Publ. Math. Debrecen
85/3-4 (2014), 467–480
DOI: 10.5486/PMD.2014.6073

Additive local invertibility preservers

By M. BENDAOUD (Meknès), M. JABBAR (Meknès) and M. SARIH (Meknès)

Abstract. Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on a complex Banach space X, and for a nonzero vector $x \in X$ and $T \in \mathcal{L}(X)$, let $\sigma_T(x)$ denote the local spectrum of T at x. We characterize additive surjective maps ϕ on $\mathcal{L}(X)$ which satisfy $0 \in \sigma_{\phi(T)}(x)$ if and only if $0 \in \sigma_{\phi(T)}(x)$ for every $x \in X$ and $T \in \mathcal{L}(X)$. Extensions of this result to the case of different Banach spaces are also established. As application, additive maps from $\mathcal{L}(X)$ onto itself that preserve the inner local spectral radius zero of operators are classified.

1. Introduction and statement of the main results

Throughout this paper, X and Y will denote complex Banach spaces and $\mathcal{L}(X, Y)$ will denote the space of all bounded linear operators from X into Y. As usual, when $X = Y$ we simply write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. The local resolvent set of an operator $T \in \mathcal{L}(X)$ at a vector $x \in X$, $\rho_T(x)$, is the set of all λ in the complex field $\mathbb{C}$ for which there exists an open neighborhood U_λ of λ in $\mathbb{C}$ and an X-valued analytic function $f : U_\lambda \to X$ such that $(\mu - T)f(\mu) = x$ for all $\mu \in U_\lambda$. Its complement in $\mathbb{C}$, denoted by $\sigma_T(x)$, is called the local spectrum of T at x, and is a compact (possibly empty) subset of the usual spectrum $\sigma(T)$

Mathematics Subject Classification: 47B49, 47A10, 47A11.
Key words and phrases: preserver problems, local spectrum, inner local spectral radius, single-valued extension property.
The authors thank very much Professor L. Molnár for his valuable comments and suggestions. The first author also thank the invitation and the hospitality of the organizers of "The 2014 Canadian Mathematical Society (CMS) Summer Meeting, University of Manitoba, Winnipeg, Canada (June 6–9, 2014)", where the main result of this paper was announced. The research of the first and the third authors was partially supported by grants from ARS-MIU projects and the Moulay Ismail University Scientific Research Actions.

of T. The inner local spectral radius of T at x, $\iota_T(x)$, is defined by

$$\iota_T(x) := \sup\{\varepsilon \geq 0 : x \in \mathcal{X}_T(\mathbb{C} \setminus D(0, \varepsilon))\},$$

where $D(0, \varepsilon)$ denotes the open disc of radius ε centered at 0 and $\mathcal{X}_T(\mathbb{C} \setminus D(0, \varepsilon))$ is the so-called glocal spectral subspace of T associated with $\mathbb{C} \setminus D(0, \varepsilon)$, that is, the set of all $x \in X$ for which there is an X-valued analytic function f on $D(0, \varepsilon)$ such that $(\lambda - T)f(\lambda) = x$ for all $\lambda \in D(0, \varepsilon)$. The local spectral radius of T at x is given by the formula $r_T(x) := \limsup_{n\to+\infty} \|T^n x\|^{\frac{1}{n}}$. The inner local (resp. local) spectral radius of T at x coincides with the minimum (resp. maximum) modulus of $\sigma_T(x)$ provided that T has the single-valued extension property; see [17] and [18]. Recall that T is said to have the single-valued extension property (abbreviated SVEP) if for every open subset U of $\mathbb{C}$, the equation $(\mu - T)f(\mu) = 0$, $(\mu \in U)$, has no nontrivial X-valued analytic solution f on U. For example, every operator $T \in \mathcal{L}(X)$ for which the interior of the set of its eigenvalues is empty enjoys this property.

Local spectra are a useful tool for analyzing operators, furnishing information well beyond that provided by classical spectral analysis. They play a very natural role in automatic continuity and in harmonic analysis, for instance in connection with the Wiener–Pitt phenomenon. The books [2], [19] and [17] give an extensive account of the local spectral theory, as well as investigations and applications in numerous fields.

On the problem of describing mappings preserving local spectra at a fixed nonzero vector, we mention: [16], where linear mappings on matrix spaces preserving the local spectrum at a fixed nonzero vector are characterized, [13], [14] concerned with the infinite dimensional case, and in [6], [7] preserver problems that have to do with locally spectrally bounded linear maps or additive local spectrum compressors on the matrix spaces and on $\mathcal{L}(X)$ are considered. While, non-linear preserver problems on the subject were studied in [4] and [8]. On the subject focused on linear or additive mappings preserving local spectra at all vectors, we mention: [12] where it was shown that the only additive map ϕ on $\mathcal{L}(X)$ satisfying $\sigma_{\phi(T)}(x) = \sigma_T(x)$ for all $x \in X$ and $T \in \mathcal{L}(X)$ is the identity, and [15] that deal with surjective linear local spectral radius zero preservers. In this paper, by strengthening the preservability condition, we consider surjective additive maps ϕ on $\mathcal{L}(X)$ that preserve the local invertibility of operators in both directions, that is, those maps ϕ such that for every $T \in \mathcal{L}(X)$ and $x \in X$ we have $0 \in \sigma_{\phi(T)}(x)$ if and only if $0 \in \sigma_T(x)$. We prove the following version of the above mentioned result [12, Theorem 1.1].

Theorem 1.1. *Let X be a complex Banach space of dimension at least two. A surjective additive map ϕ from $\mathcal{L}(X)$ into itself satisfies*

$$0 \in \sigma_{\phi(T)}(x) \Longleftrightarrow 0 \in \sigma_T(x) \quad (T \in \mathcal{L}(X),\ x \in X) \tag{1}$$

if and only if there exists a nonzero scalar c such that $\phi(T) = cT$ for all $T \in \mathcal{L}(X)$.

Remark 1.2. The following example shows that the assumption X is of dimension at least two cannot be removed in this theorem. In [1] it is proved that there exists a nowhere continuous automorphism ϕ of the field $\mathbb{C}$. Obviously, ϕ is bijective and additive, and satisfies (1). However, it is not a scalar multiple of the identity since it not continuous.

In the case of two different Banach spaces, Theorems 1.3 and 1.4 below improve [12, Theorems 1.3 and 1.5].

Theorem 1.3. *Let $A \in \mathcal{L}(X, Y)$. If $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective linear map satisfying*

$$0 \in \sigma_{\phi(T)}(Ax) \Longleftrightarrow 0 \in \sigma_T(x) \quad (T \in \mathcal{L}(X),\ x \in X),$$

then A is invertible and there exists a nonzero scalar c such that $\phi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(X)$.

Theorem 1.4. *Let $B \in \mathcal{L}(Y, X)$. If $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective linear map satisfying*

$$0 \in \sigma_{\phi(T)}(y) \Longleftrightarrow 0 \in \sigma_T(By) \quad (T \in \mathcal{L}(X),\ y \in Y),$$

then B is invertible and there exists a nonzero scalar c such that $\phi(T) = cB^{-1}TB$ for all $T \in \mathcal{L}(X)$.

The following is a variant of Theorem 1.1.

Theorem 1.5. *Let X and Y be infinite dimensional complex Banach spaces and $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map for which there exists $B \in \mathcal{L}(Y, X)$ such that for every $y \in Y$ we have*

$$0 \in \sigma_{\phi(T)}(y) \Longleftrightarrow 0 \in \sigma_T(By) \quad (T \in \mathcal{L}(X)). \tag{2}$$

Then B is invertible and there exists a nonzero scalar c such that $\phi(T) = cB^{-1}TB$ for all $T \in \mathcal{L}(X)$.

As consequences, the following theorems, extending the main result of [10], describe additive mappings that preserve the inner local spectral radius zero of operators.

Theorem 1.6. *Let X be a complex Banach space of dimension at least two. A surjective additive map ϕ from $\mathcal{L}(X)$ into itself satisfies*

$$\iota_{\phi(T)}(x) = 0 \Longleftrightarrow \iota_T(x) = 0 \quad (T \in \mathcal{L}(X),\ x \in X)$$

if and only if there exists a nonzero scalar c such that $\phi(T) = cT$ for all $T \in \mathcal{L}(X)$.

Theorem 1.7. *Let X and Y be infinite dimensional complex Banach spaces and $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map for which there exists $B \in \mathcal{L}(Y, X)$ such that for every $y \in Y$ we have*

$$\iota_{\phi(T)}(y) = 0 \Longleftrightarrow \iota_T(By) = 0 \quad (T \in \mathcal{L}(X)).$$

Then B is invertible and there exists a nonzero scalar c such that $\phi(T) = cB^{-1}TB$ for all $T \in \mathcal{L}(X)$.

The obtained results in Theorems 1.6 and 1.7 lead to inner local spectral radius versions of the main results of [15] which describe surjective linear maps on $\mathcal{L}(X)$ that are local spectral radius zero-preserving.

2. Proof of the main results

We first fix some notation and terminology. The duality between the Banach spaces X and its dual, X^*, will be denoted by $\langle .,. \rangle$. For $x \in X$ and $f \in X^*$, as usual we denote by $x \otimes f$ the rank one operator on X given by $z \mapsto \langle z, f \rangle x$. For $T \in \mathcal{L}(X)$ we will denote by T^*, $\ker(T)$, $\mathrm{range}(T)$, $\sigma_{su}(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not surjective}\}$ and $r(T)$ the adjoint, the kernel, the range, the surjectivity spectrum, and the spectral radius of T; respectively.

The following lemmas are needed for the proof of our main results. The first one relies the SVEP and the local spectrum.

Lemma 2.1. *An operator $T \in \mathcal{L}(X)$ has the SVEP if and only if for every $\lambda \in \mathbb{C}$ and every nonzero vector x in $\ker(\lambda - T)$ we have $\sigma_T(x) = \{\lambda\}$.*

Proof. See for instance [2, Theorem 2.22]. □

The second lemma is a simple consequence of [17, Proposition 1.2.16] and [2, Theorem 2.22], and its proof is therefore omitted here.

Lemma 2.2. *Let e be a fixed nonzero vector in X and let $R = x \otimes f$ be a non-nilpotent rank one operator. Then $0 \in \sigma_R(e)$ if and only if $\langle e, f\rangle = 0$ or e and x are linearly independent.*

Recall that a map $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is said to preserve the surjectivity of operators (resp. rank one operators) in both directions provided that $\phi(T)$ is surjective (resp. of rank one) if and only if T is.

The third lemma characterizes surjective additive maps from $\mathcal{L}(X)$ into $\mathcal{L}(Y)$ that preserve the surjectivity of operators in both directions.

Lemma 2.3. *Let X and Y be infinite dimensional complex Banach spaces and let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map. If ϕ preserves surjectivity of operators in both directions, then either*

(i) *there exist invertible bounded both linear or both conjugate linear operators $A : X \to Y$ and $B : Y \to X$ such that $\phi(T) = ATB$ for all $T \in \mathcal{L}(X)$, or*

(ii) *there exist invertible bounded both linear or both conjugate linear operators $A : X^* \to Y$ and $B : Y \to X^*$ such that $\phi(T) = AT^*B$ for all $T \in \mathcal{L}(X)$.*

The last case occurs only if X and Y are reflexive.

Proof. See [11, Lemma 2.1] □

The next two lemmas may be of independent interest.

Lemma 2.4. *Let ϕ be a map from $\mathcal{L}(X)$ into $\mathcal{L}(Y)$ satisfying*

$$\exists x \in X : 0 \in \sigma_{T+S}(x) \Longleftrightarrow \exists y \in Y : 0 \in \sigma_{\phi(T)+\phi(S)}(y)$$

for all $T, S \in \mathcal{L}(X)$. Then ϕ is injective.

Proof. Assume that $\phi(A) = \phi(B)$ for some $A, B \in \mathcal{L}(X)$. For every $T \in \mathcal{L}(X)$, we have

$$\begin{aligned}\exists x \in X : 0 \in \sigma_{T+A}(x) &\Longleftrightarrow \exists y \in Y : 0 \in \sigma_{\phi(T)+\phi(A)}(y)\\ &\Longleftrightarrow \exists y \in Y : 0 \in \sigma_{\phi(T)+\phi(B)}(y)\\ &\Longleftrightarrow \exists y \in Y : 0 \in \sigma_{T+B}(y).\end{aligned}$$

From this together with the fact that

$$\sigma_{su}(T) = \bigcup_{x\in X} \sigma_T(x) \tag{3}$$

for every $T \in \mathcal{L}(X)$ (see [17, Lemma 2.3]), we infer that

$$T + A \text{ is surjective } \Longleftrightarrow T + B \text{ is surjective}$$

for every $T \in \mathcal{L}(X)$. Upon replacing T by $T - A - \lambda$, we deduce that $\sigma_{su}(T + (B - A)) = \sigma_{su}(T)$ for all $T \in \mathcal{L}(X)$. As the surjectivity spectrum contains the boundary of the spectrum, we conclude that $r(T + (B - A)) = r(T)$ for all $T \in \mathcal{L}(X)$. Thus, by the Zemánek's spectral characterization of the radical, [3, Theorem 5.3.1], $A = B$ and ϕ is injective. □

We will say that a mapping $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the local invertibility of operators at a fixed nonzero vector $e \in X$ in both directions if for every $T \in \mathcal{L}(X)$ we have $0 \in \sigma_{\phi(T)}(e)$ if and only if $0 \in \sigma_T(e)$.

Lemma 2.5. *Let X be a Banach space of dimension at last two, e be a fixed nonzero vector in X and $A : X^* \to X$ and $B : X \to X^*$ be invertible bounded both linear or both conjugate linear operators. Then the anti-automorphism $\phi : T \mapsto AT^*B$ does not preserves the local invertibility of operators at e in both directions.*

Proof. We shall only deal with the case when A and B are conjugate linear, because the linear case follows analogously. First, we claim that

$$0 \in \sigma_T(e) \Longleftrightarrow 0 \in \sigma_{T^*BA}(A^{-1}e) \quad (T \in \mathcal{L}(X)). \tag{4}$$

For this, it suffice to show that for any $\varepsilon > 0$ and $T \in \mathcal{L}(X)$ we have $e \in \mathcal{X}_{AT^*B}(\mathbb{C} \setminus D(0, \varepsilon))$ if and only if $A^{-1}e \in \mathcal{X}_{T^*BA}(\mathbb{C} \setminus D(0, \varepsilon))$. To do so, assume that $A^{-1}e \in \mathcal{X}_{T^*BA}(\mathbb{C} \setminus D(0, \varepsilon))$ and let f be an X^*-valued analytic function on $D(0, \varepsilon)$ such that

$$(\mu - T^*BA)f(\mu) = A^{-1}e$$

for all $\mu \in D(0, \varepsilon)$. We have

$$(\overline{\mu} - AT^*B)Af(\mu) = e$$

for all $\mu \in D(0, \varepsilon)$; where $\overline{\mu}$ is the complex conjugate of μ. Set

$$\widetilde{f}(\overline{\mu}) := Af(\mu), \quad (\mu \in D(0, \varepsilon)),$$

and note that the map $\widetilde{f}$ is an analytic function on $D(0, \varepsilon)$ since

$$\lim_{h \to 0} \frac{\widetilde{f}(\overline{\mu} + h) - \widetilde{f}(\overline{\mu})}{h} = \lim_{h \to 0} A(\frac{f(\mu + \overline{h}) - f(\mu)}{\overline{h}}) = Af'(\mu)$$

for all $\mu \in D(0,\varepsilon)$, where $f'(\mu)$ is the derivative of f at μ. Hence $e \in \mathcal{X}_{AT^*B}(\mathbb{C} \setminus D(0,\varepsilon))$. As the reverse implication can be obtained by similarity, the claim is proved.

Next, assume by the way of contradiction that the map ϕ preserves the local invertibility of operators at e in both directions. We will prove that the condition (4) is not satisfied. If $\langle e, Be\rangle = 0$, choose a linear functional $f \in X^*$ so that $\langle e, f\rangle = 1$, and set $T = e \otimes f$. Lemma 2.1 implies that $\sigma_T(e) = \{1\}$. On the other hand, we have

$$\langle x, T^*BA(A^{-1}e)\rangle = \langle Tx, Be\rangle = \langle x, f\rangle\langle e, Be\rangle = 0$$

for every $x \in X$. This implies that $T^*BA(A^{-1}e) = 0$, and so $\sigma_{T^*BA}(A^{-1}e) = \{0\}$; which contradicts (4). If we assume that $\langle e, Be\rangle \neq 0$, then we can find a vector $w \in X$ such that e and w are linearly independent and $\langle w, Be\rangle = 1$. For $T = w \otimes A^{-1}e$, we have, by Lemma 2.2, $0 \in \sigma_T(e)$ since $e \notin \mathbb{C}w$. Observe that

$$\langle x, T^*BA(A^{-1}e)\rangle = \langle Tx, Be\rangle = \langle x, A^{-1}e\rangle\langle w, Be\rangle = \langle x, A^{-1}e\rangle$$

is true for every $x \in X$, so that $T^*BA(A^{-1}e) = A^{-1}e$. This shows that $\sigma_{T^*BA}(A^{-1}e) = \{1\}$, contradicting (4) in this case too. The proof is therefore complete. □

Remark 2.6. Just as in the proof of the above lemma one can see that when $X = \mathbb{C}^n$ $(n \geq 2)$ and $A : \mathbb{C}^n \to \mathbb{C}^n$ and $B : \mathbb{C}^n \to \mathbb{C}^n$ are invertible bounded both linear or both conjugate linear operators, the anti-automorphism $\phi : T \to AT^{tr}B$ does not preserves the local invertibility of matrices at a fixed nonzero vector in $\mathbb{C}^n$. Here T^{tr} denotes the transpose of the matrix T.

We now have collected all the necessary ingredients and are therefore in a position to prove the main results of this section.

Proof of Theorem 1.1. Checking the 'if' part is straightforward, so we will only deal with the 'only if' part. So assume that (1) holds. From the equality (3), we have

$$\begin{aligned} T \text{ is not surjective} &\Longleftrightarrow \exists x \in X : 0 \in \sigma_T(x) \\ &\Longleftrightarrow \exists x \in X : 0 \in \sigma_{\phi(T)}(x) \\ &\Longleftrightarrow \phi(T) \text{ is not surjective} \end{aligned}$$

for every $T \in \mathcal{L}(X)$. Consequently, ϕ preserves the surjectivity of operators in both directions. We consider the following two cases:

Case 1. X is an infinite dimensional Banach space.
As the map ϕ preserves the surjectivity of operators in both directions, Lemma 2.3 implies that either

(i) there exist invertible bounded both linear or both conjugate linear operators $A : X \to X$ and $B : X \to X$ such that $\phi(T) = ATB$ for all $T \in \mathcal{L}(X)$, or

(ii) there exist invertible bounded both linear or both conjugate linear operators $A : X^* \to X$ and $B : X \to X^*$ such that $\phi(T) = AT^*B$ for all $T \in \mathcal{L}(X)$.

Lemma 2.5 entails that the form of ϕ in the statement (ii) is excluded, and consequently there exist invertible bounded both linear or both conjugate linear operators $A : X \to X$ and $B : X \to X$ such that $\phi(T) = ATB$ for all $T \in \mathcal{L}(X)$. Similar argument as the one used in the proof of Lemma 2.5 allow to get that for every $x \in X$ and $T \in \mathcal{L}(X)$, we have

$$\begin{aligned} 0 \in \sigma_{B^{-1}A^{-1}\phi(T)}(B^{-1}x) &\Longleftrightarrow 0 \in \sigma_T(x) \\ &\Longleftrightarrow 0 \in \sigma_{\phi(T)}(x). \end{aligned}$$

From this together with the surjectivity of ϕ, we infer that

$$0 \in \sigma_{B^{-1}A^{-1}T}(x) \Longleftrightarrow 0 \in \sigma_T(Bx) \tag{5}$$

for all $x \in X$ and $T \in \mathcal{L}(X)$. With similarly, we also have

$$0 \in \sigma_{TB^{-1}A^{-1}}(Ax) \Longleftrightarrow 0 \in \sigma_T(x) \tag{6}$$

for all $x \in X$ and $T \in \mathcal{L}(X)$.

Now, let us show that B is a multiple of the identity operator by a nonzero scalar. Assume on the contrary that there exists a vector $x \in X$ such that x and Bx are linearly independent, and pick a linear functional f on X such that $\langle x, f\rangle = 1$ and $\langle Bx, f\rangle = 0$. For $T = ABx \otimes f$, we have $\sigma_T(Bx) = \{0\}$. However, that $B^{-1}A^{-1}T(x) = x$ implies that $\sigma_{B^{-1}A^{-1}T}(x) = \{1\}$, contradicting (5). Thus, B is a nonzero scalar multiple of the identity. So, by taking into account (6), we get

$$0 \in \sigma_{TA^{-1}}(Ax) \Longleftrightarrow 0 \in \sigma_T(x) \tag{7}$$

for all $x \in X$ and $T \in \mathcal{L}(X)$. Let us also show that A is a nonzero scalar multiple of the identity. Assume for a contradiction that there exists a vector $x \in X$ such that x and $A^{-1}x$ are linearly independent, and let $f \in X^*$ so that $\langle x, f\rangle = 1$ and $\langle A^{-1}x, f\rangle = 0$. Set $T = x \otimes f$, and note that $\sigma_T(x) = \{1\}$. However, the fact that $(TA^{-1})^2(Ax) = TA^{-1}x = 0$ implies that $\sigma_{B^{-1}A^{-1}T}(x) = \{0\}$, and contradicts (7).

Hence we must have that A and B are nonzero scalar multiple of the identity, and consequently there exists a non-null constant c such that $\phi(T) = cT$ for all $T \in \mathcal{L}(X)$.

Case 2. X is a finite dimensional space. The proof of it will be completed after checking the following two claims.

Claim 1. ϕ preserves rank one operators in both directions.

Proof. Lemma 2.4 implies that ϕ is injective. Since ϕ is assumed surjective, it is invertible. From this together with the fact that, in this case, an operator T is injective if and only if it is invertible, we infer that ϕ is a bijective map preserving invertibility in both directions. So, using the spectral characterization of rank one operators [20, Lemma 2.1] together with the same approach as in [5, Theorem 4.1] one can see that ϕ preserves rank one operators in both directions; which proves the claim.

Claim 2. There exists a nonzero scalar c such that $\phi(R) = cR$ for all non-nilpotent rank one operator R.

Proof. Let $R = x \otimes f$ be a non-nilpotent rank one operator. According to the above lemma we can find a linear functional $g \in X^*$ and a vector $y \in X$ such that $\phi(R) = y \otimes g$. The fact that $\sigma_R(x) = \{\langle x, f\rangle\}$ together with (1) imply that $0 \notin \sigma_{y\otimes g}(x) = \sigma_{\phi(R)}(x)$, and consequently, it follows, from Lemma 2.2, that x and y are linearly dependent. By absorbing a constant in the seconde term in the tensor product, one can now see that $\phi(R) = x \otimes L_{x,f}$ for some $L_{x,f} \in X^*$.

Now, let us prove that for every non-nilpotent rank one operator $R = x \otimes f$, the mapping $L : x \otimes f \mapsto L_{x,f}$ is independent of x. To do so, let z be a vector such that x and z are linearly independent and $\langle z, f\rangle \neq 0$. If $\langle x + z, f\rangle \neq 0$, then $(x + z) \otimes f$ is a non-nilpotent rank one operator and

$$x \otimes L_{x,f} + z \otimes L_{z,f} = \phi((x + z) \otimes f) = (x + z) \otimes L_{x+z,f}.$$

On the other hand, it easy to see that the operator $x \otimes L_{x,f} + z \otimes L_{z,f}$ has rank 2 whenever x and z as well as $L_{x,f}$ and $L_{z,f}$ are linearly independent. Consequently, $L_{x,f}$ and $L_{z,f}$ are linearly dependent, and so there exits a nonzero scalar α such that $L_{z,f} = \alpha L_{x,f}$. This gives that $(x + \alpha z) \otimes L_{x,f} = (x + z) \otimes L_{x+z,f}$; which implies that $\alpha = 1$ and $L_{z,f} = L_{x,f} = L_{x+z,f}$. In the case when $\langle x + z, f\rangle = 0$, we have $\langle x - z, f\rangle \neq 0$, and by similarity, we get $L_{z,f} = L_{x,f}$ in this case too. Therefore, for every non-nilpotent rank one operator $R = x \otimes f$, the mapping $L : x \otimes f \mapsto L_{x,f}$ becomes independent of x. Thus, we may denote $L_{x,f}$ simply by L_f.

Next, let us show that L_f and f are linearly dependent for all non zero $f \in X^*$. Assume on the contrary that there exists a vector $x \in X$ such that $\langle x, f\rangle = 0$ and $\langle x, L_f\rangle \neq 0$. Clearly, we have $\sigma_{x\otimes f}(x) = \{0\}$ and $\sigma_{\phi(x\otimes f)}(x) = \sigma_{x\otimes L_f}(x) = \{\langle x, L_f\rangle\}$; which leads to a contradiction. Therefore there exists a nonzero scalar c_f such that $L_f = c_f f$ for all non zero f in X^*. Moreover, we claim that the mapping c_f does not depend on f. Indeed, let $f, g \in X^*$ be linearly independent, and let $x \in X$ such that $\langle x, f\rangle \neq 0 \neq \langle x, g\rangle$ and $\langle x, f + g\rangle \neq 0$. We have

$$x \otimes c_{f+g}(f + g) = x \otimes L_{f+g} = \phi(x \otimes (f + g)) = x \otimes (c_f f + c_g g),$$

and so we get $c_f = c_g = c_{f+g}$. It follows that the mapping c_f does not depend on f. Thus, we may write c instead of c_f, and consequently $\phi(R) = cR$ for all non-nilpotent rank one operator R; which concludes the proof of the claim.

As ϕ is additive, and every nilpotent rank one operator is a sum of two non-nilpotent rank one operator, we deduce that $\phi(R) = cR$ for all rank one operator R. Since X is of finite dimensional, we conclude that $\phi(T) = cT$ for all $T \in \mathcal{L}(X)$, and the theorem follows. □

Proof of Theorem 1.5. The sufficiency condition is easily verified. To prove the necessity, assume that (2) holds. The proof of it will be completed after checking several steps.

Step 1. The mapping ϕ has one of the forms (i) and (ii) in Lemma 2.3.

Proof. Similar argument as the one used in the beginning of the proof of Theorem 1.1 allows to get that the map ϕ preserves surjectivity of operators in both directions, and so the desired conclusion follows from Lemma 2.3.

Step 2. The operator B is injective.

Proof. If $By = 0$, then (2) and the surjectivity of ϕ give $0 \notin \sigma_T(y)$ for each $T \in \mathcal{L}(Y)$, and therefore $y = 0$.

Step 3. The form (ii) of ϕ in Step 1 is excluded.

Proof. Assume for a contradiction that there exist invertible bounded both linear or both conjugate linear operators $A_1 : X^* \to Y$ and $B_1 : Y \to X^*$ such that $\phi(T) = A_1 T^* B_1$. The same argument as in the proof of Lemma 2.5 together with (2) allows to get that the equivalence

$$0 \in \sigma_T(By) \Longleftrightarrow 0 \in \sigma_{T^* B_1 A_1}(A_1^{-1} y) \tag{8}$$

holds true for any $T \in \mathcal{L}(X)$ and $y \in Y$. Pick an arbitrary non zero vector y in Y, and note that, by the above step, $By \neq 0$. We will show that the condition (8) is not satisfied. Firstly assume that $\langle By, B_1 y\rangle = 0$. Choose a linear functional f in X^* such that $\langle By, f\rangle = 1$, and set $T = By \otimes f$. We have

$$\langle x, T^* B_1 A_1 (A_1^{-1} y)\rangle = \langle Tx, B_1 y\rangle = \langle x, f\rangle\langle By, B_1 y\rangle = 0$$

for all $x \in X$. This implies that $\sigma_{T^* B_1 A_1}(A_1^{-1} y) = \{0\}$, and contradicts (8) since $\sigma_T(By) = \{1\}$.

Next, assume that $\langle By, B_1 y\rangle \neq 0$. Then we can find $w \in X$ such that By and w are linearly independent and $\langle w, B_1 y\rangle = 1$. Set $T = w \otimes A_1^{-1} y$, and note that $0 \in \sigma_T(By)$ since $By \notin \mathbb{C}w$. But, the fact that

$$\langle x, T^* B_1 A_1 (A_1^{-1} y)\rangle = \langle Tx, B_1 y\rangle = \langle x, A_1^{-1} y\rangle\langle w, B_1 y\rangle = \langle x, A_1^{-1} y\rangle$$

is true for every $x \in X$, implies that $T^* B_1 A_1 (A_1^{-1} y) = A_1^{-1} y$. Consequently, $\sigma_{T^* B_1 A_1}(A_1^{-1} y) = \{1\}$. This contradicts (8) in this case too, and achieves the proof of the step.

Step 4. The operator B is invertible.

Proof. By combining Claim 1 and Claim 3, we infer that there exist invertible bounded both linear or both conjugate linear operators $A_1 : X \to Y$ and $B_1 : Y \to X$ such that $\phi(T) = A_1 T B_1$ for all $T \in \mathcal{L}(X)$. If B were not surjective, then we could find $x \in X \setminus \operatorname{range}(B)$ and $f \in X^*$ such that $\langle B_1 A_1 x, f\rangle = 1$. Set $T = x \otimes f$. Since x and $BA_1 x$ are linearly independent, Lemma 2.2 tell us that $0 \in \sigma_T(BA_1 x)$. But, $\sigma_{\phi(T)}(A_1 x) = \{1\}$ since

$$\phi(T)(A_1 x) = A_1 x \otimes f \circ B_1 (A_1 x) = A_1 x,$$

arriving to a contradiction. Thus, B is invertible as desired.

In order to complete the proof of the theorem, define $\chi : \mathcal{L}(X) \to \mathcal{L}(X)$ by putting $\chi(T) = B\phi(T)B^{-1}$, and note that the map χ is a surjective additive map satisfying

$$\begin{aligned} 0 \in \sigma_{\chi(T)}(By) &\Longleftrightarrow 0 \in \sigma_{B\phi(T)B^{-1}}(By) \\ &\Longleftrightarrow 0 \in \sigma_{\phi(T)}(y) \end{aligned}$$

for any $y \in Y$ and $T \in \mathcal{L}(X)$. Upon replacing y by $B^{-1}x$ and by taking into account (2), we get

$$0 \in \sigma_{\chi(T)}(x) \Longleftrightarrow 0 \in \sigma_T(x)$$

for any $x \in X$ and $T \in \mathcal{L}(X)$. Theorem 1.1 implies that there exists a nonzero scalar c such that $\chi(T) = cT$ for every $T \in \mathcal{L}(X)$, and consequently $\phi(T) = cB^{-1}\phi(T)B$ for all $T \in \mathcal{L}(X)$; which achieves the proof. □

Proof of Theorem 1.4. We shall only deal with the case when X or Y is finite dimensional since otherwise the result of the theorem is a consequence of Theorem 1.5. So, assume that either X or Y is finite dimensional, and note that, by Lemma 2.4, the map ϕ is injective. The fact that, in this case, ϕ is linear and bijective implies that X and Y are both finite dimensional, having the same dimension over $\mathbb{C}$. Claim. 1 shows now that B is in fact bijective. So, as in the end of the proof of Theorem 1.5, the map $\chi : T \mapsto B\phi(T)B^{-1}$ is linear and surjective, and satisfies (1). Consequently, the result follows by applying Theorem 1.1 to the map χ in the case when X is of dimension at least two. The case when $\dim X = 1$ is a consequence of [21, Theorem 1.1], and the proof is therefore complete. □

Proof of Theorem 1.3. Lemma 2.4 shows that the map ϕ is injective. Since ϕ is assumed surjective, it is invertible. Thus, the desired conclusion follows by applying Theorem 1.4 to the map ϕ^{-1}; which achieves the proof. □

Proof of Theorems 1.6 and 1.7. As the notion of local invertibility encompasses inner spectral radius zero: for any $x \in X$ and $T \in \mathcal{L}(X)$ we have

$$0 \in \sigma_T \Longleftrightarrow \iota_T(x) = 0$$

(see [18]), Theorems 1.1 and 1.5 remain valid when the assumption "$0 \in \sigma_{.}(.)$" is replaced by "$\iota_{.}(.) = 0$"; which yield the desired conclusions in Theorems 1.6 and 1.7. □

Remark 2.7. If X is of finite dimensional space and ϕ is a linear map on $\mathcal{L}(X)$ satisfying (1), then Lemma 2.4 shows that the map ϕ is automatically surjective. It is conceivable that the surjectivity assumption in Theorem 1.1 can be removed.

3. Open problem

It is interesting to relax the additivity assumption and to know what kind of transformations ϕ on $\mathcal{L}(X)$ will leave invariant the local invertibility property at a fixed nonzero vector $e \in X$. Clearly, if one just assume that

$$0 \in \sigma_{\phi(T)}(e) \Longleftrightarrow 0 \in \sigma_T(e)$$

for every $T \in \mathcal{L}(X)$ on ϕ, the structure of ϕ can be quite arbitrary. So, it is reasonable to impose a more restrictive condition on such transformations relating the local spectra of a pair of operators. In [8], classifications were established for mappings ϕ on $\mathcal{M}_n(\mathbb{C})$, the algebra of all $n \times n$ complex matrices, satisfying $\sigma_{\phi(T)-\phi(S)}(e) = \sigma_{T-S}(e)$ for any matrices T and S. Characterizations for mappings on $\mathcal{M}_n(\mathbb{C})$ that compress or expand the local spectrum of the sum or the product of matrices at a fixed nonzero vector, and investigation of several extensions of these results were obtained in [4] and [9].

We close this paper by the following similar natural problem which suggests itself.

Problem 3.1. Let e be a fixed nonzero vector X. Characterize surjective mappings ϕ on $\mathcal{L}(X)$ satisfying

$$0 \in \sigma_{\phi(T)-\phi(S)}(e) \Longleftrightarrow 0 \in \sigma_{T-S}(e)$$

for all $T, S \in \mathcal{L}(X)$.

References

[1] J. Aczél and J. Dhombres, Functional Equations in Several Variables, *Encyclopedia Math. Appl.*, *Cambridge, U. P.*, 1989.

[2] P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, *Kluwer Academic Publishers*, *Dordrecht, Boston, London*, 2004.

[3] B. Aupetit, A primer on Spectral Theory, *Springer-Verlag*, *New York*, 1991.

[4] M. Bendaoud, Preservers of local spectra of matrix sums, *Lin. Alg. Appl.* **438** (2013), 2500–2507.

[5] M. Bendaoud, Additive maps preserving the minimum and surjectivity moduli of operators, *Arch. Math.* **92** (2009), 257–265.

[6] M. Bendaoud and M. Sarih, Additive local spectrum compressors, *Lin. Alg. Appl.* **435** (2011), 1473–1478.

[7] M. Bendaoud and M. Sarih, Locally spectrally bounded linear maps, *Math. Bohem.* **136** (2011), 81–89.

[8] M. Bendaoud, M. Douimi and M. Sarih, Maps on matrices preserving local spectra, *Lin. Multilinear Alg.* **61** (2013), 871–880.

[9] M. Bendaoud, M. Jabbar and M. Sarih, Preservers of local spectra of operator products, *Lin. Multilinear Alg.* (2014), `http://dx.doi.org/10.1080/03081087.2014.902944`.

[10] M. Bendaoud and M. Sarih, Surjective linear maps preserving certain spectral radii, *Rocky Mountain J. Math.* **41** (2011), 727–735.

[11] M. Bendaoud and M. Sarih, Additive maps preserving the inner local spectral radius, *Oper. Theo. Adv. Appl.* **236** (2014), 95–102.

[12] A. Bourhim and T. Ransford, Additive maps preserving local spectrum, *Int. Equ. Oper. Theo.* **55** (2006), 377–385.

[13] J. Bračič and V. Müller, Local spectrum and local spectral radius at a fixed vector, *Studia Math.* **194** (2009), 155–162.

[14] C. Costara, Automatic continuity for surjective linear mappings decresing the local spectral radius at some fixed vector, *Arch. Math.* **95** (2010), 567–573.

[15] C. Costara, Linear maps preserving operators of local spectral radius zero, *Integ. Equ. Oper. Theo.* **73** (2012), 7–16.

[16] M. González and M. Mbekhta, Linear maps on $M_n(\mathbb{C})$ preserving the local spectrum, *Lin. Alg. Appl.* **427** (2007), 176–182.

[17] K. B. Laursen and M. M. Neumann, An Introduction to Local Spectral Theory, *Oxford University Press*, *New York*, 2000.

[18] T. L. Miller, V. G. Miller and M. Neumann, Local spectral properties of weighted shifts, *J. Oper. Theo.* **51** (2004), 71–88.

[19] V. Müller, Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras, Vol. 139, *Oper. Theo. Adv. Appl.*, *Birkhäuser, Basel, Boston, Berlin*, 2007.

[20] M. Omladič and P. Šemrl, Spectrum-preserving additive mappings, *Lin. Alg. Appl.* **153** (1991), 67–72.

[21] A. R. Sourour, Inversibility preserving linear maps on $\mathcal{L}(X)$, *Trans. Amer. Math. Soc.* **348** (1996), 13–30.

M. BENDAOUD
DEPARTMENT OF MATHEMATICS
MOULAY ISMAIL UNIVERSITY, ENSAM
MARJANE II, B.P. 15290
AL MANSOUR, MEKNÈS
MOROCCO

E-mail: m.bendaoud@ensam-umi.ac.ma

M. JABBAR
DEPARTMENT OF MATHEMATICS
MOULAY ISMAIL UNIVERSITY, ENSAM
MARJANE II, B.P. 15290
AL MANSOUR, MEKNÈS
MOROCCO

E-mail: m.jabbar@ensam-umi.ac.ma

M. SARIH
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
BP 11201
ZITOUNE, MEKNÈS
MOROCCO

E-mail: m.sarih@fs-umi.ac.ma

(Received January 25, 2014; revised April 13, 2014)