |x-y|^p\,|K_j(x,y)|\,dy\Big)^{1/p}.
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
&\Big|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\Big| \\
&\quad \leq \omega(f,\delta)\Big\{M\sum_{j=1}^{\infty} a_{kj}^{(n)} + \frac{M^{1/q}}{\delta}\sum_{j=1}^{\infty} a_{kj}^{(n)}\Big(\int_a^b |x-y|^p\,\Big|K_j(x,y)\Big|dy\Big)^{1/p}\Big\} \\
&\qquad + C\Big|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\Big|.
\end{aligned}$$

If we apply again Hölder's inequality to the second summation on the right hand side of the above inequality, then we get

$$\begin{aligned}
&\Big|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\Big| \\
&\leq \omega(f,\delta)\Big\{M\sum_{j=1}^{\infty} a_{kj}^{(n)} + \frac{1}{\delta}\Big(M\sum_{j=1}^{\infty} a_{kj}^{(n)}\Big)^{1/q}\Big(\sum_{j=1}^{\infty} a_{kj}^{(n)}\int_a^b |x-y|^p\,|K_j(x,y)|\,dy\Big)^{1/p}\Big\} \\
&\quad + C\Big|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\Big|.
\end{aligned}$$

Now, if we choose $\delta := \delta_k^{(n)}(x,p)$ defined by (2.6), then we have

$$\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x) - f(x)\right| \leq \left(M\sum_{j=1}^{\infty} a_{kj}^{(n)} + \left(M\sum_{j=1}^{\infty} a_{kj}^{(n)}\right)^{\frac{p-1}{p}}\right)\omega\left(f,\delta_k^{(n)}(x,p)\right) + C\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0;x) - f_0(x)\right|,$$

which implies that

$$\frac{1}{c_k}\left\|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f\right\| \leq \left(M\sum_{j=1}^{\infty} a_{kj}^{(n)} + \left(M\sum_{j=1}^{\infty} a_{kj}^{(n)}\right)^{\frac{p-1}{p}}\right)\frac{\omega\left(f,\left\|\delta_k^{(n)}(\cdot,p)\right\|\right)}{b_k} + \frac{C}{a_k}\left\|\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f_0) - f_0\right\|$$

due to $c_k = \max\{a_k, b_k\}$ for every $k \in \mathbb{N}$. Now, in the last inequality, taking limit as $k \to \infty$ and also considering the hypotheses (2.2) and (2.5) the proof is completed for $p > 1$. If $p = 1$, then it is enough to choose $\delta := \delta_k^{(n)}(x,1)$ in (2.7) without any Hölder's inequality. □

Now we get an approximation in the space L_p by a regular summability process.

Theorem 2.3. *Let $\{L_j\}$ be a sequence of singular operators defined as in (1.1), and let $\mathcal{A} = \{[a_{kj}^{(n)}]\}$ $(j,k,n \in \mathbb{N})$ be a regular summability method of infinite matrices with non-negative entries. Assume that (a_k) is a sequence satisfying $a_k \geq \alpha > 0$. Assume further that the kernel K_j $(j \in \mathbb{N})$ is measurable in the square $[a,b] \times [a,b]$ and that*

$$\int_a^b |K_j(x,y)|\,dy \leq M, \tag{2.8}$$

$$\int_a^b |K_j(x,y)|\,dx \leq M \tag{2.9}$$

hold for all $j \in \mathbb{N}$ and almost all $x, y \in [a,b]$. Then, for each $f \in L_p$ $(p \geq 1)$, $\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x)$ exists for almost all $x \in [a,b]$, $k,n \in \mathbb{N}$ and is a function of the class L_p. Furthermore, if, for all f belonging to a subset $H \subset L_p$ which is everywhere dense in L_p,

$$\{L_j(f)\} \text{ is } \mathcal{A}\text{-summable to } f \text{ with the rate of } o(a_k), \tag{2.10}$$

then this is also true for any $f \in L_p$.

PROOF. We first assume that $f \in L_p$ with $p > 1$. Then, we may write that

$$\sum_{j=1}^{\infty} a_{kj}^{(n)} \left| L_j(f;x) \right| \leq \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_a^b \left| f(y) \right| \left| K_j(x,y) \right| dy.$$

Now applying Hölder's inequality to the integral on the right hand side of the above inequality and also considering (2.8), we get

$$\begin{aligned} \sum_{j=1}^{\infty} a_{kj}^{(n)} \left| L_j(f;x) \right| &\leq \sum_{j=1}^{\infty} a_{kj}^{(n)} \left(\int_a^b \left| f(y) \right|^p \left| K_j(x,y) \right| dy \right)^{1/p} \left(\int_a^b \left| K_j(x,y) \right| dy \right)^{1/q} \\ &\leq M^{1/q} \sum_{j=1}^{\infty} a_{kj}^{(n)} \left(\int_a^b \left| f(y) \right|^p \left| K_j(x,y) \right| dy \right)^{1/p}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Again using Hölder's inequality on the last summation, we have

$$\begin{aligned} \sum_{j=1}^{\infty} a_{kj}^{(n)} \left| L_j(f;x) \right| &\leq M^{1/q} \left(\sum_{j=1}^{\infty} a_{kj}^{(n)} \int_a^b \left| f(y) \right|^p \left| K_j(x,y) \right| dy \right)^{1/p} \left(\sum_{j=1}^{\infty} a_{kj}^{(n)} \right)^{1/q} \\ &\leq \left(M \sum_{j=1}^{\infty} a_{kj}^{(n)} \right)^{1/q} \left(\sum_{j=1}^{\infty} a_{kj}^{(n)} \int_a^b \left| f(y) \right|^p \left| K_j(x,y) \right| dy \right)^{1/p}. \qquad (2.11) \end{aligned}$$

On the other hand, since the following inequality

$$\begin{aligned} \int_a^b \left(\int_a^b \left| f(y) \right|^p \left| K_j(x,y) \right| dy \right) dx &= \int_a^b \left| f(y) \right|^p \left(\int_a^b \left| K_j(x,y) \right| dx \right) dy \\ &\leq M \int_a^b \left| f(y) \right|^p dy = M \| f \|_p^p \qquad (2.12) \end{aligned}$$

holds for almost all $x \in [a,b]$, the integral $\int_a^b |f(y)|^p |K_j(x,y)| dy$ exists for almost all $x \in [a,b]$. Combining this fact with (2.11), we conclude that $\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x)$ exists for almost all $x \in [a,b]$ and $k, n \in \mathbb{N}$. Furthermore, it follows from (2.11) and (2.12) that

$$\begin{aligned} &\int_a^b \left(\sum_{j=1}^{\infty} a_{kj}^{(n)} \left| L_j(f;x) \right| \right)^p dx \\ &\quad \leq \left(M \sum_{j=1}^{\infty} a_{kj}^{(n)} \right)^{p/q} \int_a^b \left(\sum_{j=1}^{\infty} a_{kj}^{(n)} \int_a^b \left| f(y) \right|^p \left| K_j(x,y) \right| dy \right) dx \end{aligned}$$

$$\leq \left(M \sum_{j=1}^{\infty} a_{kj}^{(n)} \right)^{p/q} M\|f\|_p^p \sum_{j=1}^{\infty} a_{kj}^{(n)} = \left(M \sum_{j=1}^{\infty} a_{kj}^{(n)} \right)^{p} \|f\|_p^p. \tag{2.13}$$

Taking supremum over n on the both sides of the last inequality, we immediately observe that $\sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f)$ belongs to L_p with $p > 1$. Then, we get from (2.13) that

$$\left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) \right\|_p \leq M\|f\|_p \sum_{j=1}^{\infty} a_{kj}^{(n)} \tag{2.14}$$

holds for every $k, n \in \mathbb{N}$. Since H is everywhere dense in L_p, for $\varepsilon > 0$, there exists an $h \in H$ such that $\|f - h\|_p < \varepsilon$. Hence, using (2.14) and considering the linearity of the operators L_j, it follows from Minkowski's inequality that

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\|_p &\leq \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f-h) \right\|_p + \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(h) - h \right\|_p + \|f-h\|_p \\ &\leq \left(M \sum_{j=1}^{\infty} a_{kj}^{(n)} + 1 \right) \varepsilon + \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(h) - h \right\|_p \end{aligned}$$

which yields that

$$\frac{1}{a_k} \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\|_p \leq \left(M \sum_{j=1}^{\infty} a_{kj}^{(n)} + 1 \right) \frac{\varepsilon}{\alpha} + \frac{1}{a_k} \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(h) - h \right\|_p .$$

Now, taking limit as $k \to \infty$, the proof follows from (2.10) for $p > 1$. The case $p = 1$ is very similar to this process, but in this case, we do not need the condition (2.8). □

From Theorems 2.1 and 2.3, we obtain the following result.

Corollary 2.4. *Let $\{L_j\}$ be a sequence of singular operators defined as in (1.1), and let $\mathcal{A} = \{[a_{kj}^{(n)}]\}$ $(j, k, n \in \mathbb{N})$ be a regular summability method of infinite matrices with non-negative entries. Let the sequences (a_k) and (b_k) be the same as in Theorem 2.1. Assume that the kernel K_j $(j \in \mathbb{N})$ is measurable in the square $[a, b] \times [a, b]$ and (2.8), (2.9) hold for all $j \in \mathbb{N}$ and all $x, y \in [a, b]$. Furthermore, if (2.2) and (2.3) are satisfied, then, for all $f \in L_p$ $(p \geq 1)$, we have*

$\{L_j(f)\}$ is $\mathcal{A}$-summable to f (in L_p) with the rate of $o(c_k)$,

where (c_k) is the same as in Theorem 2.1.

Proof. By hypotheses, we deduce from Theorem 2.1 that, for all $f \in C[a,b]$,

$$\left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\| = o(c_k) \text{ as } k \to \infty, \text{ uniformly in } n. \tag{2.15}$$

It is well-known that the uniform convergence on $[a,b]$ implies the convergence in L_p. Hence, (2.15) gives that, for all $f \in C[a,b]$,

$$\left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\|_p = o(c_k) \text{ as } k \to \infty, \text{ uniformly in } n.$$

Also, we know that $C[a,b]$ is everywhere dense in L_p. Now choosing $H = C[a,b]$ in Theorem 2.3, the proof follows immediately. □

With a similar manner, one can get the following result by using Theorems 2.2 and 2.3, at once.

Corollary 2.5. *Let $\{L_j\}$ be a sequence of singular operators defined as in (1.1), and let $\mathcal{A} = \{[a_{kj}^{(n)}]\}$ $(j,k,n \in \mathbb{N})$ be a regular summability method of infinite matrices with non-negative entries. Let (a_k) and (b_k) be the same as in Theorem 2.1. Assume that the kernel K_j $(j \in \mathbb{N})$ is measurable in the square $[a,b] \times [a,b]$ and (2.8), (2.9) hold for all $j \in \mathbb{N}$ and all $x,y \in [a,b]$. Furthermore, if (2.2) and (2.5) are satisfied, then, for all $f \in L_p$ $(p \geq 1)$, we have*

$$\{L_j(f)\} \text{ is } \mathcal{A}\text{-summable to } f \text{ (in } L_p) \text{ with the rate of } o(c_k),$$

where (c_k) is the same as in Theorem 2.1.

3. Concluding remarks

In this section we display a sequence of singular operators, which satisfy all conditions in the preceding results.

Now we define $\mathcal{A} = \{[a_{kj}^{(n)}]\}$ as in (1.2). In this case, we know that $\mathcal{A}$-summability process reduces to the almost convergence in (1.2). Consider the sequences $u = \{u_j\}$ and $v = \{v_j\}$ given respectively by

$$u_j := \begin{cases} -1, & \text{if } j \text{ is odd} \\ 3 & \text{if } j \text{ is even} \end{cases}$$

and

$$v_j = \int_{-1}^{1} (1-t^2)^j dt.$$

Then observe that $F - \lim u = 1$ (that is, $\{u_j\}$ is almost convergent to 1), but u is non-convergent. Now define the operators T_j by

$$T_j(f;x) := \frac{u_j}{v_j} \int_0^1 f(y) \left(1-(y-x)^2\right)^j dy, \quad j \in \mathbb{N} \text{ and } x \in [0,1], \tag{3.1}$$

where $f \in C[0,1]$. If we take

$$K_j(x,y) = \frac{u_j}{v_j} \left(1-(y-x)^2\right)^j, \quad x,y \in [0,1], \tag{3.2}$$

then the operators T_j given by (3.1) has the form of (1.1) with $a=0$ and $b=1$. Furthermore, this kernel K_j satisfies all conditions of Theorem 2.1. Indeed, first observe that

$$\int_0^1 |K_j(x,y)|\, dy = \frac{|u_j|}{v_j} \int_0^1 \left(1-(y-x)^2\right)^j dy \le \frac{3}{v_j} \int_{-x}^{1-x} \left(1-t^2\right)^j dt \le 3,$$

which gives (2.1) with $M=3$. Also, we know from [19] (see also [20]) that

$$\lim_{j\to\infty} \frac{1}{v_j} \int_0^1 \left(1-(y-x)^2\right)^j dy = 1, \text{ uniformly in } x \in [0,1].$$

Then, since $F - \lim_{j\to\infty} u = 1$, we conclude that

$$F - \lim_{j\to\infty} \int_0^1 K_j(x,y) = 1, \text{ uniformly in } x \in [0,1].$$

which means

$$F - \lim_{j\to\infty} T_j(f_0;x) = f_0(x) = 1, \text{ uniformly in } x \in [0,1],$$

and hence (2.2) is valid with the choice, for example, $a_k = 1$ for every $k \in \mathbb{N}$. Finally, from [19], [20], we know that, for each $\delta > 0$,

$$\lim_{j\to\infty} \frac{1}{v_j} \int_{|y-x|\ge\delta;\ y\in[0,1]} \left(1-(y-x)^2\right)^j dy = 0, \text{ uniformly in } x \in [0,1],$$

which implies that

$$F - \lim_{j\to\infty} \int_{|y-x|\ge\delta; y\in[0,1]} |K_j(x,y)|\, dy = 0, \text{ uniformly in } x \in [0,1]$$

The last almost limit shows that (2.3) is also valid with the choice of $b_k = 1$ for every $k \in \mathbb{N}$. Hence, by Theorem 2.1, we get, for all $f \in C[0,1]$,

$$F - \lim_{j\to\infty} T_j(f;x) = f(x), \text{ uniformly in } x \in [0,1]. \tag{3.3}$$

Also, it follows from Corollary 2.4 that the almost convergence in (3.3) is also valid for all $f \in L_p$, $p > 1$.

Observe now that, in (3.3), we cannot replace the almost convergence by the ordinary convergence. Furthermore, our operators T_j are not positive for all $j \in \mathbb{N}$. This application shows that using summability methods in the approximation theory provides us more applicable and powerful results than the classical aspects. Furthermore, observe that the statistical approximation process of the operators T_j is not valid by the fact that the sequence $\{u_j\}$ is not statistically convergent.

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