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# Groups, partitions and representation functions

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**Abstract.** Let $X$ be a semigroup written additively and $h \geq 2$ a fixed integer. Let $x$ be an element of $X$ and $A_1, \dots, A_h$ be nonempty subsets of $X$. Let $R_{A_1+\dots+A_h}(x)$ denote the number of solutions of the equation $a_1 + \dots + a_h = x$, where $a_i \in A_i$. In this paper for $X = \mathbb{N}$ we give a necessary and sufficient condition such that the equality $R_{A_1+A_2}(n) = R_{X\setminus A_1+X\setminus A_2}(n)$ holds from a certain point on. We study similar questions when $X = \mathbb{Z}_m$ and in general when $X = G$, where $G$ is a finite additive group.

## 1. Introduction

Let $X$ be a semigroup, written additively. Let $A_1, \dots, A_h$ be nonempty subsets of $X$ and let $x$ be an element of $X$. We denote by $|A|$ the cardinality of the set $A$. We define the ordered representation function

$$R_{A_1+\dots+A_h}(x) = |\{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h : a_1 + \dots + a_h = x\}|.$$

If $A_i = A$ for $i = 1, \dots, h$, then we write

$$R^{(1)}_{A,h}(x) = |\{(a_1, \dots, a_h) : a_i \in A,\ a_1 + \dots + a_h = x\}|.$$

Let $X$ be an abelian semigroup, written additively. For $A \subset X$, let $A^h$ denote the set of all $h$-tuples of $A$. Two $h$-tuples $(a_1, \dots, a_h) \in A^h$ and $(a'_1, \dots, a'_h) \in A^h$ are equivalent if there is a permutation $\alpha : \{1, \dots, h\} \to \{1, \dots, h\}$ such that

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$a_{\alpha(i)} = a'_i$ for $i = 1, \dots, h$. Two other representation functions arise often and naturally in additive number theory. The unordered representation function $R^{(2)}_{A,h}(x)$ counts the number of equivalence classes of $h$-tuples $(a_1, \dots, a_h)$ such that $a_1 + \ldots + a_h = x$. The unordered restricted representation function $R^{(3)}_{A,h}(x)$ counts the number of equivalence classes of $h$-tuples $(a_1, \dots, a_h)$ of pairwise distinct elements of $A$ such that $a_1 + \cdots + a_h = x$. It is easy to see that the definitions of the unordered and the unordered restricted representation functions make sense only in Abelian groups.

Alternative definitions for $R^{(2)}_{A,2}(x)$ and $R^{(3)}_{A,2}(x)$ are the following. Denote by

$$D_A(x) = |\{a : a \in A,\ a + a = x\}|$$

then

$$R^{(2)}_{A,2}(x) = \frac{1}{2} R^{(1)}_{A,2}(x) + \frac{1}{2} D_A(x) \tag{1}$$

and

$$R^{(3)}_{A,2}(x) = \frac{1}{2} R^{(1)}_{A,2}(x) - \frac{1}{2} D_A(x). \tag{2}$$

Let $\mathbb{N}$ be the set of nonnegative integers. Let $X = \mathbb{N}$. Sárközy asked if there exist two sets $A$ and $B$ with $|A \Delta B| = \infty$ such that $R^{(i)}_{A,2}(n) = R^{(i)}_{B,2}(n)$, for $i = 1, 2, 3$ and for all sufficiently large $n$. For $i = 2$ Dombi [3] proved that the answer is positive and for $i = 1$ the answer is negative. For $i = 3$ Chen and Wang [1] proved that the set of natural numbers can be partitioned into two subsets $A$ and $B$ such that $R^{(3)}_{A,2}(n) = R^{(3)}_{B,2}(n)$ for every $n$ large enough. Lev [5] and independently Sándor [6] characterized all subsets $A \subset \mathbb{N}$ such that $R^{(2)}_{A,2}(n) = R^{(2)}_{\mathbb{N}\setminus A,2}(n)$ or $R^{(3)}_{A,2}(n) = R^{(3)}_{\mathbb{N}\setminus A,2}(n)$ for big enough $n$. The precise theorems are the following.

**Theorem** (Lev, Sándor, 2004)**.** *Let $X = \mathbb{N}$. Let $N$ be a positive integer. The equality $R^{(2)}_{A,2}(n) = R^{(2)}_{\mathbb{N}\setminus A,2}(n)$ holds for $n \ge 2N-1$ if and only if $|A \cap [0, 2N-1]| = N$ and $2m \in A \Leftrightarrow m \in A$, $2m+1 \in A \Leftrightarrow m \notin A$ for $m \ge N$.*

**Theorem** (Lev, Sándorn, 2004)**.** *Let $X = \mathbb{N}$. Let $N$ be a positive integer. The equality $R^{(3)}_{A,2}(n) = R^{(3)}_{\mathbb{N}\setminus A,2}(n)$ holds for $n \ge 2N-1$ if and only if $|A \cap [0, 2N-1]| = N$ and $2m \in A \Leftrightarrow m \notin A$, $2m+1 \in A \Leftrightarrow m \in A$ for $m \ge N$.*

Tang [4] gave an elementary proofs of Lev and Sándor's results. In [7], [8], [9], [10]. Chen and Yang studied related problems about weighted representation functions. Similar statement to the above theorems can not be formulated for the representation function $R^{(1)}_{A,2}(n)$ because $R^{(1)}_{A,2}(n)$ is odd if and only if $\frac{n}{2} \in A$, therefore either $R^{(1)}_{A,2}(2m)$ or $R^{(1)}_{\mathbb{N}\setminus A,2}(2m)$ is odd. A nontrivial result is the following in this direction.

**Theorem 1.** *Let $X = \mathbb{N}$. The equality $R^{(1)}_{A+B}(n) = R^{(1)}_{\mathbb{N}\setminus A+\mathbb{N}\setminus B}(n)$ holds from a certain point on if and only if $|\mathbb{N} \setminus (A \cup B)| = |A \cap B| < \infty$.*

The modular questions were solved by Chen and Yang [3].

**Theorem** (Chen, Yang, 2012)**.** *Let $X = \mathbb{Z}_m$. The equality $R^{(1)}_{A,2}(n) = R^{(1)}_{\mathbb{Z}_m\setminus A,2}(n)$ holds for all $n \in \mathbb{Z}_m$ if and only if $m$ is even and $|A| = m/2$.*

**Theorem** (Chen, Yang, 2012)**.** *Let $X = \mathbb{Z}_m$. For $i \in \{2, 3\}$, the equality $R^{(i)}_{A}(n) = R^{(i)}_{\mathbb{Z}_m\setminus A}(n)$ holds for all $n \in \mathbb{Z}_m$ if and only if $m$ is even and $t \in A \Leftrightarrow t + m/2 \notin A$ for $t = 0, 1, \dots, m/2 - 1$.*

We extend the first theorem to arbitrary finite group $G$ and the second theorem to finite Abelian group.

**Theorem 2.** *Let $X = G$ be a finite group. Then*

(i) *If there exists a $g \in G$ for which the equality $R_{A+B}(g) = R_{G\setminus A+G\setminus B}(g)$ holds, then $|A| + |B| = |G|$.*

(ii) *If $|A| + |B| = |G|$, then the equality $R_{A+B}(g) = R_{G\setminus A+G\setminus B}(g)$ holds for all $g \in G$.*

We generalize Chen and Yang's theorems in the following way.

**Theorem 3.** *Let $X = G$ be a finite group and $h \geq 2$ a fixed integer.*

(i) *If the equality $R^{(1)}_{A,h}(g) = R^{(1)}_{G\setminus A,h}(g)$ holds for all $g \in G$, then $|G|$ is even and $|A| = |G|/2$.*

(ii) *If $h$ is even and $|A| = |G|/2$ then $R^{(1)}_{A,h}(g) = R^{(1)}_{G\setminus A,h}(g)$ holds for all $g \in G$.*

The case when $h$ is odd is still open.

**Problem.** *Let $h > 1$ be a fixed odd positive integer. Let $G$ be an Abelian group and $A \subset G$ be a nonempty subset. Does there exist a $g \in G$ such that $R^{(1)}_{A,h}(g) \neq R^{(1)}_{G\setminus A,h}(g)$?*

When $h$ is odd we can only prove the following weaker result.

**Theorem 4.** *Let $X = \mathbb{Z}_m$ and $h > 2$ be a fixed odd integer. If $A \subset \mathbb{Z}_m$ such that $|A| = m/2$ then there exists a $g \in \mathbb{Z}_m$ such that $R^{(1)}_{A,h}(g) \neq R^{(1)}_{\mathbb{Z}_m\setminus A,h}(g)$.*

It would be interesting to characterize all that partitions of a finite Abelian group $G$ such that $R^{(1)}_{A_i,h}(g) = R^{(1)}_{A_j,h}(g)$ for every $g \in G$.

**Problem.** *Let $G$ be an Abelian group and $h \geq 2$. Characterize all the partitions of $G$ into pairwise disjoint sets $A_1, A_2, \dots, A_h$ such that for every $g \in G$ and for every $1 \leq i, j \leq h$, $R^{(1)}_{A_i,h}(g) = R^{(1)}_{A_j,h}(g)$.*

For the two other representation functions we have the following result.

**Theorem 5.** *Let $X = G$ be a finite Abelian group. For $i \in \{2,3\}$ the equality $R_A^{(i)}(g) = R_{G\setminus A,2}^{(i)}(g)$ holds for every $g \in G$, if and only if $D_A(g) = D_{G\setminus A}(g)$ for every $g \in G$.*

## 2. Proofs

Proof of Theorem 1. Let $A(x) = \sum_{a\in A} x^a$ be the generating function of the set $A$ and let $B(x) = \sum_{b\in B} x^b$ be the generating function of the set $B$. It is easy to see that

$$A(x)B(x) = \Big(\sum_{a\in A} x^a\Big)\Big(\sum_{b\in B} x^b\Big) = \sum_{n=0}^{\infty} R_{A+B}(n)x^n.$$

Since

$$\frac{1}{1-x} - A(x) = \sum_{n=0}^{\infty} x^n - \sum_{a\in A} x^a = \sum_{a\in \mathbb{N}\setminus A} x^a,$$

and similarly

$$\frac{1}{1-x} - B(x) = \sum_{n=0}^{\infty} x^n - \sum_{b\in B} x^b = \sum_{b\in \mathbb{N}\setminus B} x^b,$$

it follows that

$$\Big(\frac{1}{1-x} - A(x)\Big)\Big(\frac{1}{1-x} - B(x)\Big) = \sum_{n=0}^{\infty} R_{\mathbb{N}\setminus A+\mathbb{N}\setminus B}(n)x^n.$$

Hence the condition $R_{A+B}(n) = R_{\mathbb{N}\setminus A+\mathbb{N}\setminus B}(n)$ holds from a certain point on is equivalent to

$$A(x)B(x) - \Big(\frac{1}{1-x} - A(x)\Big)\Big(\frac{1}{1-x} - B(x)\Big) = p(x),$$

where $p(x)$ is a polynomial with integral coefficients. This is equivalent to

$$A(x) + B(x) = \frac{1}{1-x} + p(x)(1-x). \tag{3}$$

Let

$$\frac{1}{1-x} + p(x)(1-x) = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{N} d_n x^n = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = 0, 1$ or 2. As $\sum_{n=0}^{N} d_n = 0$, it follows that the equation (3) holds if and only if $c_n = 1$ except for finitely many integer $n$ and the number of $n$ for which $c_n = 0$ is equal to the number of $n$ for which $c_n = 2$. This is equivalent to the condition $|\mathbb{N} \setminus (A \cup B)| = |A \cap B| < \infty$. $\square$

Proof of Theorem 2. For a given $S \subset G$ we denote by $\chi_S$ its characteristic function, that is $\chi_S(g) = 1$ if $g \in S$ and $\chi_S(g) = 0$ if $g \notin S$ for every $g \in G$. It follows that $\chi_{G\setminus S}(g) = 1$ if $g \in G \setminus S$, i.e., $g \notin S$ and $\chi_{G\setminus S}(g) = 0$ if $g \notin G \setminus S$, i.e., $g \in S$. Thus we have

$$\chi_{G\setminus S} = 1 - \chi_S. \tag{4}$$

It is easy to see that

$$R_{A+B}(g) = \sum_{\substack{c+d=g \\ c,d\in G}} \chi_A(c)\chi_B(d) = \sum_{c\in G} \chi_A(c)\chi_B(-c+g),$$

$$R_{G\setminus A+G\setminus B}(g) = \sum_{\substack{c+d=g \\ c,d\in G}} \chi_{G\setminus A}(c)\chi_{G\setminus B}(d) = \sum_{c\in G} \chi_{G\setminus A}(c)\chi_{G\setminus B}(-c+g).$$

It follows from (4) that for every $g \in G$ we have

$$\begin{aligned} R_{A+B}(g) - R_{G\setminus A+G\setminus B}(g) &= \sum_{c\in G} \chi_A(c)\chi_B(-c+g) - \sum_{c\in G} \chi_{G\setminus A}(c)\chi_{G\setminus B}(-c+g) \\ &\sum_{c\in G} \chi_A(c)\chi_B(-c+g) - \sum_{c\in G} (1-\chi_A(c))(1-\chi_B(-c+g)) \\ &= \sum_{c\in G} \chi_A(c) + \sum_{c\in G} \chi_B(-c+g) - \sum_{c\in G} 1 = |A| + |B| - |G|. \end{aligned}$$

Hence if there exists a $g \in G$ such that $R_{A+B}(g) = R_{G\setminus A+G\setminus B}(g)$ then $|A|+|B| = |G|$ and if $|A| + |B| = |G|$ then $R_{A+B}(g) = R_{G\setminus A+G\setminus B}(g)$ for every $g \in G$. $\square$

Proof of Theorem 3.

$$R^{(1)}_{A,h}(g) = R^{(1)}_{G\setminus A,h}(g) \tag{5}$$

holds for all $g \in G$. It is clear that

$$\sum_{g\in G} R^{(1)}_{A,h}(g) = |A|^h,$$

and

$$\sum_{g\in G} R^{(1)}_{G\setminus A,h}(g) = |G \setminus A|^h.$$

It follows from (5) that $|A|^h = |G\setminus A|^h$ which implies $|A| = |G|/2$, which proves (i).

In the next step we prove (ii). Assume that $h$ is even. For $h = 2$ the statement follows immediately from Theorem 2. Let $h > 2$. Then we have

$$\begin{aligned}
R^{(1)}_{A,h}(g) &= |\{(a_1, a_2, \dots, a_h) : a_i \in A,\ a_1 + \dots + a_h = g\}| \\
&= \sum_{\substack{g_1,\dots,g_{h/2}\in G\\ g_1+\dots+g_{h/2}=g}} |\{(a_1, a_2, \dots, a_h) : a_i \in A, a_1 + a_2 = g_1, \dots \\
&\qquad \dots, a_{h-1} + a_h = g_{h/2}, a_1 + \dots + a_h = g\}| \\
&= \sum_{\substack{g_1,\dots,g_{h/2}\in G\\ g_1+\dots+g_{h/2}=g}} |\{(a_1, a_2)\in A : a_1+a_2=g_1\}| \cdots |\{(a_{h-1}, a_h)\in A : a_{h-1}+a_h=g_{h/2}\}| \\
&= \sum_{\substack{g_1,\dots,g_{h/2}\in G\\ g_1+\dots+g_{h/2}=g}} R^{(1)}_{A,2}(g_1) \cdots R^{(1)}_{A,2}(g_{h/2}) = \sum_{\substack{g_1,\dots,g_{h/2}\in G\\ g_1+\dots+g_{h/2}=g}} R^{(1)}_{G\setminus A,2}(g_1) \cdots R^{(1)}_{G\setminus A,2}(g_{h/2}) \\
&= \sum_{\substack{g_1,\dots,g_{h/2}\in G\\ g_1+\dots+g_{h/2}=g}} |\{(a_1, a_2) \in G \setminus A : a_1 + a_2 = g_1\}| \cdots \\
&\qquad \cdots |\{(a_{h-1}, a_h) \in G \setminus A : a_{h-1} + a_h = g_{h/2}\}| \\
&= \sum_{\substack{g_1,\dots,g_{h/2}\in G\\ g_1+\dots+g_{h/2}=g}} |\{(a_1, a_2, \dots, a_h) : a_i \in G \setminus A,\ a_1 + a_2 = g_1, \dots \\
&\qquad \dots, a_{h-1} + a_h = g_{h/2}, a_1 + \dots + a_h = g\}| \\
&= |\{(a_1, a_2, \dots, a_h) : a_i \in G \setminus A,\ a_1 + \dots + a_h = g\}| = R^{(1)}_{G\setminus A,h}(g). \qquad \square
\end{aligned}$$

Proof of Theorem 4. We prove by contradiction. Assume that for every $g \in \mathbb{Z}_m$, $R^{(1)}_{A,h}(g) = R^{(1)}_{\mathbb{Z}_m\setminus A,h}(g)$. It is easy to see that

$$\begin{aligned}
R^{(1)}_{A,h}(g) &= |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} \equiv g \bmod m\}| \\
&= \sum_{j=0}^{h-1} |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} = g + jm\}|,
\end{aligned}$$

thus we have

$$\sum_{g=0}^{m-1} R^{(1)}_{A,h}(g) z^g = \sum_{g=0}^{m-1} \left( \sum_{j=0}^{h-1} |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} = g + jm\}| \right) z^g.$$

It is clear that

$$A^h(z) = \sum_{j=0}^{h-1} \sum_{g=0}^{m-1} |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} = g + jm\}| z^{g+jm}$$

$$= \sum_{g=0}^{m-1} \sum_{j=0}^{h-1} |\{(i_1, i_2, \ldots, i_h) : a_{i_1} + \ldots + a_{i_h} = g + jm\}| z^{g+jm}.$$

It follows that there exist $p_1(z)$ and $p_2(z)$ polynomials with integral coefficients such that

$$A^h(z) = (1 - z^m)p_1(z) + \sum_{g=0}^{m-1} R^{(1)}_{A,h}(g) z^g,$$

and similarly

$$(1 + \ldots + z^{m-1} - A(z))^h = (1 - z^m)p_2(z) + \sum_{g=0}^{m-1} R^{(1)}_{\mathbb{Z}_m \setminus A,h}(g) z^g.$$

Thus we have

$$(A(z))^h - \Big(\frac{1-z^m}{1-z} - A(z)\Big)^h = (1-z^m)(p_1(z) - p_2(z)) + \sum_{g=0}^{m-1} (R^{(1)}_{A,h}(g) - R^{(1)}_{\mathbb{Z}_m \setminus A,h}(g)) z^g.$$

As $R^{(1)}_{A,h}(g) = R^{(1)}_{\mathbb{Z}_m \setminus A,h}(g)$ for every $g \in \mathbb{Z}_m$, we have

$$1 - z^m | (A(z))^h - \left(\frac{1 - z^m}{1 - z} - A(z)\right)^h.$$

Since $\frac{1-z^m}{1-z} | 1 - z^m$ and $h$ is odd, we have

$$\frac{1 - z^m}{1 - z} | 2(A(z))^h.$$

As the polynomial $\frac{1-z^m}{1-z} = 1 + \ldots + z^{m-1}$ has no multiple roots, we have

$$\frac{1 - z^m}{1 - z} | A(z),$$

which is a contradiction. □

Proof of Theorem 5. We only prove the case $i = 2$, because the proof of case $i = 3$ is very similar. For every fixed $g \in G$ we have

$$\begin{aligned}|A| &= |\{(a, y) : a \in A, y \in G,\ a + y = g\}| \\ &= |\{(a, y) : a \in A,\ y \in A, a + y = g\}| + |\{(a, y) : a \in A,\ y \in G \setminus A,\ a + y = g\}| \\ &= R_{A+A}(g) + R_{A+G\setminus A}(g) = 2R^{(2)}_{A,2}(g) - D_A(g) + R_{A+G\setminus A}(g),\end{aligned}$$

thus

$$R^{(2)}_{A,2}(g)=\frac{1}{2}|A|+\frac{1}{2}D_A(g)-\frac{1}{2}R_{A+G\setminus A}(g).$$

Replace $A$ by $G\setminus A$ and using also the fact that $R_{A+G\setminus A}(g)=R_{G\setminus A+G\setminus(G\setminus A)}(g)$ we get

$$R^{(2)}_{G\setminus A,2}(g)=\frac{1}{2}|G\setminus A|+\frac{1}{2}D_{G\setminus A}(g)-\frac{1}{2}R_{A+G\setminus A}(g).$$

Hence for every $g\in G$ we have

$$\begin{aligned}R^{(2)}_{A,2}(g)-R^{(2)}_{G\setminus A,2}(g)&=\frac{1}{2}|A|+\frac{1}{2}D_A(g)-\left(\frac{1}{2}|G\setminus A|+\frac{1}{2}D_{G\setminus A}(g)\right)\\&=|A|-\frac{1}{2}|G|+\frac{1}{2}(D_A(g)-D_{G\setminus A}(g)).\end{aligned}$$

Suppose that $R^{(2)}_{A,2}(g)=R^{(2)}_{G\setminus A,2}(g)$ for every $g\in G$. Then we have

$$\binom{|A|+1}{2}=\sum_{g\in G}R^{(2)}_{A,2}(g)=\sum_{g\in G}R^{(2)}_{G\setminus A,2}(g)=\binom{|G\setminus A|+1}{2},$$

therefore $|A|=|G\setminus A|$, that is $|A|=|G|-|A|$. Hence we get that $D_A(g)=D_{G\setminus A}(g)$ for every $g\in G$.

Finally, suppose that $D_A(g)=D_{G\setminus A}(g)$ for every $g\in G$. Then we have

$$\begin{aligned}|A|&=|\cup_{g\in G}\{a:a\in A,\ a+a=g\}|=\sum_{g\in G}|\{a:a\in A,\ a+a=g\}|\\&=\sum_{g\in G}D_A(g)=\sum_{g\in G}D_{G\setminus A}(g)=|G\setminus A|,\end{aligned}$$

therefore $|A|=|G|-|A|$, thus we get that $R^{(2)}_{A,2}(g)=R^{(2)}_{G\setminus A,2}(g)$ for every $g\in G$, which completes the proof. □

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