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A note on the corank of abelian groups

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Abstract. The corank of abelian groups is a concept dual to the well established notion of the rank of such groups. Both of them can be considered as group analogs of the linear dimension. In [7] abelian groups whose rank satisfies counterparts of some fundamental properties of the linear dimension were described. In this paper we study similar questions related to the corank of abelian groups.

1. Introduction and preliminaries

The *rank* $r(G)$ of an abelian group G is defined as the supremum of the cardinalities α such that G contains a direct sum of α non-zero subgroups. It can be considered in a more general setting of lattices with 0. Namely $r(G)$ is the Goldie dimension of the lattice $L(G)$ of subgroups of G (cf. [4], [5]). One can define the corank $c(G)$ of G as the Goldie dimension of the lattice $L^0(G)$ dual to $L(G)$. It coincides with the corank of G (regarded as a $\mathbb{Z}$-module, where $\mathbb{Z}$ is the ring of integers) introduced for modules in another way in [9]. One can extend to lattices the concept of a basis of a linear space, understood as a maximal linearly independent set, and show that the cardinality of every basis of a modular lattice is equal to its Goldie dimension [4], [5]. Thus the Goldie dimension of modular lattices (possessing bases) satisfies a counterpart of an important property of the linear dimension. Many papers (cf. [1], [2], [6], [7]) studied other similarities among the linear and Goldie dimensions. In [7] a lattice counterpart of the concept

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of a generating set of a linear space was introduced and there were characterized modular lattices L satisfying properties

P1 : minimal generating sets of L are bases of L;

P2 : the cardinality of every minimal generating set of L is equal to the Goldie dimension of L.

For every abelian group G both $L(G)$ and $L^0(G)$ have bases, so such questions are of particular interest in this case. In [7] abelian groups G such that $L(G)$ satisfies P1 or P2 were described (none of them is satisfied for all abelian groups). In this note we study these questions for $L^0(G)$. We will show that $L^0(G)$ satisfies P2 for every abelian group G. This is an elementary but to some extend surprising fact as it shows that the corank is an invariant of abelian groups more close to the linear dimension than the rank, which might be useful in some studies. In that context it is interesting to examine also P1 for $L^0(G)$. Unfortunately it does not hold for all abelian groups. We classify abelian groups for which it is satisfied.

We begin with establishing the notation and recalling some concepts and results. This is quite important as the standard terminologies for modules and abelian groups are not always compatible.

Throughout the paper the term "group" means abelian group written additively. For undefined terms and facts on abelian groups we refer to [3].

To denote that N is a subgroup of a group G we write $N \leq G$. A subgroup N of a group G is said to be *essential* in G, which we denote $N \leq_e G$, if for every non-zero $K \leq M$, $K \cap N \neq 0$.

Applying Zorn's lemma one gets that for a given subgroup H of a group G there exists a subgroup N of G maximal with respect to $H \cap N = 0$. Then $H \simeq (H+N)/N \leq_e G/N$. Every such subgroup N will be called an e-complement of H.

In the following lemma we collect some well-known properties of essential subgroups for later use.

Lemma 1.1. (i) *If $A \leq_e B$ and $B \leq_e C$, then $A \leq_e C$;*

(ii) *If $f : A \to B$ is a homomorphism of groups and $C \leq_e B$, then $f^{-1}(C) \leq_e A$;*

(iii) *For arbitrary groups $A \leq_e B$ and C, $A \oplus C \leq_e B \oplus C$.*

For a given group G and a prime p, we denote by G_p the *p-component* of G, i.e., the subgroup of G consisting of all elements of G whose orders are non-negative powers of p.

In what follows $\mathbb{Z}$ denotes the additive group of integers, $\mathbb{Q}$ - the additive group of rational numbers, $\mathbb{Z}_n$ - the cyclic group of order n for an integer $n \geq 2$ and $\mathbb{Z}_{p^\infty}$ - the Prüfer p-group for a prime p.

A fundamental role in studies of the dual Goldie dimension (corank) of modules is played by hollow modules. A module is called *hollow* if it cannot be presented as a sum of two proper submodules. From [8], Corollary 2.4, it follows that an abelian group, treated as a $\mathbb{Z}$-module, is hollow if and only if it is isomorphic to $\mathbb{Z}_{p^n}$ or $\mathbb{Z}_{p^\infty}$ for a prime p and a positive integer n.

The term "hollow" is not much used in the theory of abelian groups. However, [3], Theorem 3.1, gives that an abelian group is hollow if and only if it is cocycilc or, equivalently, subdirectly irreducible, i.e., the intersection of all its non-zero subgropus is non-zero. In the sequel we will use the common in the theory of abelian groups term "cocyclic group".

Let us add in passing that it is not true that for all rings R the classes of hollow and subdirectly irreducible R-modules (called also cocyclic modules [10]) coincide.

We need the following result, which is an exercise in [3]. We include its proof for completeness.

Proposition 1.2 (cf. [3], Exercise 8.10)**.** *Every subgroup S of the group $G = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ is isomorphic to $A \oplus B$, where A, B are subgroups of $\mathbb{Z}_{p^\infty}$.*

Proof. If S is finite, then it is a direct sum of cyclic p-groups and, since $r(S) \leq 2$, the number of summands is ≤ 2 and we are done. Thus suppose that S is infinite. Let π be the canonical projection of G onto the second component of $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ and φ be its restriction to S. The kernel K of φ is equal to the intersection of S with the first component of $G = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$. Hence, if K is infinite it is equal to $\mathbb{Z}_{p^\infty}$. Consequently $S = K \oplus B$, where $B \simeq \mathrm{Im}\varphi \leq \mathbb{Z}_{p^\infty}$. Thus in this case the result holds. If K is finite, then the image of φ is infinite, which implies that it is equal to $\mathbb{Z}_{p^\infty}$. Consequently $p(S/K) = S/K$, and hence $S = pS + K$. Since K is finite, there exists a positive integer n such that $p^n K = 0$. Now $p^n S = p^{n+1} S + p^n K = p^{n+1} S$, so $p^n S$ is a divisible subgroup of S. Since $S/K \simeq \mathbb{Z}_{p^\infty}$, we have $p^n S \neq 0$. Thus S contains a subgroup isomorphic to $\mathbb{Z}_{p^\infty}$ and hence $S \simeq \mathbb{Z}_{p^\infty} \oplus B$ for a subgroup B of S. Obviously $r(B) \leq 1$, so B is isomorphic to a subgroup of $\mathbb{Z}_{p^\infty}$ and we are done. □

A set X of proper subgroups of a group G is called *coindependent* if for arbitrary distinct subgroups $N_1, \ldots, N_k \in X$, $N_1 + \bigcap_{2 \leq i \leq k} N_i = G$. The *corank* or the *dual Goldie dimension* of G, denoted $c(G)$, is defined as the supremum of cardinalities of coindependent sets of G.

A maximal coindependent set X of subgroups of a group G such that for every $H \in X$, G/H is a cocyclic group, is called a *cobasis* of G.

From Zorn's lemma it follows that every coindependent set X of subgroups of G such that for every $H \in X$, the group G/H is cocyclic can be extended to a cobasis of G.

If G is a non-zero group, then either G is divisible and then G can be homomorphically mapped onto $\mathbb{Z}_{p^\infty}$ for a prime p, or there is a prime q such that $G \neq qG$, so G can be homomorphically mapped onto $\mathbb{Z}_q$. Thus G contains a subgroup H such that G/H is a cocyclic group. Hence there is a cobasis of G containing H. From [5], Theorem 1, it follows that the cardinality of an arbitrary cobasis of G is equal to $c(G)$.

A set X of subgroups of a group G is called a *cogenerating* set of G if for all $H \in X$, G/H are cocyclic groups and for every proper subgroup S of G there are $H_1, H_2, \dots, H_n \in X$ such that $S + \bigcap_i H_i \neq G$.

It is clear that every cobasis of G is a minimal cogenerating set of G. The converse does not hold in general. In Theorem 2.5 we describe all groups with that property.

2. Results

We start with showing that for every group G the lattice $L^0(G)$ satisfies P2.

Theorem 2.1. *For every abelian group G the cardinality of an arbitrary minimal cogenerating set of G is a equal to $c(G)$.*

Proof. Theorem 5.6 in [7] applied to abelian groups gives that to get the result it suffices to show that if A and B are subgroups of G such that G/A and G/B are cocyclic groups, then $c(G/A\cap B) \leq 2$. The group $G/A\cap B$ is isomorphic to a subgroup of $G/A\oplus G/B$. Since both G/A and G/B are cocyclic groups, there are primes p, q such that G/A and G/B are isomorphic to subgroups of $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_{q^\infty}$, respectively. Hence the group $G/A \cap B$ is isomorphic to a subgroup S of $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$. If $p \neq q$, then $S = S_p \oplus S_q$ and S_p, S_q are isomorphic to subgroups of $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_{q^\infty}$, respectively. Consequently $c(S) \leq 2$. If $p = q$, then from Proposition 1.2 it follows that G is isomorphic to a group $A\oplus B$, where both A, B are subgroups of $\mathbb{Z}_{p^\infty}$. Hence each of A and B is a cocyclic group or equal to 0 and we get that $c(G) \leq 2$. □

Now we pass to study abelian groups in which minimal cogenerating sets are cobases. Denote this class of groups by $\mathcal{C}$. Our studies will be based on the following result, which is Theorem 5.5 in [7] applied to abelian groups.

Theorem 2.2. *Every minimal cogenerating set of a group G is a cobasis of G if and only if G cannot be homomorphically mapped onto a group $A \oplus B$, where A is a cocyclic group and B a non-trivial subgroup of A.*

The following corollary follows directly from Theorem 2.2 and Proposition 1.2.

Corollary 2.3. (i) *A group G belongs to $\mathcal{C}$ if and only if for every prime p neither $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ nor $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ is a homomorphic image of G. In particular, the class $\mathcal{C}$ is homomorphically closed.*

(ii) *A subgroup of $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ is in $\mathcal{C}$ if and only if it is either a divisible or an elementary p-group.*

We will also need the following properties of groups in the class $\mathcal{C}$.

Proposition 2.4. *Suppose that $G \in \mathcal{C}$ and $pG \neq G$ for a prime p.*

(i) *If pG contains a subgroup $H \simeq \mathbb{Z}_{p^2}$, then $pG \leq_e G$;*

(ii) *pG cannot contain a subgroup isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$.*

Proof. (i) Let N be an e-complement of H in pG. It is clear that $p(G/N) = pG/N \neq G/N$. Applying Lemma 1.1 it suffices to show that $p(G/N) \leq_e G/N$. Hence, since $\mathcal{C}$ is homomorphically closed, we can factor out N and assume without loss of generality that $N = 0$ and $H \leq_e pG$. If pG is not essential in G, then G contains a subgroup S of order p such that $S \cap pG = 0$. Let D be a divisible closure of G and D_1 the divisible closure of $pG + S = pG \oplus S$ in D. Then $D = D_1 \oplus D_2$ for a subgroup D_2 of D. If π is the projection of D onto D_1, then $pG = \pi(pG) = p(\pi(G))$ and $S = \pi(S)$. Hence $0 \neq p\pi(G) \neq \pi(G)$ and the group $\pi(G)$ is neither divisible nor elementary. Since $H \leq_e pG$, we have $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \simeq H \oplus S \leq_e pG \oplus S$ by Lemma 1.1 (iii). Hence $D_1 \simeq \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$. Consequently $\pi(G)$ is isomorphic to a subgroup of $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$. Applying Corollary 2.3 (ii) we get that $\pi(G) \notin \mathcal{C}$, a contradiction.

(ii) Suppose on the contrary that pG contains a subgroup $H \simeq \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. Let N be an e-complement of H in pG. Clearly $p(G/N) = pG/N \neq G/N$ and $p(G/N)$ contains a subgroup $H' \simeq H$ such that $H' \leq_e p(G/N)$. From (i) it follows that $p(G/N) \leq_e G/N$. Consequently G/N is isomorphic to a subgroup of $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$, which is neither divisible nor elementary. Applying Corollary 2.3 we get that $G \notin \mathcal{C}$, a contradiction. □

Now we are ready to describe the groups in $\mathcal{C}$.

Theorem 2.5. 1. *A torsion group G belongs to $\mathcal{C}$ if and only if for every prime p, the group G_p is divisible or elementary or cyclic;*

2. *A torsion free group G belongs to $\mathcal{C}$ if and only if G is divisible or is isomorphic to a subgroup of $\mathbb{Q}$;*

3. *A group G which is neither torsion-free nor torsion, belongs to $\mathcal{C}$ if and only if G is isomorphic to a group $A \oplus D$, where D is a non-zero divisible group and A is a subgroup of $\mathbb{Q}$ such that $\mathbb{Z} \leq A$ and if $D_p \neq 0$ for a prime p, then $(A/\mathbb{Z})_p \simeq \mathbb{Z}_{p^\infty}$ or, equivalently, $p^{-k} \in A$ for every positive integer k.*

Proof. 1. The "if" part is clear. It is also clear that to get the "only if" part it suffices to show it for p-groups, where p is a prime. Thus let G be a p-group and suppose that G is neither divisible nor elementary. Then $0 \neq pG \neq G$. If $p^2G = 0$, then G is a direct sum of cyclic groups by [3], Theorem 17.2. If $r(G) = 1$, then G is a cyclic group. Otherwise G is a direct sum of at least two non-zero cyclic groups of order $\leq p^2$ and since $pG \neq 0$ at least one of them is of order p^2. Thus G contains a subgroup isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$, which contradicts Proposition 2.4 (ii). Consequently G is a cyclic group. If $p^2G \neq 0$, then pG contains a subgroup $H \simeq \mathbb{Z}_{p^2}$ and by Proposition 2.4 (i), $pG \leq_e G$. Note that $H \leq_e pG$ as otherwise pG would contain a subgroup isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$, which is impossible by Proposition 2.4 (ii). Hence $\mathbb{Z}_{p^2} \simeq H \leq_e G$, so G is isomorphic to a subgroup of $\mathbb{Z}_{p^\infty}$ and since G is not divisible, G is cyclic.

2. If G is divisible, then clearly $G \in \mathcal{C}$. It is well known that for every non-zero subgroup H of $\mathbb{Q}$, $\mathbb{Q}/H$ is a direct sum of p-groups S_p, where p runs over the set of primes and for each p, S_p is a cyclic group or a group isomorphic to $\mathbb{Z}_{p^\infty}$. This implies that no subgroup of $\mathbb{Q}$ can be homomorphically mapped onto $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$, where p is a prime. Hence all subgroups of $\mathbb{Q}$ are in $\mathcal{C}$. This proves the "if" part. Suppose now that $G \in \mathcal{C}$ and G is not divisible. Then there is a prime p such that $pG \neq G$. Since G is torsion-free, $pG \leq_e G$. If $r(pG) \geq 2$, then pG contains a subgroup $S \simeq \mathbb{Z} \oplus \mathbb{Z}$. Obviously S contains a subgroup H such that $S/H \simeq \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. Now $p(G/H) \neq G/H$ and $p(G/H)$ contains a subgroup isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$, which contradicts Proposition 2.4 (ii). Consequently $r(pG) = 1$. Since $pG \leq_e G$, we get that $r(G) = 1$, so G is isomorphic to a subgroup of $\mathbb{Q}$.

3. Suppose that $G \in \mathcal{C}$ and the torsion part of G is D. We claim that D is divisible. If not, then for a prime p, $D \neq pD$. Since D/pD is an elementary p-group, D contains a subgroup H such that $pD \subseteq H$ and $D' = D/H$ is a group of order p. Since $G'/D' \simeq G/D$ is a torsion-free group, the torsion part of the group $G' = G/H$ is D'. The map $x \to px$ is an epimorphism of G' onto pG' with the kernel D'. Consequently $pG' \simeq G'/D'$ is a torsion-free subgroup of G' and $pG' \oplus D' \leq G'$. Since pG' is a torsion-free group it contains a subgroup

S isomorphic to $\mathbb{Z}$. Now for $G'' = G'/p^2S$ we have that $pG'' \neq G''$ and pG'' contains a subgroup isomorphic to $\mathbb{Z}_{p^2}$. However pG'' is not essential in G'', which contradicts Proposition 2.4 (i). Thus D is a divisible group and consequently $G \simeq A \oplus D$ for a torsion-free group A. Now $r(A) = 1$ as otherwise A would contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and A could be homomorphically mapped onto a group T containing a subgroup isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. Hence G could be homomorphically mapped onto $T \oplus D$, which contradicts Proposition 2.4 (ii). Thus we can assume that $\mathbb{Z} \leq A \leq \mathbb{Q}$. If $D_p \neq 0$, then $pA = A$ as otherwise A could be homomorphically mapped onto $\mathbb{Z}_p$ and G could be homomorphically mapped onto $\mathbb{Z}_p \oplus \mathbb{Z}_{p^\infty}$. This obviously implies that $p^{-k} \in A$ for every positive integer k and concludes the proof of the "only if" part.

To prove the other implication, suppose that $f : A \oplus D \to C$ is a group homomorphism, where C is a p-group for a prime p. Clearly $f(D) = f(D_p)$ and $f(A) \simeq A/Kerf$ is a p-group. Since $\mathbb{Z} \leq A$, $\mathbb{Z} \cap Kerf = p^k\mathbb{Z}$ for an integer $k \geq 0$. Now $(\mathbb{Q}/p^k\mathbb{Z})_p \simeq \mathbb{Z}_{p^\infty}$, so $A/Kerf$ is isomorphic to a subgroup of $\mathbb{Z}_{p^\infty}$. Consequently $f(A)$ is a cyclic p-group or is isomorphic to $\mathbb{Z}_{p^\infty}$. By the assumption the latter holds if $D_p \neq 0$. Thus in this case $f(A \oplus D)$ is a divisible group. If $D_p = 0$, then $f(A \oplus D)$ is a cyclic p-group or is a group isomorphic to $\mathbb{Z}_{p^\infty}$. Thus C is neither $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ nor $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$. Consequently $G \in \mathcal{C}$, which proves the "if" part. □

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