

On a pure ternary exponential Diophantine equation

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Abstract. Let r be a positive integer with $r > 1$ and m a positive even integer. Let $a = |V(m, r)|$, $b = |U(m, r)|$, and $c = m^2 + 1$, where $V(m, r) + U(m, r)\sqrt{-1} = (m + \sqrt{-1})^r$. In this paper we prove that if $m > \max\{10^{15}, 2r^3\}$, then the equation $a^x + b^y = c^z$ has only the positive integer solution $(x, y, z) = (2, 2, r)$.

1. Introduction

Let \mathbb{Z} , \mathbb{N} be the sets of all integers and positive integers respectively. Let a , b , c be fixed coprime positive integers with $\min\{a, b, c\} > 1$. In 1933, K. MAHLER [21] used his p -adic analogue of the Thue–Siegel method to prove that the ternary exponential Diophantine equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N} \tag{1.1}$$

has only finitely many solutions (x, y, z) . His method is ineffective in the sense that it gives no indication on the number of possible solutions. An effective result for solutions of (1.1) was given by A. O. GEL'FOND [7]. In analogy to a conjecture of L. JEŚMANOWICZ concerning Pythagorean triple (see [10]), N. TERAJ

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[24] conjectured that (1.1) always has at most one solution (x, y, z) . Simple counterexamples to this statement have been found by Z.-F. CAO [2], who suggested that the condition $\max\{a, b, c\} > 7$ should be added to the hypotheses of Terai's conjecture. However, it turns out that this condition is not sufficient to ensure the uniqueness. A family of counterexamples has been found by M.-H. LE [12], and he also stated the following form of Terai's conjecture:

Conjecture 1. *For any fixed a, b , and c , (1.1) has at most one solution (x, y, z) with $\min\{x, y, z\} > 1$.*

This problem relates to the famous generalized Fermat conjecture for fixed a , b and c (See Problem B19 of [8] and its references). It was verified for some special cases. But, in general, the problem is not yet solved. It seems that Conjecture 1 is a very difficult problem.

Let r be a positive integer with $r > 1$ and m a positive even integer. Let us define $U(m, r)$ and $V(m, r)$ by

$$V(m, r) + U(m, r)\sqrt{-1} = (m + \sqrt{-1})^r. \quad (1.2)$$

Then $V(m, r)$ and $U(m, r)$ are coprime nonzero integers satisfying

$$V^2(m, r) + U^2(m, r) = (m^2 + 1)^r. \quad (1.3)$$

Most known results on Conjecture 1.1 relate to the case

$$a = |V(m, r)|, \quad b = |U(m, r)|, \quad c = m^2 + 1. \quad (1.4)$$

Obviously, using (1.3), if a , b , and c satisfy (1.4), then equation (1.1) has the solution $(x, y, z) = (2, 2, r)$. Recently, T. MIYAZAKI [22] stated the following conjecture which amounts to Conjecture 1 in this case.

Conjecture 2. *If a , b , and c satisfy (1.4), then equation (1.1) has the solution $(x, y, z) = (2, 2, r)$.*

Many authors have verified Conjecture 2 under certain assumptions on r and m (see [2]–[6], [9], [11]–[18], [22], [24]–[30]). In the above papers, the conjecture was established mainly under one of the following additional hypotheses:

- (i) (M.-H. LE [18]). $r \equiv 5 \pmod{8}$ and either $m > r^2$, $r < 11500$ or $m > \frac{2r}{\pi}$, $r > 11500$.
- (ii) (M. CIPU and M. MIGNOTTE [5]). $r \equiv 3 \pmod{4}$ and $m \equiv 2 \pmod{4}$.
- (iii) (T. MIYAZAKI [22]). $r \equiv 4 \pmod{8}$ or $r \equiv 6 \pmod{8}$ and $\frac{m^2}{\log(m^2+1)} \geq \frac{r^3}{\log 2}$.

To the long list of papers that treated various particular cases, we will add two recent papers. F. LUCA [19] considered the system (1.1) and (1.4) and proved that if $m \geq 2$ is an even integer and $r \geq 1$ is an integer, the system admits a solution $(x, y, z) \neq (2, 2, r)$ only in finitely many instances (m, r) and that there exists a computable constant c_0 such that all such solutions satisfy $\max\{m, r, x, y, z\} \leq c_0$. Also, the second author, S. YANG and B. HE [31] studied equations (1.1) and (1.4). They mainly proved that if $m > 24650 r^2 (\log r)^2$, then equation (1.1) has only the positive integer solution $(x, y, z) = (2, 2, r)$. They also considered some particular cases. In fact, they showed that if $r \equiv 5 \pmod{8}$ or $r \equiv 3, 19 \pmod{24}$ and $m \geq 2r$, or if r is a prime, $r \equiv 5 \pmod{8}$ or $r \equiv 19 \pmod{24}$, and $m \geq \frac{2}{\pi} r$, then equation (1.1) has only the positive integer solution $(x, y, z) = (2, 2, r)$. In this paper, using the Gel'fond–Baker method, we prove the following more general result. It is good to specify that the method used in the present paper is slightly different to that used in [31].

Theorem 1. *For any r , if a, b and c satisfy (1.4) with $m > \max\{10^{15}, 2r^3\}$, then (1.1) has only the solution $(x, y, z) = (2, 2, r)$.*

Our theorem proves Conjecture 2, for any fixed r , except for a finite number of m . We organize this paper as follows. In Section 2, we only recall some useful results particularly a result of LE [12] on lower bounds of linear forms in two logarithms and a result of BUGEAUD [1] on upper bounds of linear forms in p -adic logarithms. In Section 3, we prove all necessary results that we will need to get our main result. The proof of Theorem 1 will be done in Section 4 by the means of the results cited or proved in Sections 2 and 3.

2. Preliminaries

In this section, we will only recall some important results obtained by other authors that will be useful for the proof of our main result.

Lemma 1 ([32]). *Let n be a positive integer and let X, Y be coprime nonzero integers with $X \neq Y$. If every prime divisor p of $\frac{X^n - Y^n}{X - Y}$ satisfies $p \mid (X - Y)$, then $n \leq 6$.*

Lemma 2 ([17], Lemma 2). *If $m > \frac{2r}{\pi}$, then $U(m, r)$ and $V(m, r)$ are positive integers.*

Lemma 3 ([23], pp. 12–13). *Every solution (X, Y, Z) of the equation*

$$X^2 + Y^2 = Z^2, \quad X, Y, Z \in \mathbb{N}, \quad \gcd(X, Y) = 1, \quad 2 \mid Y, \quad Z > 0 \quad (2.1)$$

can be expressed as

$$X = f^2 - g^2, \quad Y = 2fg, \quad Z = f^2 + g^2, \quad f, g \in \mathbb{N}, \quad f > g, \quad \gcd(f, g) = 1, \quad 2 \mid fg.$$

Lemma 4 ([12], Lemma 5). *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be positive integers with α_1, α_2 multiplicatively independent and $\min\{\alpha_1, \alpha_2\} > 10^3$. Further let*

$$\Lambda = \beta_1 \log \alpha_1 - \beta_2 \log \alpha_2.$$

If $\Lambda \neq 0$, then

$$\log |\Lambda| > -17.61(\log \alpha_1)(\log \alpha_2)(1.78 + B)^2,$$

where

$$B = \max \left\{ 8.45, 0.23 + \log \left(\frac{\beta_1}{\log \alpha_2} + \frac{\beta_2}{\log \alpha_1} \right) \right\}.$$

Lemma 5 ([1], Theorem 2). *Let α_1, α_2 be positive odd integers with $\min\{\alpha_1, \alpha_2\} > 1$ and let β_1, β_2 be positive integers. Further let*

$$\Lambda' = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}$$

and let $v_2(\Lambda')$ denote the order of 2 in Λ' , namely $2^{v_2(\Lambda')} \parallel \Lambda'$. If $\Lambda' \neq 0$ and $\alpha_1 \equiv 1 \pmod{4}$, then

$$v_2(\Lambda') < 208(\log \alpha_1)(\log \alpha_2)(\log B')^2,$$

where

$$\log B' = \max \left\{ 10, 0.04 + \log \left(\frac{\beta_1}{\log \alpha_2} + \frac{\beta_2}{\log \alpha_1} \right) \right\}.$$

3. Further lemmas on exceptional solutions of (1.1)

In this section, we assume that $r \geq 7$, $m > \frac{4r}{\pi}$ and a, b, c satisfy (1.4). Using Lemma 2, we have

$$a = V(m, r), \quad b = U(m, r), \quad c = m^2 + 1, \quad (3.1)$$

and by (1.3), we get

$$a^2 + b^2 = c^r. \quad (3.2)$$

Now we suppose that (x, y, z) is a solution of equation (1.1) with $(x, y, z) \neq (2, 2, r)$, called an *exceptional solution* of (1.1). We will prove the following sequence of results.

Lemma 6. *If $r \geq 3$ and $m > \frac{4r}{\pi}$, then $U^{\frac{3}{2}}(m, r) > V(m, r) > U(m, r) > 0$.*

PROOF. Let $\alpha = m + \sqrt{-1}$ and $\beta = m - \sqrt{-1}$. Then we have

$$\alpha = \sqrt{m^2 + 1}e^{\theta\sqrt{-1}}, \quad \beta = \sqrt{m^2 + 1}e^{-\theta\sqrt{-1}}, \tag{3.3}$$

where θ is a real number satisfying

$$0 < \theta = \arctan \frac{1}{m} < \frac{\pi}{2}. \tag{3.4}$$

Since $m \geq 2$, equation (3.4) implies

$$\frac{11}{12m} \leq \frac{1}{m} \left(1 - \frac{1}{3m^2}\right) < \theta = \frac{1}{m} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)m^{2i}} < \frac{1}{m}. \tag{3.5}$$

By (1.2) and (3.3), we get

$$V(m, r) = (m^2 + 1)^{\frac{r}{2}} \cos(r\theta), \quad U(m, r) = (m^2 + 1)^{\frac{r}{2}} \sin(r\theta). \tag{3.6}$$

Using $m > \frac{4r}{\pi}$ and Lemma 2, we have $V(m, r) > 0$ and $U(m, r) > 0$. Further, by (3.5), we get

$$0 < r\theta < \frac{r}{m} < \frac{\pi}{4}. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we deduce that $\frac{U(m,r)}{V(m,r)} = \tan(r\theta) < \tan\left(\frac{\pi}{4}\right) = 1$ and $V(m, r) > U(m, r) > 0$.

We now assume that $V(m, r) \geq U^{\frac{3}{2}}(m, r) > 0$. So $V^2(m, r) \geq U^3(m, r)$ and we use (3.6) to get

$$1 \geq \cos^2(r\theta) \geq (m^2 + 1)^{\frac{r}{2}} \sin^3(r\theta). \tag{3.8}$$

Since $0 < r\theta < \frac{\pi}{4}$, we have

$$\sin(r\theta) = r\theta \sum_{i=0}^{\infty} \frac{(-1)^i (r\theta)^{2i}}{(2i+1)!} > r\theta \left(1 - \frac{(r\theta)^2}{3!}\right) > r\theta \left(1 - \frac{\pi^2}{3!4^2}\right) > 0.89r\theta. \tag{3.9}$$

Therefore, as $r \geq 3$, from (3.5), (3.8), and (3.9), we get

$$\begin{aligned} 1 &> m^r \sin^3(r\theta) > m^r (0.89r\theta)^3 > m^r \left(\frac{0.89 \times 11r}{12m}\right)^3 \\ &> 0.53m^{r-3}r^3 \geq 0.53r^3 \geq 0.53 \times 3^3 > 1. \end{aligned} \tag{3.10}$$

This is a contradiction. Thus we have $U^{\frac{3}{2}}(m, r) > V(m, r)$. The lemma is proved. □

Lemma 7. $rx \neq 2z$ and $ry \neq 2z$.

PROOF. We first consider the case that $2 \mid r$. Since $2 \mid m$ and $2 \mid r$, we see from (1.2) and (3.1) that $2 \nmid a$ and $2 \mid b$. Therefore, applying Lemma 3 to (3.2), we get

$$\begin{aligned} a &= f^2 - g^2, \quad b = 2fg, \quad c^{\frac{r}{2}} = f^2 + g^2, \quad f, g \in \mathbb{N}, \\ f &> g, \quad \gcd(f, g) = 1, \quad 2 \mid fg. \end{aligned} \quad (3.11)$$

Thus we obtain

$$c^{\frac{r}{2}} + a = 2f^2. \quad (3.12)$$

Equation (3.12) implies

$$c^{\frac{r}{2}} \equiv -a \pmod{2f}. \quad (3.13)$$

If $rx = 2z$, then $z = \frac{rx}{2}$. Hence, by (1.1), we get

$$b^y = c^z - a^x = c^{\frac{rx}{2}} - a^x = (c^{\frac{r}{2}})^x - a^x. \quad (3.14)$$

In (3.11) we know that $2f \mid b$, so from (3.13) and (3.14) we see that

$$0 \equiv b^y \equiv (c^{\frac{r}{2}})^x - a^x \equiv ((-1)^x - 1)a^x \pmod{2f}. \quad (3.15)$$

Since $2f > 2$ and $\gcd(a, 2f) = \gcd(a, b) = 1$, congruence (3.15) implies that x must be even. Therefore, we use (3.2) and (3.14) to get

$$b^y = (c^r)^{\frac{x}{2}} - (a^2)^{\frac{x}{2}} = (c^r - a^2) \left(\frac{(c^r)^{\frac{x}{2}} - (a^2)^{\frac{x}{2}}}{c^r - a^2} \right) = b^2 \left(\frac{(c^r)^{\frac{x}{2}} - (a^2)^{\frac{x}{2}}}{c^r - a^2} \right). \quad (3.16)$$

Thus we obtain $y \geq 2$ and

$$b^{y-2} = \frac{(c^r)^{\frac{x}{2}} - (a^2)^{\frac{x}{2}}}{c^r - a^2}. \quad (3.17)$$

When $y = 2$, equation (3.17) implies that $x = 2$ and $z = \frac{rx}{2} = r$, which contradicts the definition of exceptional solutions of (1.1).

When $y > 2$, equations (3.2) and (3.17) give

$$0 \equiv b^{y-2} \equiv \frac{1}{b^2} \left((a^2 + b^2)^{\frac{x}{2}} - (a^2)^{\frac{x}{2}} \right) \equiv \frac{x}{2} a^{x-2} \pmod{b}. \quad (3.18)$$

As $\gcd(a, b) = 1$, we get from (3.18) that $b \mid \frac{x}{2}$ and

$$b \leq \frac{x}{2}. \quad (3.19)$$

On the other hand, since $c^r - a^2 = b^2$, applying Lemma 1 to (3.17), we get $\frac{x}{2} \leq 6$. Hence by (3.19), we have $b \leq 6$. But, since $r \geq 7$ and $m > \frac{4r}{\pi}$, by Lemma 6, we get $b^3 > a^2$ and $252 = 6^3 + 6^2 \geq b^3 + b^2 > a^2 + b^2 = c^r \geq c^7 > 5^7 = 78125$, which is a contradiction. Thus we have $rx \neq 2z$.

Now suppose that $ry = 2z$. Then we have $z = \frac{ry}{2}$ and

$$a^x = c^z - b^y = c^{\frac{ry}{2}} - b^y = (c^{\frac{r}{2}})^y - b^y. \tag{3.20}$$

From (3.11), we deduce $(f + g) \mid a$ and

$$c^{\frac{r}{2}} + b = (f + g)^2. \tag{3.21}$$

Hence (3.20) and (3.21) imply

$$0 \equiv a^x \equiv (c^{\frac{r}{2}})^y - b^y \equiv ((-1)^y - 1)b^y \pmod{f + g}. \tag{3.22}$$

Thus, since $f + g > 2$ and $\gcd(f + g, b) = \gcd(a, b) = 1$, we see that y must be even. Therefore, by (3.2) and (3.20), we obtain

$$a^x = (c^r)^{\frac{y}{2}} - (b^2)^{\frac{y}{2}} = (c^r - b^2) \left(\frac{(c^r)^{\frac{y}{2}} - (b^2)^{\frac{y}{2}}}{c^r - b^2} \right). \tag{3.23}$$

But, using the same method as in the proof of the conclusion that $rx \neq 2z$, we can prove that (3.23) is impossible. It implies that $ry \neq 2z$. Thus the lemma holds for $2 \mid r$.

We next consider the case that $2 \nmid r$. If $rx = 2z$, then

$$x = 2n, \quad z = rn, \quad n \in \mathbb{N}. \tag{3.24}$$

When $n = 1$, from (1.1), (3.2), and (3.24) we see that $(x, y, z) = (2, 2, r)$, which is a contradiction.

When $n > 1$, by (1.1), (3.2) and (3.24), we have

$$b^y = c^z - a^x = c^{rn} - a^{2n} = (c^r - a^2) \left(\frac{c^{rn} - a^{2n}}{c^r - a^2} \right) = b^2 \left(\frac{c^{rn} - a^{2n}}{c^r - a^2} \right). \tag{3.25}$$

Thus we get $y > 2$ and

$$b^{y-2} = \frac{c^{rn} - a^{2n}}{c^r - a^2}. \tag{3.26}$$

Since $\frac{c^{rn} - a^{2n}}{c^r - a^2} \equiv na^{2n-2} \pmod{b^2}$, we deduce that $b \mid n$ and

$$b \leq n. \tag{3.27}$$

On the other hand, applying Lemma 1 to (3.26), we get $n \leq 6$. Hence, by (3.27), we have $b \leq 6$. But, from Lemma 6 we deduce $b^3 > a^2$ and

$$252 = 6^3 + 6^2 \geq b^3 + b^2 > a^2 + b^2 = c^r \geq c^7 > 5^7 = 78125,$$

which is a contradiction. Thus, we have $rx \neq 2z$. Using the same analysis, we can prove that $ry \neq 2z$ for $2 \nmid r$. The lemma is proved. \square

Lemma 8. *If $a^x > b^y$ or $b^y > a^x$, then $rx > 2z$ or $ry > 2z$, respectively.*

PROOF. If $a^x > b^y$, then from (1.1) we get $2a^x > c^z$. Hence, by (3.6) and (3.1), we have

$$2c^{\frac{rx}{2}} \cos^x(r\theta) > c^z. \tag{3.28}$$

Now we assume that $rx \leq 2z$. By Lemma 7, we have $rx \neq 2z$. This implies that $rx < 2z$ and $2z - rx \geq 1$. Therefore, by (3.28), we get $2 \geq 2 \cos^x(r\theta) > \sqrt{c} = \sqrt{m^2 + 1} > m \geq 2$, which is a contradiction. Thus, we have $rx > 2z$.

Using the same method, one can prove that if $b^y > a^x$, then $ry > 2z$. The lemma is proved. \square

Lemma 9. *If $2 \mid r$ and $m > 2r^3$, then*

$$z + \frac{1}{2}r(r-1)x \geq mr^3. \tag{3.29}$$

PROOF. Since $2 \mid r$, by (1.2) and (3.1), we have

$$\begin{aligned} a &= V(m, r) = (-1)^{\frac{r}{2}} \sum_{i=0}^{\frac{r}{2}} (-1)^i \binom{r}{2i} m^{2i}, \\ b &= U(m, r) = (-1)^{\frac{r}{2}-1} m \sum_{i=0}^{\frac{r}{2}-1} (-1)^i \binom{r}{2i+1} m^{2i}. \end{aligned} \tag{3.30}$$

Hence, we use (3.1) and (3.30) to get

$$a^x \equiv (-1)^{\frac{rx}{2}} \left(1 - \binom{r}{2} m^2 x \right) \pmod{m^4}, \quad c^z \equiv 1 + m^2 z \pmod{m^4}. \tag{3.31}$$

The substitution of (3.31) into (1.1) gives

$$b^y \equiv c^z - a^x \equiv \left(1 - (-1)^{\frac{rx}{2}} \right) + \left(z + (-1)^{\frac{rx}{2}} \binom{r}{2} x \right) m^2 \pmod{m^4}. \tag{3.32}$$

Equations (3.30) help to see that $2m \mid b$, so congruence (3.32) implies that $\frac{rx}{2}$ must be even and it becomes

$$b^y \equiv \left(z + \frac{1}{2}r(r-1)x \right) m^2 \pmod{m^4}. \tag{3.33}$$

When $y \geq 4$, since $b^y \equiv 0 \pmod{m^4}$, from (3.33) we see that

$$z + \frac{1}{2}r(r-1)x \equiv 0 \pmod{m^2}.$$

This implies that

$$z + \frac{1}{2}r(r-1)x \geq m^2. \tag{3.34}$$

As $m > 2r^3$, we immediately get inequality (3.29).

When $y = 3$, we use (3.30) and (3.33) to have

$$(-1)^{\frac{r}{2}-1}mr^3 \equiv z + \frac{1}{2}r(r-1)x \pmod{m^2}. \tag{3.35}$$

If $2 \mid \frac{r}{2}$, then from (3.35) we get

$$\left(z + \frac{1}{2}r(r-1)x \right) + mr^3 \geq m^2 > 2mr^3. \tag{3.36}$$

Hence, we immediately obtain (3.29). If $2 \nmid \frac{r}{2}$, then we have

$$mr^3 \equiv z + \frac{1}{2}r(r-1)x \pmod{m^2}. \tag{3.37}$$

Since $mr^3 < m^2$, from (3.37) we see that (3.29) is true.

When $y = 2$, from (3.30) and (3.33) we deduce that

$$r^2 \equiv z + \frac{1}{2}r(r-1)x \pmod{m^2}. \tag{3.38}$$

As $(x, y, z) \neq (2, 2, r)$, we use (3.2) to get $x \neq 2$. If $x = 1$, then

$$a + b^2 = c^z. \tag{3.39}$$

Comparing (3.2) with (3.39), we have $z < r$ and $a(a-1) \equiv 0 \pmod{c^z}$. Furthermore, as $\gcd(a, c) = 1$, we get $a-1 \equiv 0 \pmod{c^z}$. Since $a > 1$, we have $a-1 \geq c^z$. Equation (3.39) gives $c^z = a + b^2 > a \geq c^z + 1$, which is a contradiction. If $x \geq 3$ and as $r \geq 7$, then we have $\frac{r(r-1)x}{2} \geq \frac{3r(r-1)}{2} > r^2$. Therefore, the congruence (3.38) implies $z + \frac{r(r-1)x}{2} \geq m^2 + r^2 > m^2 > mr^3$ and (3.29) holds.

When $y = 1$, we use (3.30) and (3.33) to have

$$(-1)^{\frac{r}{2}-1}r \equiv \left(z + \frac{r(r-1)x}{2} \right) m \pmod{m^2}. \tag{3.40}$$

Therefore, we get $m \mid r$ and $r \geq m > 2r^3$. This gives a contradiction and completes the proof of Lemma 9. \square

Lemma 10. *If $2 \nmid r$ and $m > 2r^3$, then*

$$z + \frac{1}{2}r(r-1)y \geq mr^3. \quad (3.41)$$

PROOF. The proof of this lemma is similar to that of Lemma 9. Since $2 \nmid r$, by (1.2) and (3.1), we have

$$\begin{aligned} a &= V(m, r) = \sum_{i=0}^{\frac{r-1}{2}} (-1)^i \binom{r}{2i} m^{2i}, \\ b &= U(m, r) = m \sum_{i=0}^{\frac{r-1}{2}} (-1)^i \binom{r}{2i+1} m^{2i}. \end{aligned} \quad (3.42)$$

Using (3.1) and (3.42), we get

$$b^y \equiv (-1)^{\frac{(r-1)y}{2}} \left(1 - \binom{r}{2} m^2 y \right) \pmod{m^4}, \quad c^z \equiv 1 + m^2 z \pmod{m^4}. \quad (3.43)$$

We substitute (3.43) into (1.1) to obtain

$$a^x \equiv c^z - b^y \equiv \left(1 - (-1)^{\frac{(r-1)y}{2}} \right) + \left(z + (-1)^{\frac{(r-1)y}{2}} \binom{r}{2} y \right) m^2 \pmod{m^4}. \quad (3.44)$$

As $m > 2r^3$ and $m \mid a$ (see (3.42)), then the congruence (3.44) implies that $\frac{(r-1)y}{2}$ must be even. Hence, (3.44) becomes

$$a^x \equiv \left(z + \frac{1}{2}r(r-1)y \right) m^2 \pmod{m^4}. \quad (3.45)$$

When $x \geq 4$, since $a^x \equiv 0 \pmod{m^4}$, we get from (3.45) that $z + \frac{1}{2}r(r-1)y \equiv 0 \pmod{m^2}$ and $z + \frac{1}{2}r(r-1)y \geq m^2 > mr^3$.

When $x = 3$, equations (3.42) and (3.45) give

$$(-1)^{\frac{r-1}{2}} mr^3 \equiv z + \frac{r(r-1)y}{2} \pmod{m^2}. \quad (3.46)$$

As $m > 2r^3$ and $m^2 > mr^3$, from (3.46) we see that (3.41) is true.

When $x = 2$, we have

$$r^2 \equiv z + \frac{r(r-1)y}{2} \pmod{m^2}. \quad (3.47)$$

From the beginning we consider $(x, y, z) \neq (2, 2, r)$, so $y \neq 2$. If $y = 1$, then from (1.1) we get

$$a^2 + b = c^z. \quad (3.48)$$

Comparing (3.2) and (3.48) implies $z < r$, $b(b-1) \equiv 0 \pmod{c^z}$ and $b-1 \geq c^z$.

Hence, by (3.48), we get $c^z = a^2 + b > b \geq c^z + 1$, which is a contradiction. If $y \geq 3$, since $r \geq 7$, then we have $z + \frac{r(r-1)y}{2} > r^2$. Therefore, (3.47) implies that (3.41) is true.

When $x = 1$, we have

$$(-1)^{\frac{r-1}{2}} r \equiv \left(z + \frac{r(r-1)y}{2} \right) m \pmod{m^2}. \tag{3.49}$$

But, since $2 \nmid r$ and $2 \mid m$, congruence (3.49) is impossible. This completes the proof of Lemma 10. \square

Lemma 11. *Let $\min\{a, b, c\} > 10^3$. If $a^x > b^{\frac{3y}{2}}$ or $b^y > a^{\frac{3x}{2}}$, then $x < 7215 \log c$ or $y < 7215 \log c$, respectively.*

PROOF. If $a^x > b^{\frac{3y}{2}}$, then from (1.1) we get

$$\begin{aligned} z \log c = \log(a^x + b^y) &= x \log a + \frac{2b^y}{2a^x + b^y} \sum_{i=0}^{\infty} e^{-\frac{1}{2i+1}} \left(\frac{b^y}{2a^x + b^y} \right)^{2i} \\ &< x \log a + \frac{1}{a^{\frac{x}{3}}} \sum_{i=0}^{\infty} \frac{1}{2i+1} \left(\frac{1}{2a^{\frac{x}{3}}} \right)^{2i} < x \log a + \frac{2}{a^{\frac{x}{3}}}. \end{aligned} \tag{3.50}$$

Let

$$\Lambda = z \log c - x \log a.$$

We see from (3.50) that $0 < \Lambda < \frac{2}{a^{\frac{x}{3}}}$ and

$$\log |\Lambda| < \log 2 - \frac{x}{3} \log a. \tag{3.51}$$

On the other hand, by Lemma 4, we have

$$\log |\Lambda| > -17.61(\log c)(\log a)(1.78 + B)^2, \tag{3.52}$$

where

$$B = \max \left\{ 8.45, 0.23 + \log \left(\frac{z}{\log a} + \frac{x}{\log c} \right) \right\}. \tag{3.53}$$

Combining (3.51) and (3.52) yields

$$\log 2 + 17.61(\log c)(\log a)(1.78 + B)^2 > \frac{x}{3} \log a. \tag{3.54}$$

Since $\min\{a, b, c\} > 10^3$, from (3.54) we deduce

$$52.85(\log c)(1.78 + B)^2 > x. \tag{3.55}$$

When $8.45 \geq 0.23 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right)$, we immediately get $B = 8.45$ and $x < 5550 \log c$.

When $8.45 < 0.23 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right)$, we have

$$52.85 \left(2.01 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right) \right)^2 > \frac{x}{\log c}. \quad (3.56)$$

Since $\frac{z}{\log a} < \frac{x}{\log c} + \frac{2}{a^{\frac{x}{3}}(\log a)(\log c)} < \frac{6x}{5 \log c}$, we see that

$$52.85 \left(2.80 + \log\left(\frac{x}{\log c}\right) \right)^2 > \frac{x}{\log c}. \quad (3.57)$$

A straightforward computation gives $x < 7215 \log c$.

By using the same method, we can prove that if $b^y > a^{\frac{3x}{2}}$, then $y < 7215 \log c$. Therefore, the lemma is proved. \square

Lemma 12. *If $2 \mid r$, then*

$$y < 104(\log c)(\log a)(\log B')^2, \quad (3.58)$$

where

$$\log B' = \max \left\{ 10, 0.04 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right) \right\}. \quad (3.59)$$

PROOF. Let

$$\Lambda' = c^z - a^x.$$

Then we have $\Lambda' = b^y \neq 0$. From (3.30), we know that $4 \mid b$, so $2y \leq v_2(\Lambda')$. Thus, as $c \equiv 1 \pmod{4}$, applying Lemma 5, we immediately obtain the result. \square

Using the same method, we can get the following lemma.

Lemma 13. *If $2 \nmid r$, then*

$$x < 208(\log c)(\log b)(\log B'')^2, \quad (3.60)$$

where

$$\log B'' = \max \left\{ 10, 0.04 + \log\left(\frac{z}{\log b} + \frac{y}{\log c}\right) \right\}. \quad (3.61)$$

4. Proof of Theorem

Now we suppose that (1.1) has an exceptional solution (x, y, z) . By the results of [4], [20], and [22], this solution does not exist for $2 \leq r \leq 6$. It will suffice to assume that $r \geq 7$. Moreover, we suppose that $m > \max\{10^{15}, 2r^3\}$.

Then, $c = m^2 + 1 > 10^{30} b^{\frac{3}{2}} > a > b$ (see (3.1)). Hence (3.2) implies $a > \sqrt{\frac{c^r}{2}} \geq \sqrt{\frac{c^7}{2}} > \sqrt{\frac{(10^{30})^7}{2}} > 10^{104}$ and $b > a^{\frac{2}{3}} > 10^{69}$.

First we consider the case that $2 \mid r$ and $r \geq 8$. We will separately study the following four cases.

Case I: $a^x > b^{\frac{3y}{2}}$.

In this case, Lemma 8 helps to see that $rx > 2z$. Therefore, with Lemma 9, we get $\frac{r^2x}{2} > z + \frac{r(r-1)x}{2} > mr^3$ and

$$x > 2mr. \tag{4.1}$$

On the other hand, Lemma 11 implies

$$x < 7215 \log c. \tag{4.2}$$

The combination of (4.1) and (4.2) yields

$$m < 450.9375 \log(m^2 + 1). \tag{4.3}$$

So $m < 8119$. This is a contradiction to the fact that $m > 10^{15}$.

Case II: $b^{\frac{3y}{2}} > a^x > b^y$.

Since $a^x > b^y$ and as in Case I, x satisfies (4.1). Using Lemma 12, one can see that y satisfies (3.58).

If $10 \geq 0.04 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right)$, then (3.58) and (3.59) give

$$y < 10400(\log a)(\log c). \tag{4.4}$$

Since $a^x < b^{\frac{3y}{2}}$, we have

$$2x \log a < 3y \log b. \tag{4.5}$$

Therefore, combining inequalities (4.4) and (4.5) we get

$$x < 15600(\log b)(\log c). \tag{4.6}$$

Moreover, from (3.2) we know that $\max\{a^2, b^2\} < c^r$, so we have

$$\max\{2 \log a, 2 \log b\} < r \log c. \tag{4.7}$$

Therefore, we get

$$x < 7800r(\log c)^2. \tag{4.8}$$

The combination of (4.1) and (4.8) yields

$$m < 3900 (\log(m^2 + 1))^2. \tag{4.9}$$

Then we have $m < 3.6 \cdot 10^6$, which contradicts the condition $m > 10^{15}$.

If $10 < 0.04 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right)$, then we have

$$y < 104(\log a)(\log c) \left(0.04 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right)\right)^2. \quad (4.10)$$

As $a^x > b^y$, we get $2a^x > a^x + b^y = c^z$ and

$$\frac{z}{\log a} < \frac{x}{\log c} + \frac{\log 2}{(\log a)(\log c)} < \frac{2x}{\log c}. \quad (4.11)$$

Therefore, we obtain

$$y < 104(\log a)(\log c) \left(1.14 + \log\left(\frac{x}{\log c}\right)\right)^2. \quad (4.12)$$

Combining (4.5) and (4.12), we get

$$\frac{x}{\log c} < 156(\log b) \left(1.14 + \log\left(\frac{x}{\log c}\right)\right)^2. \quad (4.13)$$

Let $X = \frac{x}{\log c}$, $f(X) = X - 156(\log b)(1.14 + \log X)^2$, and $X_0 = 3120(\log b)(\log \log b)^2$. Since $b > 10^{69}$, we have $f(X_0) > 0$ and $f'(X) = 1 - \frac{312(\log b)(1.14 + \log X)}{X} > 0$ for $X \geq X_0$. Therefore, we obtain

$$\frac{x}{\log c} < 3120(\log b)(\log \log b)^2. \quad (4.14)$$

As $b > 10^{69}$, we have $\log b > (\log \log b)^2$ and

$$x < 3120(\log b)^2(\log c). \quad (4.15)$$

Then (4.7) and (4.15) imply

$$x < 780r^2(\log c)^3. \quad (4.16)$$

With (4.1), we get

$$m < 390r(\log c)^3. \quad (4.17)$$

Using $m > 2r^3$, we have $m^{\frac{1}{3}} > r$ and

$$m^{\frac{2}{3}} < 390(\log(m^2 + 1))^3. \quad (4.18)$$

A straightforward computation gives $m < 3 \cdot 10^{11}$. This contradicts $m > 10^{15}$.

Case III: $a^{\frac{3x}{2}} > b^y > a^x$.

Since $a > b$ (see Lemma 6) and $b^y > a^x$, we get $y > x$. In addition, by Lemma 8, we have $ry > 2z$. Therefore, Lemma 9 helps us to see that

$$y > 2mr. \tag{4.19}$$

On the other hand, by Lemma 12, y satisfies (3.58).

Now, if $10 \geq 0.04 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right)$, then y satisfies (4.4). The combination of (4.4) and (4.19) yields

$$mr < 5200(\log a)(\log c). \tag{4.20}$$

Inequality (4.7) helps to get

$$m < 2600(\log(m^2 + 1))^2. \tag{4.21}$$

This implies that $m < 2.3 \cdot 10^6$, which is a contradiction to the condition $m > 10^{15}$.

If $10 < 0.04 + \log\left(\frac{z}{\log a} + \frac{x}{\log c}\right)$, then y satisfies (4.10). Again here, we see that the following inequalities $a > b$, $b^y > a^x$, $2b^y > c^z$, and $\log 2 + y \log a > \log 2 + y \log b > z \log c$ are true. This implies that

$$\frac{z}{\log a} < \frac{z}{\log b} < \frac{y}{\log c} + \frac{\log 2}{(\log b)(\log c)} < \frac{2y}{\log c}. \tag{4.22}$$

Recall that $y > x$. So (4.10) and (4.22) yield

$$\frac{y}{\log c} < 104(\log a) \left(1.14 + \log\left(\frac{y}{\log c}\right)\right)^2. \tag{4.23}$$

Similarly, we get

$$y < 2080(\log a)(\log \log a)^2(\log c) < 2080(\log a)^2(\log c). \tag{4.24}$$

Also using (4.7) and (4.24), we have

$$y < 520r^2(\log c)^3. \tag{4.25}$$

Since $m > 2r^3$, we use (4.19) and (4.25) to obtain

$$m^{\frac{2}{3}} < 260(\log(m^2 + 1))^3. \tag{4.26}$$

Thus $m < 3 \cdot 10^{11}$. Again we have a contradiction to the fact that $m > 10^{15}$.

Case IV: $b^y > a^{\frac{3x}{2}}$.

Since $b^y > a^x$, then $ry > 2z$ (see Lemma 8). Moreover, $b^y > a^{\frac{3x}{2}}$ and $a > b$ imply $y > \frac{3x}{2}$. Therefore, by Lemma 9, y must satisfy (4.19).

On the other hand, $b^y > a^{\frac{3x}{2}}$ and Lemma 11, give

$$y < 7215 \log c. \quad (4.27)$$

Again here, we use (4.19) and (4.27) to come to a contradiction. Thus, the theorem is proved for $2 \mid r$.

Finally, we consider the case $2 \nmid r$. Using a method similar to that in the proof of the case $2 \mid r$ and replacing Lemmas 9 and 12 by Lemmas 10 and 13 respectively, we come to the same conclusion. This completes the proof of the theorem.

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