Publ. Math. Debrecen
85/3-4 (2014), 453–465
DOI: 10.5486/PMD.2014.6033

A note on the arithmetic properties of Stern Polynomials

By MACIEJ GAWRON (Kraków)

Abstract. We investigate the Stern polynomials defined by $B_0(t) = 0$, $B_1(t) = 1$, and for $n \geq 1$ by the recurrence relations $B_{2n}(t) = tB_n(t)$, $B_{2n+1}(t) = B_n(t) + B_{n+1}(t)$. We prove that all possible rational roots of that polynomials are $0, -1, -1/2, -1/3$. We give complete characterization of n such that $\deg(B_n) = \deg(B_{n+1})$ and $\deg(B_n) = \deg(B_{n+1}) = \deg(B_{n+2})$. Moreover, we present some results concerning reciprocal Stern polynomials.

1. Introduction

We consider the sequence of Stern polynomials $(B_n(t))_{n\geq 0}$ defined by the following formula

$$B_0(t) = 0, \quad B_1(t) = 1, \quad B_n(t) = \begin{cases} tB_{n/2}(t) & \text{for } \ n \text{ even}, \\ B_{(n+1)/2}(t) + B_{(n-1)/2}(t) & \text{for } \ n \text{ odd}. \end{cases}$$

This sequence was introduced in [2] as a polynomial analogue of the Stern [5] sequence $(s_n)_{n\in\mathbb{N}}$, where $s_n = B_n(1)$. Arithmetic properties of the Stern polynomials and the sequence of degrees of Stern polynomials $e(n) = \deg B_n(t)$ were considered in [6], [7]. Reducibility properties of Stern polynomials were considered in [4]. The aim of this paper is to resolve some open problems and conjectures given in [6].

In Section 2 we prove that the only possible rational roots of the Stern polynomials are in the set $\left\{0, -1, -\frac{1}{2}, -\frac{1}{3}\right\}$. We prove that each of those numbers

---

Mathematics Subject Classification: 11B83.
Key words and phrases: Stern polynomials, Stern diatomic sequence, Automatic Sequences.
This research was supported by Grant UMO-2012/07/E/ST1/00185.

is a root of infinitely many Stern polynomials. We also give some characterization of n such that $B_n(t)$ has one of these roots.

It was proven in [6] that there are no four consecutive Stern polynomials with equal degrees. In Section 3 we characterize the set of n for which $e(n) = e(n+1)$, i.e. those n for which two consecutive Stern polynomials has the same degree. We also characterize the set of n for which $e(n) = e(n+1) = e(n+2)$.

In Section 4 we investigate reciprocal Stern polynomials, i.e. polynomials such that $B_n(t) = t^{e(n)} B_n\left(\frac{1}{t}\right)$. It was observed in [6] that $B_n(t)$ is reciprocal for each n of the form $2^m - 1$ or $2^m - 5$. We give two infinite sequences $(u_n), (v_n)$ such that $B_n(t)$ is reciprocal for each n of the forms $2^m - u_n$, and $2^m - v_n$.

For $n \in \mathbb{N}$ we denote by $\mathrm{bin}(n)$ the sequence of its binary digits without leading zeros. Let $w = (w_0, w_1, \ldots, w_n)$ be a finite sequence of zeros and ones, we denote by $\overline{w} = \overline{w_0 w_1 \ldots w_n}$ the number $w_0 \cdot 2^n + w_1 \cdot 2^{n-1} + \cdots + w_n$. For a finite sequence w of zeros and ones, we denote by w^k concatenation of k copies of w. For a set $A \subset \mathbb{Z}$ and an integer n, we denote by $A + n$ the set $\{a + n \mid a \in A\}$. For a subset $A \subset \mathbb{N}$ of non-negative integers we say that it has lower density δ if $\liminf_{n\to\infty} \frac{\#(A\cap[1,n])}{n} = \delta$. We say that it has upper density δ if $\limsup_{n\to\infty} \frac{\#(A\cap[1,n])}{n} = \delta$. If lower and upper density are both equal δ then we say that our set has density δ. The symbol log means always binary logarithm. We use standard Landau symbols.

2. Rational roots of Stern polynomials

We prove the following theorem, which can be found as a conjecture in [6, Conjecture 6.1].

Theorem 2.1. *If $a \in \mathbb{Q}$ and there exists a positive integer n such that $B_n(a) = 0$ then $a \in \left\{0, -1, -\frac{1}{2}, -\frac{1}{3}\right\}$.*

Proof. Let $k \geq 4$ be an integer. We define a sequence $\{b_n\}_{n=0}^{\infty}$ by the formula $b_n = B_n\left(-\frac{1}{k}\right)$. We prove that the following inequality holds

$$b_{2n+1} > \frac{1}{2} \max\{|b_n|, |b_{n+1}|\} > 0 \tag{1}$$

for every integer $n \geq 0$. In order to prove (1) we use induction. For $n = 0$ we get $1 = b_1 > \frac{1}{2} \max\{|b_0|, |b_1|\} = \frac{1}{2} > 0$. Suppose that the statement is true for all $m < n$. Let us consider two cases $n = 2s + 1$ and $n = 2s$.

If $n = 2s$ then we have

$$\begin{aligned} b_{4s+1} = b_{2s} + b_{2s+1} = -\frac{1}{k}b_s + b_{2s+1} \geq b_{2s+1} - \frac{1}{4}|b_s| > b_{2s+1} \\ - \frac{1}{2}b_{2s+1} = \frac{1}{2}b_{2s+1}. \end{aligned} \tag{2}$$

Moreover, $b_{2s+1} \geq \frac{1}{2}|b_s|$ and thus $\max\{|b_{2s}|, |b_{2s+1}|\} = \max\left\{\frac{1}{k}|b_s|, |b_{2s+1}|\right\} = b_{2s+1} > 0$. So in this case our inequality is proved.

If $n = 2s + 1$ then we have

$$\begin{aligned} b_{4s+3} = b_{2s+1} + b_{2s+2} = b_{2s+1} - \frac{1}{k}b_{s+1} \geq b_{2s+1} - \frac{1}{4}|b_{s+1}| > b_{2s+1} \\ - \frac{1}{2}b_{2s+1} = \frac{1}{2}b_{2s+1}. \end{aligned} \tag{3}$$

Moreover, $b_{2s+1} \geq \frac{1}{2}|b_{s+1}|$ and thus

$$\max\{|b_{2s+1}|, |b_{2s+2}|\} = \max\left\{\frac{1}{k}|b_{s+1}|, |b_{2s+1}|\right\} = b_{2s+1} > 0.$$

So in this case our inequality is also proved.

Now let $a \in \mathbb{Q}$ be a root of the Stern polynomial $B_n(t)$. If $a \neq 0$ then we can assume that n is odd. Moreover $B_{2s+1}(0) = 1$ for each integers $s \geq 0$, so a is of the form $\pm\frac{1}{k}$ for some integer k. But all coefficients of Stern polynomials are non-negative, so $a = -\frac{1}{k}$. Now our theorem follows from inequality (1) . □

Now for each $a \in \left\{0, -1, -\frac{1}{2}, -\frac{1}{3}\right\}$ we want to characterize those n for which $B_n(a) = 0$. Let us denote $R_a = \{n \in \mathbb{N} \mid B_n(a) = 0\}$. One can check that $B_n(0) = n \pmod 2$ and $B_n(-1) = (n+1) \pmod 3 - 1$, so in this case our problem is easy and we get $R_0 = 2\mathbb{N}$ and $R_{-1} = 3\mathbb{N}$. The problem is much more complicated when $a \in \left\{-\frac{1}{2}, -\frac{1}{3}\right\}$.

Theorem 2.2. *The sets $R_{-\frac{1}{2}}, R_{-\frac{1}{3}}$ satisfies the following properties*

(a) $2R_{-\frac{1}{2}} \subset R_{-\frac{1}{2}}, 2R_{-\frac{1}{3}} \subset R_{-\frac{1}{3}}$,

(b) $R_{-\frac{1}{2}} \subset 5\mathbb{N}$, $R_{-\frac{1}{3}} \subset 21\mathbb{N}$,

(c) *If $5n \in R_{-\frac{1}{2}}$, and $2^k > 5n$ then $5(2^k \pm n) \in R_{-\frac{1}{2}}$.*

(d) *If $21n \in R_{-\frac{1}{3}}$, and $2^k > 21n$ then $21(2^k \pm n) \in R_{-\frac{1}{3}}$.*

Proof. Part (a) is obvious. We can look only for odd elements of $R_{-\frac{1}{2}}$, $R_{-\frac{1}{3}}$. Looking at the reductions modulo 5 we get $B_n\left(-\frac{1}{2}\right) \equiv B_n(2) = n \pmod 5$ so $R_{-\frac{1}{2}} \subset 5\mathbb{N}$. By looking modulo 7 we get $B_n\left(-\frac{1}{3}\right) \equiv B_n(2) = n \pmod 7$. By looking modulo 2 we get $B_n\left(-\frac{1}{3}\right) \equiv B_n(1) \pmod 2$. One can easily prove that $B_n(1)$ is even if and only if n is divisible by 3, so $R_{-\frac{1}{3}} \subset 21\mathbb{N}$. Let us prove part (c). We prove by induction on n that $B_{5\cdot 2^k \pm n}\left(-\frac{1}{2}\right) = \mp\frac{1}{4}B_n\left(-\frac{1}{2}\right)$, when $2^k \geq n$. For $n = 1$ then we have to prove $B_{5\cdot 2^k \pm 1}\left(-\frac{1}{2}\right) = \mp\frac{1}{4}$, which can be easily proven by induction on k. Now if $n = 2s$ then

$$B_{5\cdot 2^k \pm 2s}\left(-\frac{1}{2}\right) = -\frac{1}{2}B_{5\cdot 2^{k-1}\pm s}\left(-\frac{1}{2}\right) = -\frac{1}{2}\left(\mp\frac{1}{4}B_s\left(-\frac{1}{2}\right)\right) = \mp\frac{1}{4}B_{2s}\left(-\frac{1}{2}\right).$$

If $n = 2s+1$ then

$$\begin{aligned} B_{5\cdot 2^k \pm (2s+1)}\left(-\frac{1}{2}\right) &= B_{5\cdot 2^{k-1}\pm s}\left(-\frac{1}{2}\right) + B_{5\cdot 2^{k-1}\pm (s+1)}\left(-\frac{1}{2}\right) \\ &= \mp\frac{1}{4}\left(B_s\left(-\frac{1}{2}\right) + B_{s+1}\left(-\frac{1}{2}\right)\right) = \mp\frac{1}{4}B_{2s+1}\left(-\frac{1}{2}\right). \end{aligned}$$

Which completes the induction step, and part (c) follows. In the same manner one can prove that for $2^k > n$ we have $B_{21\cdot 2^k \pm n}\left(-\frac{1}{3}\right) = \mp\frac{1}{27}B_n\left(-\frac{1}{3}\right)$ which implies part (d). $\square$

Using previous theorem we can check that no other reductions of our sequences are eventually periodic.

Corollary 2.3. (a) *For each prime number $p \neq 2, 5$ the sequence $\left(B_n\left(-\frac{1}{2}\right) \pmod p\right)_{n\geq 0}$ is not eventually periodic.*

(b) *For each prime number $p \neq 2, 3, 7$ the sequence $\left(B_n\left(-\frac{1}{3}\right) \pmod p\right)_{n\geq 0}$ is not eventually periodic.*

Proof. Let us prove part (a). Suppose that for $n > N$ we have $B_n\left(-\frac{1}{2}\right) \equiv B_{n+2^u(2v+1)}\left(-\frac{1}{2}\right) \pmod p$. Let us take n such that $B_{5n}\left(-\frac{1}{2}\right) \equiv 0 \pmod p$. We choose k such that $2^k - 1 = (2v+1)s$ for some integer s, and $2^{k+u} > 5\cdot 2^u n$. From previous theorem we have that $B_{5(2^u n + 2^{k+u})}\left(-\frac{1}{2}\right) \equiv 0 \pmod p$, so $0 \equiv B_{5(2^u n+2^{k+u})-5(2^u(2v+1)s)}\left(-\frac{1}{2}\right) \equiv B_{5(2^u n + 2^u)}\left(-\frac{1}{2}\right) \equiv B_{5(n+1)}\left(-\frac{1}{2}\right) \pmod p$, hence all numbers M divisible by 5 greater than N satisfy $B_M\left(-\frac{1}{2}\right) \equiv 0 \pmod p$. But one can check that $B_{15+5\cdot 2^n}\left(-\frac{1}{2}\right) = -\frac{5}{32}$ for $n > 3$ and this number is not 0 $\pmod p$. We get a contradiction. Proof of part (b) can be done in the same manner. $\square$

Using reductions modulo prime numbers we prove that the sets $R_{-\frac{1}{2}}$, $R_{-\frac{1}{3}}$ have lower density 0. Before we stete the next result we recall definiotion of a 2-automatic sequence [1],

Definition 2.4. *A* 2-automaton *consists of*

- *a finite set $S = \{s_1 = I, s_2, \ldots, s_n\}$, which is called the set of states. One of the states is denoted by I and called the initial state,*
- *two maps from S to itself F_0, F_1,*
- *a map (the output function), say π from S to a set Y.*

The automaton generates a sequence $(u_n)_{n\geq 0}$ with values in Y (called a 2-automatic sequence*) as follows: to compute the term u_n one expands n in base 2, say $n = \sum_{j=0}^{k} n_j 2^j$, then u_n is defined by $u_n = \pi(F_{n_k}(F_{n_{k-1}}(\ldots(F_{n_0}(I))\ldots)))$.*

Now we are ready to prove the following

Theorem 2.5. *Let $p > 3$ be a prime number, t be a fixed constant from the set $\{-\frac{1}{2}, -\frac{1}{3}\}$, and $d_{n,p} = \frac{\#\{0\leq k<n \mid B_k(t)\equiv 0 \pmod{p}\}}{n}$. We have that*

$$\lim_{n\to\infty} \frac{d_{2^0,p} + d_{2^1,p} + \cdots + d_{2^n,p}}{n+1} \leq \frac{2}{\log p}.$$

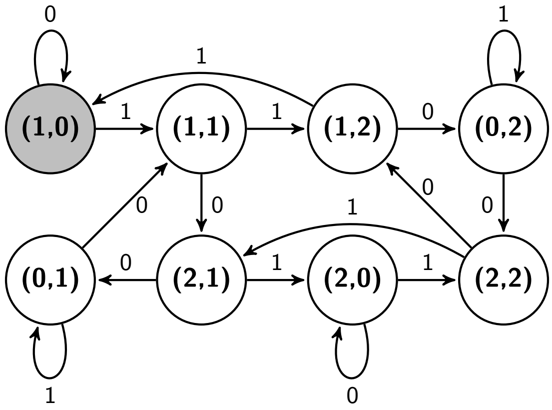
Proof. We denote by b_n the sequence $B_n(t) \pmod{p}$. Let us consider two operators $E_0, E_1 : (\mathbb{N} \to \mathbb{F}_p) \to (\mathbb{N} \to \mathbb{F}_p)$ defined by formulas $E_0((a_n)_{n\geq 0}) = (a_{2n})_{n\geq 0}$, and $E_1((a_n)_{n\geq 0}) = (a_{2n+1})_{n\geq 0}$. Let us observe that the set $\{(\alpha b_n + \beta b_{n+1})_{n\geq 0} \mid \alpha, \beta \in \mathbb{F}_p\}$ is closed under action of E_0, E_1 because

$$E_0((\alpha b_n + \beta b_{n+1})) = (\alpha b_{2n} + \beta b_{2n+1}) = ((t\alpha + \beta) b_n + \beta b_{n+1})$$

and

$$E_1((\alpha b_n + \beta b_{n+1})) = (\alpha b_{2n+1} + \beta b_{2n+2}) = (\alpha b_n + (\alpha + t\beta) b_{n+1}).$$

Therefore (b_n) has a finite orbit under actions of E_0, E_1 so our sequence is 2-automatic. We construct automaton with states labeled with (α, β) for each $\alpha, \beta \in \mathbb{F}_p$, and edges $(\alpha, \beta) \xrightarrow{0} (t\alpha + \beta, \beta)$ and $(\alpha, \beta) \xrightarrow{1} (\alpha, t\beta + \alpha)$. The initial state is $(1, 0)$. For each state (α, β) we define its projection $\pi(\alpha, \beta) = \beta$. One can observe that if $\mathrm{bin}(k) = w_0 w_1 \ldots w_s$ then $b_k = ((E_{w_0} \circ E_{w_1} \circ \cdots \circ E_{w_s})((b_n)))_0$, so to compute b_k we start at initial state then go through the path $w_s w_{s-1} \ldots w_0$ and finally use projection function.



Automaton for $t = -\frac{1}{2}$ and $p = 3$.

We call a state reachable if it is reachable from initial state. Let us observe that the state $(0,1)$ is reachable. We have

$$(1,0) \xrightarrow{1} (1,1) \xrightarrow{0} (t+1,1) \xrightarrow{0} (t^2+t+1,1) \xrightarrow{0} \dots \xrightarrow{0} (t^{p-2}+\dots+t+1,1) = (0,1).$$

Now we can see that the state (α,β) is reachable if and only if the state (β,α) is reachable (just go to $(0,1)$ and then use opposite arrows) . Now observe that if for some $u \in \mathbb{F}_p^*$ the state $(u,0)$ is reachable then the following different states are also reachable

$$\begin{aligned}(0,u) \xrightarrow{0} (u,u) \xrightarrow{0} (u(t+1),u) \xrightarrow{0} \\ \xrightarrow{0} (u(t^2+t+1),u)) \xrightarrow{0} \dots \xrightarrow{0} (u(t^{\operatorname{ord}(t)-1}+\dots+t+1),u)\end{aligned}$$

Where $\operatorname{ord}(t)$ is the order of t in group $\mathbb{F}_p^*$. It is not hard to see that $\operatorname{ord}(t) \geq \log_3 p \geq \frac{1}{2}\log p$, whence at most $\frac{2K}{\log p}$ states are of the form $(u,0)$. Here K denotes the number of those states which are reachable from the initial state. Moreover from each reachable state there is a path to initial state. Indeed, from state (α,β) we can go to $\left(\begin{smallmatrix} t & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ or to $\left(\begin{smallmatrix} 1 & 0 \\ 1 & t \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$, those matrices has finite order in $GL_2(\mathbb{F}_p)$ so we can go backwards.

Let M be adjacency matrix of strongly connected component of our automaton as a directed graph containing initial vertex. Let $A = \frac{1}{2}M$. Using a straightforward application of Frobenius–Perron theorem (see for example [3, Example 8.3.2 , p. 677]) we get that there is a limit

$$\lim_{k\to\infty} \frac{I + A + A^2 + \dots + A^k}{k+1} = G. \tag{4}$$

Of course $AG = GA = G$. As every vertex of our strongly connected component has inner and outer degree 2, one can observe that each element of the matrix G, namely $g_{i,j}$, is an arithmetic mean of g_{i,j_0} and g_{i,j_1} where $j \xrightarrow{0} j_0$ and $j \xrightarrow{1} j_1$, and on the other hand an arithmetic mean of $g_{i_0,j}$ and $g_{i_1,j}$ where $i_0 \xrightarrow{0} i$ and $i_1 \xrightarrow{1} i$. Hence using strongly connectedness we get that

$$G = \frac{1}{K}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

where K is the cardinality of our component.

Number $d_{2^n,p}$ is the number of paths of length n from initial state to one of states of the form $(u, 0)$ divided by 2^n. We know that $A^n_{s_1,s_2}$ is the number of paths of length n from s_1 to s_2, divided by 2^n. Therefore from equation (4) we get that

$$\lim_{n\to\infty} \frac{d_{0,p} + d_{2^0,p} + d_{2^1,p} + \cdots + d_{2^n,p}}{n+1} = \frac{\#\{\text{ reachable states of the form } (u,0)\}}{K} \leq \frac{2}{\log p}.$$

Our theorem is proved. □

The above result implies the following

Corollary 2.6. *The sets* $R_{-\frac{1}{2}}, R_{-\frac{1}{3}}$ *has lower density* 0.

Proof. Let t be a fixed constant from the set $\{-\frac{1}{2}, -\frac{1}{3}\}$. Let

$$d_n = \frac{\#\{0 \leq k < n \mid B_k(t) = 0\}}{n}.$$

Of course $d_{n,p} \geq d_n$ for each p. From our lemma we get

$$\limsup_{n\to\infty} \frac{d_{2^0} + d_{2^1} + \cdots + d_{2^n}}{n+1} \leq \frac{2}{\log p},$$

therefore

$$\lim_{n\to\infty} \frac{d_{2^0} + d_{2^1} + \cdots + d_{2^n}}{n+1} = 0,$$

and finally

$$\liminf_{n\to\infty} d_n = 0.$$ □

We expect that $R_{-\frac{1}{2}}, R_{-\frac{1}{3}}$ has upper density 0.

Conjecture 2.7. *The sets* $R_{-\frac{1}{2}}, R_{-\frac{1}{3}}$ *has density* 0.

3. Consecutive polynomials with equal degrees

In this section we will characterize those n for which $e(n) = e(n+1)$. Resolving conjecture posted in [6, Conjecture 6.3]. We also characterize those n for which $e(n) = e(n+1) = e(n+2)$. We recall the recurrence satisfied be the sequence $(e_n)_{n\in\mathbb{N}}$ of degrees of Stern Polynomials

$$\begin{cases} e(0) = 1 \\ e(2n) = e(n) + 1 \\ e(4n+1) = e(n) + 1 \\ e(4n+3) = e(n+1) + 1. \end{cases}$$

Let us the define sequences $(p_n)_{n\mathbb{N}}, (q_n)_{n\mathbb{N}}$ as follows

$$p_n = \frac{4^n - 1}{3}, \quad q_n = \frac{5 \cdot 4^n - 2}{3}, \quad n \geq 0.$$

Theorem 3.1. *We have the following equality*

$$\begin{aligned} \{n \mid e(n) = e(n+1)\} = {} & \{2^{2k+1}n + p_k \mid k, n \in \mathbb{N}_+\} \\ & \cup \{2p_k \mid k \in \mathbb{N}_+\} \cup \{2^{2k+1}n + q_k \mid k, n \in \mathbb{N},\ k > 0\}. \end{aligned}$$

Proof. We will use binary representations of integers. We write $\mathrm{bin}(m)$ for sequence of digits of m in base two. Let us observe that $\mathrm{bin}(p_k) = 1(01)^{k-1}$, and $\mathrm{bin}(q_k) = 1(10)^k$ for each $k > 0$. We consider eight cases depending on the binary expansion of n.

(i) If $n = 4k$ then we have

$$e(n) = e(4k) = e(2k) + 1 = e(k) + 2$$

and

$$e(n+1) = e(4k+1) = e(k) + 1.$$

In particular $e(n) \neq e(n+1)$ in this case.

(ii) If $n = 4k+3$ then we have

$$e(n) = e(4k+3) = e(k+1) + 1$$

and

$$e(n+1) = e(4k+4) = e(k+1) + 2.$$

We thus get $e(n) \neq e(n+1)$.

(iii) If $n = \overline{\mathrm{bin}(m)0(01)^k}$, where $m, k \in \mathbb{N}_+$ then we have

$$e(n) = e(\overline{\mathrm{bin}(m)0(01)^k}) = e(\overline{\mathrm{bin}(m)0}) + k = e(m) + k + 1$$

and

$$\begin{aligned} e(n+1) &= e(\overline{\mathrm{bin}(m)0(01)^{k-1}10}) = e(\overline{\mathrm{bin}(m)00(10)^{k-2}11}) + 1 \\ &= e(\overline{\mathrm{bin}(m)00(10)^{k-3}11}) + 2 = \cdots = e(\overline{\mathrm{bin}(m)01}) + k = e(m) + k + 1. \end{aligned}$$

Our computation implies $e(n) = e(n+1)$.

(iv) If $n = \overline{\mathrm{bin}(m)11(01)^k}$, where $k, m \in \mathbb{N}$, $k > 0$, then we have

$$e(n) = e(\overline{\mathrm{bin}(m)11(01)^k}) = k + e(\overline{\mathrm{bin}(m)11}) = k + 1 + e(\overline{\mathrm{bin}(m+1)})$$

and

$$\begin{aligned} e(n+1) &= e(\overline{\mathrm{bin}(m)11(01)^{k-1}10}) \\ &= e(\overline{\mathrm{bin}(m)1(10)^{k-1}11}) + 1 = e(\overline{\mathrm{bin}(m)1(10)^{k-2}11}) + 2 \\ &= e(\overline{\mathrm{bin}(m)111}) + k = e(\overline{\mathrm{bin}(m+1)0}) + k + 1 = e(\overline{\mathrm{bin}(m+1)}) + k + 2. \end{aligned}$$

We get $e(n) \neq e(n+1)$ in this case.

(v) If $n = \overline{(01)^k}$, where $k \in \mathbb{N}_+$, then we have

$$e(n) = e(\overline{(01)^k}) = e(\overline{1}) + k - 1 = k - 1$$

and

$$\begin{aligned} e(n+1) &= e(\overline{(01)^{k-1}10}) = e(\overline{(01)^{k-1}1}) + 1 = e(\overline{(01)^{k-2}011}) + 1 \\ &= e(\overline{(01)^{k-2}1}) + 2 = e(\overline{11}) + k - 1 = k, \end{aligned}$$

and thus $e(n) \neq e(n+1)$.

(vi) If $n = \overline{\mathrm{bin}(m)00(10)^k}$, where $m, k \in \mathbb{N}_+$, then we have

$$e(n) = e(\overline{\mathrm{bin}(m)00(10)^k}) = k + 2 + e(\overline{\mathrm{bin}(m)})$$

and

$$\begin{aligned} e(n+1) &= e(\overline{\mathrm{bin}(m)0(01)^k1}) = e(\overline{\mathrm{bin}(m)0(01)^{k-1}1}) + 1 \\ &= e(\overline{\mathrm{bin}(m)01}) + k = e(\overline{\mathrm{bin}(m)}) + k + 1. \end{aligned}$$

Similarly as in the previous case we get $e(n) \neq e(n+1)$.

(vii) If $n = \overline{(10)^k}$ where $k \in \mathbb{N}_+$, then we have

$$e(n) = e(\overline{(10)^k}) = e(\overline{(01)^k}) + 1 = e(1) + k = k$$

and

$$e(n+1) = e(\overline{(10)^{k-1}11}) = e(\overline{(10)^{k-2}11}) + 1 = e(\overline{11}) + k - 1 = k,$$

and thus $e(n) = e(n+1)$.

(viii) If $n = \overline{\mathrm{bin}(m)1(10)^k}$ where $m, k \in \mathbb{N}$, $k > 0$, then we have

$$e(n) = e(\overline{\mathrm{bin}(m)1(10)^k}) = e(\overline{\mathrm{bin}(m)11(01)^{k-1}}) + 1 = e(\overline{\mathrm{bin}(m+1)}) + k + 1$$

and

$$\begin{aligned} e(n+1) &= e(\overline{\mathrm{bin}(m)1(10)^{k-1}11}) = e(\overline{\mathrm{bin}(m)1(10)^{k-2}11}) + 1 \\ &= e(\overline{\mathrm{bin}(m)111}) + k - 1 = e(\overline{\mathrm{bin}(m+1)}) + k + 1. \end{aligned}$$

and thus $e(n) = e(n+1)$.

All numbers of the form $4k+1$ can be written uniquely in one of the forms (iii), (iv), (v), and all numbers of the form $4k+2$ can be can be written uniquely in one of the forms (vi), (vii), (viii). Therefore all cases were considered. In cases (iii), (vii), (viii) we have that $e(n) = e(n+1)$, and one can observe that the numbers in this cases are exactly the elements of the set $\{2^{2k+1}n + p_k \mid k, n \in \mathbb{N}_+\} \cup \{2p_k \mid k \in \mathbb{N}_+\} \cup \{2^{2k+1}n + q_k \mid k, n \in \mathbb{N},\ k > 0\}$. So our theorem is proved. $\square$

In the next theorem we characterize the set of n for which $e(n) = e(n+1) = e(n+2)$.

Theorem 3.2. *We have the following equality*

$$\begin{aligned} \{n \mid e(n) = e(n+1) = e(n+2)\} = \{&2^{2k+1}n + p_k \mid k, n \in \mathbb{N}_+,\ k \geq 2\} \\ &\cup \{2p_k - 1 \mid k \geq 2\} \cup \{2^{2k+1}n + q_k - 1 \mid k, n \in \mathbb{N},\ k \geq 2\}. \end{aligned}$$

Proof. Let $A = \{2^{2k+1}n + p_k \mid k, n \in \mathbb{N}_+\}$, $B = \{2p_k \mid k \in \mathbb{N}_+\}$, and $C = \{2^{2k+1}n + q_k \mid k, n \in \mathbb{N},\ k > 0\}$. We want to compute

$$((A \cup B \cup C) \cap ((A \cup B \cup C) + 1)) - 1.$$

But as every element of A equals $1 \pmod 4$ and every element of $B \cup C$ equals $2 \pmod 4$, it is enough to compute $((A+1) \cap B) \cup ((A+1) \cap C)$. First let us compute $(A+1) \cap B$, we have

$$2^{2k+1}n + p_k + 1 = 2p_l, \quad \text{for } n, k, l \geq 1.$$

By a simple computations we get

$$3 \cdot 2^{2k+1} n + 4^k = 2 \cdot 4^l - 4.$$

Looking modulo 4 we get that $k = 1$ or $l = 1$. When $k = 1$ then our equality became $12n + 4 = 4^l$. For each l we have exactly one n such that this equality holds, but as $n \geq 2$ we have $l \geq 2$. If $l = 1$ then our equality became $2^{2k+1}n + p_k + 1 = 2$ which contradicts $n, k \geq 1$. Summarizing $(A+1) \cap B = \{2p_l - 1 \mid l \geq 2\}$. Let us compute $(A+1) \cap C$. We have

$$2^{2k+1}n + p_k + 1 = 2^{2l+1}m + q_l, \qquad \text{for } n, k, l \geq 1,\ m \geq 0.$$

By a simple computations we get

$$2^{2k+1} \cdot 3n + 4^k = 2^{2l+1} \cdot 3m + 5 \cdot 4^l - 4.$$

Looking $\pmod 8$ we get that $k = 1$ or $l = 1$. In the first case we get $8n + 2 = 2^{2l+1}m + q_l$. For each l, m such that $l \geq 2$ we have exactly one n such that this equality holds. So we get $\{2^{2l+1}m + q_l \mid m,\ l \in \mathbb{N},\ l \geq 2\}$. In the second case we get $\{2^{2k+1}n + p_k + 1 \mid k, n \in \mathbb{N}_+,\ k \geq 2\}$ in the same way. Summarizing we get $(A+1) \cap C = \{2^{2l+1}m + q_l \mid m, l \in \mathbb{N},\ l \geq 2\} \cup \{2^{2k+1}n + p_k + 1 \mid k, n \in \mathbb{N}_+,\ k \geq 2\}$. And our result follows. $\square$

4. Reciprocal Stern polynomials

Let us recall that the polynomial $B_n(t)$ is called reciprocal if it satisfies equality $B_n(t) = t^{e(n)} B_n\left(\frac{1}{t}\right)$. It was observed in [6] that for each n polynomials $B_{2^n-1}(t)$ and $B_{2^n-5}(t)$ are reciprocal. We define sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ as follows

$$\begin{cases} u_0 = 1 & u_n = 2^{8n-2}u_{n-1} + 2^{4n} - 1, \\ v_0 = 5 & v_n = 2^{8n+2}v_{n-1} - 2^{4n+2} + 1, \end{cases}$$

and show that whenever $2^k > u_n$ (or $2^k > v_n$) then $B_{2^k-u_n}$ (or $B_{2^k-v_n}$) is reciprocal.

Theorem 4.1. *For each integer $n \geq 0$ all polynomials in the sequences $(B_{2^k-v_n})_{k \geq 4n^2+6n+3}$, and $(B_{2^k-u_n})_{k \geq 4n^2+2n+1}$ are reciprocal.*

Proof. Let us observe that $\operatorname{bin}(u_n) = 10^2 1^4 0^6 \ldots 0^{4n-2} 1^{4n}$ and $\operatorname{bin}(v_n) = 10^2 1^4 \ldots 1^{4n} 0^{4n+1} 1$. Using recursive formula for $e(n)$, binary representations and induction we will get that $e(2^k - u_n) = k - 2n - 1$ and $e(2^k - v_n) = k - 2n - 2$. Let us prove our theorem. We proceed by induction. For $u_0 = 1$ we have $B_{2^k-1} = t^{k-1} + t^{k-2} + \cdots + t + 1$ and our conditions are satisfied. For $v_0 = 5$ we have

$$\begin{aligned} B_{2^k-5}(t) &= (1+t)B_{2^{k-2}-1}(t) + tB_{2^{k-3}-1}(t) \\ &= (1+t)t^{k-3}B_{2^{k-2}-1}(t^{-1}) + t^{k-3}B_{2^{k-3}-1}(t^{-1}) \\ &= t^{k-2}\left((1+t^{-1})B_{2^{k-2}-1}(t^{-1}) + t^{-1}B_{2^{k-3}-1}(t^{-1})\right) = t^{k-2}B_{2^k-5}(t^{-1}). \end{aligned}$$

Let us compute

$$\begin{aligned} B_{2^k-u_n}(t) &= B_{2^k-\overline{10^21^40^6\ldots0^{4n-2}1^{4n}}}(t) \\ &= B_{2^{k-1}-\overline{10^21^40^6\ldots0^{4n-2}1^{4n-1}}}(t) + B_{2^{k-1}-\overline{10^21^40^6\ldots0^{4n-3}10^{4n-1}}}(t) \\ &= B_{2^{k-1}-\overline{10^21^40^6\ldots0^{4n-2}1^{4n-1}}}(t) + t^{4n-1}B_{2^{k-4n}-\overline{10^21^40^6\ldots0^{4n-3}1}}(t) \\ &= B_{2^{k-1}-\overline{10^21^40^6\ldots0^{4n-2}1^{4n-1}}}(t) + t^{4n-1}B_{2^{k-4n}-v_{n-1}}(t) \\ &= \ldots \\ &= B_{2^{k-4n}-\overline{10^21^40^6\ldots0^{4n-2}}}(t) + (t^{4n-1} + \cdots + t + 1)B_{2^{k-4n}-v_{n-1}}(t) \\ &= t^{4n-2}B_{2^{k-8n+2}-u_{n-1}}(t) + (t^{4n-1} + \cdots + t + 1)B_{2^{k-4n}-v_{n-1}}(t) \\ &= (*) \end{aligned}$$

Using induction hypothesis we get

$$\begin{aligned} (*) &= t^{k-6n+1}B_{2^{k-8n+2}-u_{n-1}}(t^{-1}) + (t^{4n-1} + \cdots + t + 1)t^{k-6n}B_{2^{k-4n}-v_{n-1}}(t^{-1}) \\ &= t^{k-2n-1}\left(t^{2-4n}B_{2^{k-8n+2}-u_{n-1}}(t^{-1}) + \left(\frac{t^{4n}-1}{t-1}\right)t^{1-4n}B_{2^{k-4n}-v_{n-1}}(t^{-1})\right) \\ &= t^{k-2n-1}B_{2^k-u_n}(t^{-1}). \end{aligned}$$

So $B_{2^k-u_n}$ is reciprocal. In the same way one can prove that $B_{2^k-v_n}$ is reciprocal. We omit the details. □

Corollary 4.2. *Let* $\mathrm{Rec} = \{n \mid B_n(t) = t^{e(n)}B_n(t^{-1})\}$. *We have that* $\#(\mathrm{Rec} \cap [1,n]) = \Omega(\log(n)^{3/2})$.

Proof. Let $n > 2^k$ we have that

$$\#(\mathrm{Rec} \cap [1,n]) \geq \#\{(i,m) \mid u_m < 2^i,\ i < k\} = \sum_{u_m < 2^k}(k - 1 - \lfloor\log(u_m)\rfloor)$$

$$\sum_{4m^2+2m+1<k}(k - (4m^2+2m)) = \frac{1}{2}k\sqrt{k} - \frac{4}{3}\left(\frac{\sqrt{k}}{2}\right)^3 + o(k\sqrt{k}) = \theta(k\sqrt{k}),$$

and our result follows. □

We expect the following conjecture about density of the set Rec

Conjecture 4.3. *The function* $f(n) = \#(\mathrm{Rec} \cap [1, n])$ *is* $O(\log(n)^k)$ *for some constant* k.

References

[1] J-P. Allouche and J. Shallit, Automatic Sequences, *Cambridge University Press*, 2003.

[2] S. Klavžar, U. Milutinović and C. Petr, Stern polynomials, *Adv. Appl. Math.* **39** (2007), 86–95.

[3] C. Meyer, Matrix analysis and applied linear algebra, *Society for Industrial and Applied Mathematics*, 2000.

[4] A. Schinzel, On the factors of Stern polynomials (Remarks on the preceding paper of M. Ulas), *Publ. Math. Debrecen* **79** (2011), 83–88.

[5] M. A. Stern, Ueber eine zahlentheoretische Funktion, Vol. 55, *J. Reine Angew. Math.*, 1858, 193–220.

[6] M. Ulas, On certain arithmetic properties of Stern polynomials, Vol. 79, *Publ. Math. Debrecen*, 2011, 55–81.

[7] M. Ulas, Arithmetic properties of the sequence of degrees of Stern polynomials and related results, Vol. 8, *Int. J. Number Theory*, 2012, 669–687.

MACIEJ GAWRON
JAGIELLONIAN UNIVERSITY
INSTITUTE OF MATHEMATICS
LOJASIEWICZA 6
30-348 KRAKÓW
POLAND

E-mail: maciej.gawron@uj.edu.pl

(Received December 10, 2013; revised March 20, 2014)