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Asymtotic behavior of solutions of forced nonlinear delay differential equations

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1. Introduction

This paper is concerned with the asymptotic behavior of solutions of nonlinear forced delay differential equations of the form

$$(1) \qquad x'(t) + \sum_{i=1}^{n} p_i(t) f(x(t-\tau_i)) = r(t), \quad t \geq t_0,$$

where $p_i, r \in C([t_0, \infty), R), \tau_i \geq 0$, $i = 1, 2, \dots, n$, $f \in C(R, R), xf(x) > 0$ for $x \neq 0$. The nonoscillatory and oscillatory properties of (1) and the related equations have been studied by many authors; we mention here the work of Győri, Ladas and Pakula [3]. Kulenovic, Ladas and Meimaridou [4], [5] and the references cited therein.

As is customary, a solution is called oscillatory if is has arbitrarily large zeros; otherwise, it is called nonoscillatory.

Recently, Kulenovic, Ladas and Meimaridou [4] have obtained interesting sufficient conditions for the asymptotic stability of the trivial solution of the delay differential equation

$$(1') \qquad x'(t) + \sum_{i=1}^{n} p_i(t) x(t-\tau_i) = 0, \qquad t \geq t_0.$$

Their approach is based on dividing the set of solutions of $(1')$ into oscillatory and nonoscillatory solutions and then examining the asymtotic properties of each class. Our aim in this paper is to obtain sufficient conditions for the asymtotic behavior of all solutions of (1). Here the approach in [4] will be used. The results obtained extend and improve some of the results of [4].

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In the sequel, for convenience, we will assume that inequalities concerning values of functions are satisfied eventually, that is for all large t.

2. Main Results

Without loss of generality, we will assume throughout this paper that $0 \le \tau_1 < \tau_2 < \cdots < \tau_n$.

We introduce the following conditions:

$$|f(x)| \le M|x| \quad \text{for all } x. \tag{2}$$

where M is a positive constant, and

$$R(t) = \int_t^\infty r(s)ds \quad \text{exists on } [t_0, \infty). \tag{3}$$

Theorem 1. *Assume that* (2) *and* (3) *hold and that there exist positive constants C_1 and C_2 such that the following conditions are satisfied for sufficiently large t*

$$|p_i(t)| \le C_1 \quad \text{for } i = 1, 2, \dots n, \tag{4}$$

$$\sum_{i=1}^n p_i(t - \tau_n + \tau_i) \ge C_2, \tag{5}$$

and

$$\sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} p_i(s + \tau_i)ds \le \frac{1}{M}, \tag{6}$$

where $p_i(t)_- = \min\{p_i(t), 0\}$. *Then every nonoscillatory solution of* (1) *tends to zero as* $t \to \infty$.

Theorem 2. *Assume that* (2) *and* (3) *hold and that for sufficiently large t*

$$\sum_{i=1}^n p_i(t - \tau_n + \tau_i) \ne 0, \tag{7}$$

$$2 \limsup_{t\to\infty} Q_1(t) + \limsup_{t\to\infty} Q_2(t) < \frac{1}{M}, \tag{8}$$

where

$$Q_1(t) = \sum_{i=1}^n \int_{t-\tau_n}^{t-\tau_i} |p_i(s + \tau_i)|ds,$$

and

$$Q_2(t) = \sum_{i=1}^{n} \int_{t-\tau_n}^{t} |p_i(s - \tau_n + \tau_i)| ds.$$

Then every oscillatory solution of (1) *tends to zero as* $t \to \infty$.

Combining Theorems 1 and Theorem 2, we obtain the following

Theorem 3. *Assume that* (2)–(5) *and* (8) *are satisfied. Then all solutions of* (1) *tend to zero as* $t \to \infty$.

Remark 1. From (2) and (3) we see that our results hold for linear and for nonlinear equations, forced equations and associated unforced equations.

Remark 2. Our results can be extended to more general equations of the form

$$x'(t) + \sum_{i=1}^{n} p_i(t) f_i(x(t - \tau_i)) = r(t),$$

which involve different functions f_i each of which satisfies the corresponding conditions. When $r(t) \equiv 0$ and $p_i(t)$ are constants, (1) reduces to

$$x'(t) + \sum_{i=1}^{n} p_i f(x(t - \tau_i)) = 0. \tag{9}$$

The following corollaries are immediate consequences of Theorems 1,2 and 3.

Corollary 1. *Assume that* (2) *holds and that*

$$\sum_{i=1}^{n} p_i > 0, \tag{10}$$

and

$$\sum_{i=1}^{n-1} (\tau_n - \tau_i) p_{i-} \le \frac{1}{M}. \tag{11}$$

where $p_{i-} = \min\{p_i, 0\}$. *Then every nonoscillatory solution of* (9) *tends to zero as* $t \to \infty$.

Corollary 2. *Assume that* (2) *holds and that*

$$\sum_{i=1}^{n} p_i \neq 0, \tag{12}$$

$$\sum_{i=1}^{n} (3\tau_n - 2\tau_i)|p_i| < \frac{1}{M}. \tag{13}$$

Then every oscillatory solution of (9) *tends to zero as* $t \to \infty$.

Corollary 3. *Assume that* (2), (10) *and* (13) *hold. Then all solutions of* (9) *tend to zero as* $t \to \infty$.

For illustration we consider the following

Example. Consider the nonlinear delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i \frac{x(t-\tau_i)}{1+|x(t-\tau_i)|^{\beta}} = 0. \tag{14}$$

where $p_i, \tau_i,\ i = 1, 2, \ldots, n$, are constants and β is a positive constant. The delay equation (14) with $n = 1$ has appeared in connection with physiological control theory; see Chapin and Nussbaum [1] and Kulenovic, Ladas and Meimaridou [4].

By Corollaries 1,2 and 3, we have the following conclusions:

(i) Assume that $\sum_{i=1}^{n} p_i > 0$ and $\sum_{i=1}^{n-1} (\tau_n - \tau_i) p_{i-} \leq 1$, then every nonoscillatory solution of (14) tends to zero as $t \to \infty$;

(ii) Assume that $\sum_{i=1}^{n} p_i \neq 0$ and $\sum_{i=1}^{n} (3\tau_n - 2\tau_i)|p_i| < 1$, then every oscillatory solution of (14) tends to zero as $t \to \infty$;

(iii) Assume that $\sum_{i=1}^{n} p_i > 0$ and $\sum_{i=1}^{n} (3\tau_n - 2\tau_i)|p_i| < 1$, then all solutions of (14) tend to zero $t \to \infty$.

Remark 3. By Lemma 1 in [1] and Theorem 6 in [5] all solutions of (14) are oscillatory if and only if the equation

$$\lambda + \sum_{i=1}^{n} p_i e^{\lambda \tau_i} = 0,$$

has no real roots.

3. Proofs of the Theorems

Proof of Theorem 1. Let $x(t)$ be a solution of (1). Set

$$z(t) = x(t) + \sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} p_i(s+\tau_i) f(x(s)) ds + R(t), \tag{15}$$

with the convention that when $n = 1$ the above sum is zero. Since the negative of a solution of (1) is also a solution, we will suppose that $x(t) > 0$. From (1) and (15), we have

$$z'(t) = -\sum_{i=1}^{n-1} p_i(t-\tau_n+\tau_i) f(x(t-\tau_n)). \tag{16}$$

From (16) and (5) it follows that

$$z'(t) \le -C_2 f(x(t-\tau_n)). \tag{17}$$

which implies that $z(t)$ is a strictly decreasing function. Set $L = \lim_{t\to\infty} z(t)$. We claim that $L \in R$. Otherwise $L = -\infty$ and because of (6), $x(t)$ must be unbounded. In fact, suppose that there exists a constant C such that $x(t) \le C$. Then, from (15), (6) and (2), we have

$$\begin{aligned} z(t) &\ge x(t) - \sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} p_i(s+\tau_i)_- f(x(s)) ds + R(t) \\ &\ge x(t) - CM \sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} p_i(s+\tau_i)_- ds + R(t) \\ &\ge x(t) - CM \sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} p_i(s+\tau_i)_- ds + R(t) \ge -C^*, \end{aligned}$$

where C^* is a constant, which contradicts that $L = -\infty$. Thus $x(t)$ is unbounded. Choose a $t_1 \ge t_0 + \tau_n$ in such a way that (6) is satisfied for $t \ge t_1$, $z(t_1) - R(t_1) < 0$ and $x(t_1) = \max_{t_0 \le s \le t_1} x(s)$. Clearly, this choice of t_1, is possible because $x(t)$ is unbounded and $\lim_{t\to\infty} \langle z(t) - R(t) \rangle = -\infty$.

Then, from (15), (6) and (2), we have

$$\begin{aligned} 0 > z(t_1) - R(t_1) &= x(t_1) + \sum_{i=1}^{n-1} \int_{t_1-\tau_n}^{t_1-\tau_i} p_i(s+\tau_i) f(x(s)) ds \\ &\geq x(t_1) - \sum_{i=1}^{n-1} \int_{t_1-\tau_n}^{t_1-\tau_i} p_i(s+\tau_i)_- M x(s) ds \\ &\geq x(t_1) \langle 1 - M \sum_{i=1}^{n-1} \int_{t_1-\tau_n}^{t_1-\tau_i} p_i(s+\tau_i)_- ds \rangle \geq 0, \end{aligned}$$

which is a contradiction. Thus $L \in R$.

We are now in a position to prove that

$$\lim_{t\to\infty} x(t) = 0. \tag{18}$$

In fact, integrating (17) from t_1 to t for t_1 sufficiently large and letting $t \to \infty$, we find

$$L - z(t_1) \leq -C_2 \int_{t_1}^{\infty} f(x(s-\tau_n)) ds.$$

Hence $f(x(t)) \in L^1[t_1, \infty)$ and $\liminf_{t\to\infty} f(x(t)) = 0$. Since $x(t)$ is bounded, it follows that

$$\liminf_{t\to\infty} x(t) = 0. \tag{19}$$

Integrating (1) from t_1 to t and letting $t \to \infty$, we obtain

$$\lim_{t\to\infty} (x(t) - x(t_1)) = -\sum_{i=1}^{n} \int_{t_1}^{\infty} p_i(s) f(x(s-\tau_i)) ds + R(t_1) < \infty, \tag{20}$$

where we have used (3) and (4). Combining (20) with (19) we obtain (18) as claimed. The proof is complete.

Proof of Theorem 2. Let $x(t)$ be an oscillatory solution of (1). Firts we will prove that $x(t)$ is bounded. Suppose that $x(t)$ is unbounded. Choose a $t_1 \geq t_0 + \tau_n$ such that (7) holds for $t \geq t_1$ and also

$$\max_{t_1 \leq s \leq t} |x(s)| \geq \max_{t-\tau_n \leq s \leq t-\tau_1} |x(s)|, \quad \text{for } t \geq t_1.$$

Clearly, this choice of t_1 is possible because $x(t)$ is unbounded. Then, from (15), we have

$$|z(t)| \geq |x(t)| - \sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} |p_i(s+\tau_i)|\,|f(x(s))|ds - |R(t)|$$
$$\geq |x(t)| - M(\max_{t_1 \leq s \leq t} |x(s)|)Q_1(t) - |R(t)|,$$

which implies that

$$\max_{t_1 \leq s \leq t} |z(s)| \geq \max_{t_1 \leq s \leq t} |x(s)| - M(\max_{t_1 \leq s \leq t} |x(s)|) \max_{t_1 \leq s \leq t} Q_1(s)$$
$$- \max_{t_1 \leq s \leq t} |R(s)|$$
$$\geq \max_{t_1 \leq s \leq t} |x(s)| \langle 1 - M \max_{t_1 \leq s \leq t} Q_1(s) \rangle - \max_{t_1 \leq s \leq t} |R(t)|. \tag{21}$$

By (21), (8) and the fact that $\lim_{t\to\infty} R(t)$=0, we find that $z(t)$ is unbounded. Also, from (16), we see that $z'(t)$ oscillates. Thus, there exists a sequence of points $\{\xi_k\}$ such that $\xi_k \geq t_1$ for $k = 1, 2, \ldots$, $\lim_{k\to\infty} \xi_k = \infty$, $\lim_{k\to\infty} |z(\xi_k)| = \infty$, $z'(\xi_k) = 0$ for $k = 1, 2, \ldots$, and

$$|z(\xi_k)| = \max_{t_1 \leq s \leq \xi_k} |z(s)|.$$

Form (16), using (7) and the fact that $z'(\xi_k) = 0$, we see that $x(\xi_k - \tau_n) = 0$ for $k = 1, 2, \ldots$, and so (15) yields

$$z(\xi_k - \tau_n) = \sum_{i=1}^{n-1} \int_{\xi_k - 2\tau_n}^{\xi_k - \tau_n - \tau_i} p_i(s+\tau_i) f(x(s))ds + R(\xi_k - \tau_n). \tag{22}$$

Integrating (16) from $\xi_k - \tau_n$ to ξ_k and using (22) we obtain

$$z(\xi_k) = \sum_{i=1}^{n-1} \int_{\xi_k - 2\tau_n}^{\xi_k - \tau_n - \tau_i} p_i(s+\tau_i) f(x(s))ds$$
$$- \int_{\xi_k - \tau_n}^{\xi_k} \langle \sum_{n=1}^{n} p_i(s - \tau_n + \tau_i) \rangle f(x(s-\tau_n))ds + R(\xi_k - \tau_n). \tag{23}$$

Thus we get

$$|x(\xi_k)| \leq \max_{t_1 \leq s \leq \xi_k} |x(s)| M Q_1(\xi_k - \tau_n) + \max_{t_1 \leq s \leq \xi_k} |x(s)| M Q_2(\xi_k)$$
$$+ |R(\xi_k - \tau_n)|, \tag{24}$$

and, in view of (21)

$$\langle 1 - M \max_{t_1 \le s \le \xi_k} Q_1(s)\rangle \max_{t_1 \le s \le \xi_k} |x(s)|$$

$$\le M\langle Q_1(\xi_k - \tau_n) + Q_2(\xi_k)\rangle \max_{t_1 \le s \le \xi_k} |x(s)| + |R(\xi_k - \tau_n)| + \max_{t_1 \le s \le \xi_k} |R(s)|,$$

that is,

$$0 \le -1 + M\langle \max_{t_1 \le s \le \xi_k} Q_1(s) + Q_1(\xi_k - \tau_n) + Q_2(\xi_k)\rangle$$
$$+ (\max_{t_1 \le s \le \xi_k} |x(s)|)^{-1}(|R(\xi_k - \tau_n)| + \max_{t_1 \le s \le \xi_k} |R(s)|).$$

Let $k \to \infty$, then we find

$$0 \le -1 + M(2\bar{Q}_1 + \bar{Q}_2),$$

where $\bar{Q}_1 = \limsup_{t\to\infty} Q_1(t)$ and $\bar{Q}_2 = \limsup_{t\to\infty} Q_2(t)$, which contradicts (8) and proves our claim.

Next, we prove that every bounded oscillatory solution $x(t)$ of (1) tends to zero as $t \to \infty$. Indeed, assume that

$$\mu = \limsup_{t\to\infty} |x(t)| > 0.$$

Then for any $\varepsilon > 0$ there exists a $t_2 \ge t_1$ such that

$$|x(t)| < \mu + \varepsilon \quad \text{for } t \ge t_2.$$

Form (15) we have

$$|z(t) > |x(t)| - \sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} |p_i(s+\tau_i)|\,|f(x(s))|ds - |R(t)|$$
$$\ge |x(t)| - (\mu+\varepsilon)MQ_1(t) - R(t), \quad t \ge t_2.$$

Thus

$$\alpha = \limsup_{t\to\infty} |z(t)| \ge \mu - (\mu+\varepsilon)M\bar{Q}_1.$$

As ε is arbitrary, it follows that

$$\alpha \ge \mu(1 - M\bar{Q}_1). \tag{25}$$

Since $z'(t)$ oscillates, there exists a sequence of points $\{\zeta_k\}$ such that $\zeta_k \ge t_2$ for $k = 1, 2, \dots, \lim_{k\to\infty} \zeta_k = \infty$, $z'(\zeta_k) = 0$ for $k = 1, 2, \dots,$ and

$$\lim_{t\to\infty} |z(\zeta_k)| = \alpha.$$

Also, (22) and so (23) is true with ξ_k repalced by ζ_k. Hence, from (24),

$$|z(\zeta_k)| \leq M(\mu+\varepsilon)\langle Q_1(\zeta_k-\tau_n)+Q_2(\zeta_k)\rangle + |R(\zeta_k-\tau_n)|.$$

Letting $k \to \infty$, we obtain

$$\alpha \leq M(\mu+\varepsilon)(\bar{Q}_1+\bar{Q}_2).$$

As ε is arbitrary, it follows that

$$\alpha \leq M\mu(\bar{Q}_1+\bar{Q}_2).$$

By (25), we have

$$\mu(1-M\bar{Q}_1) \leq M\mu(\bar{Q}_1+\bar{Q}_2),$$

or

$$1 \leq 2(\bar{Q}_1+\bar{Q}_2)M,$$

which contradicts the hypothesis (8) and the proof is complete.

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