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Common expansions in noninteger bases

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Dedicated to Professor Zoltán Daróczy on his 75th birthday

Abstract. In this paper we study the existence of simultaneous representations of real numbers in bases $p > q > 1$ with the digit set $A = \{-m, \dots, 0, \dots, m\}$. We prove among others that if $q < (1+\sqrt{8m+1})/2$, then there is a continuum of sequences $(c_i) \in A^\infty$ satisfying $\sum_{i=1}^\infty c_i q^{-i} = \sum_{i=1}^\infty c_i p^{-i}$. On the other hand, if $q \ge m+1+\sqrt{m(m+1)}$, then only the trivial sequence $(c_i) = 0^\infty$ satisfies the former equality.

1. Introduction

Given a finite *alphabet* or *digit set* A of real numbers and a real *base* $q > 1$, by an *expansion* of a real number x we mean a sequence $c = (c_i) \in A^\infty$ satisfying the equality

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

This concept was introduced by Rényi [10] as a generalization of the radix representation of integers.

Given two different bases p, q we wonder whether there exist real numbers having the same expansions in both bases:

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x = \sum_{i=1}^{\infty} \frac{c_i}{p^i}. \tag{1}$$

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In case $0 \in A$ a trivial example is $x = 0$ with $(c_i) = 0^\infty$. If the alphabet A contains no pair of digits with opposite signs, then this is the only such example. Indeed, if for instance all digits are nonnegative and $0^\infty \neq (c_i) \in A^\infty$, then for $p > q$ we have

$$\sum_{i=1}^{\infty} \frac{c_i}{p^i} < \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

by an elementary monotonicity argument.

Even if the alphabet A contains digits of opposite signs, the existence of *common expansions* (1) seems to be a rare event.

Similar phenomena appears with the common radix representation. Indlekofer, Kátai and Racskó [4] called $\mathbf{a} \in \mathbb{Z}^d$ *simultaneously representable* by $\mathbf{q} \in \mathbb{Z}^d$, if there exist integers $0 \leq m_0, \ldots, m_\ell < Q := |q_1 \cdots q_d|$ such that

$$a_i = \sum_{j=0}^{\ell} m_j q_i^j, \quad i = 1, \ldots, d.$$

If $q_1, \ldots, q_d > 0$ then apart from the zero vector no integer vectors are simultaneously representable by $\mathbf{q}$. If, however, some of the base numbers are negative, then simultaneous representations may appear. For example take $q_1 = -2$ and $q_2 = -3$ then we have $(101)_{10} = (1431335045)_{-2} = (1431335045)_{-3}$. Changing the sign of the "digits" with odd position we get a common representation of 101 in bases 2 and 3 with digits from $\{-6, \ldots, 0, \ldots, 6\}$. Pethő [8] gave a criterion of simultaneous representability on the one hand with the Chinese reminder theorem and, on the other hand with CNS polynomials. A similar result was proved by Kane [7].

No results on simultaneous representability of real numbers in non-integer bases seem to have appeared in the literature. In this paper we start such a study by investigating the case of the special alphabets $A = \{-m, \ldots, 0, \ldots, m\}$ for some given integer $m \geq 1$. Let us denote by $C(p, q)$ the set of sequences $(c_i) \in A^\infty$ satisfying

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = \sum_{i=1}^{\infty} \frac{c_i}{p^i}. \tag{2}$$

We call $C(p, q)$ *trivial* if its only element is the null sequence.

Our main result is the following:

Theorem 1. *Let $p > q > 1$.*

(i) *If $q < (1 + \sqrt{8m+1})/2$, then $C(p, q)$ has the power of continuum.*

(ii) *If $(1 + \sqrt{8m+1})/2 \leq q \leq m + 1$, then $C(p, q)$ is infinite.*

(iii) *Let* $m+1<q\le 2m+1$.

(a) *If*

$$p\le \frac{(m+1)(q-1)}{q-m-1}, \tag{3}$$

then $C(p,q)$ *is nontrivial.*

(b) *If*

$$p> \frac{(m+1)(q-1)}{q-m-1}, \tag{4}$$

then $C(p,q)$ *is trivial.*

(iv) *Let* $2m+1<q<m+1+\sqrt{m(m+1)}$.

(a) $C(p,q)$ *is a finite set.*

(b) *There is a continuum of values* $p>q$ *for which* $C(p,q)$ *is nontrivial.*

(c) *If* $p>q$ *satisfies* (4), *then* $C(p,q)$ *is trivial.*

(v) *If* $q\ge m+1+\sqrt{m(m+1)}$, *then* $C(p,q)$ *is trivial.*

Remark 2.

(i) The proof of (iii) (a) will also show that if $m+1<q\le 2m+1$, and

$$\frac{1}{m}\le \frac{1}{q-1}+\left(\frac{1}{q-1}-\frac{1}{p-1}\right)\frac{q^n}{p^n-q^n}$$

for some positive integer n, then $C(p,q)$ has at least $n+1$ elements. For $n=1$ this condition reduces to (3).

Furthermore, we show in Remark 7 that the right side of this inequality is a decreasing function of p, so that the solutions p of the inequality form a half-closed interval $(q,p_n]$. (We have clearly $p_1>p_2>\cdots$.)

(ii) The proof of (iv) (a) will show more precisely that if $q>m+1$ and

$$\frac{1}{2m}> \frac{1}{q-1}+\left(\frac{1}{q-1}-\frac{1}{p-1}\right)\frac{q^n}{p^n-q^n}$$

for some positive integer n, then $C(p,q)$ has at most $(2m+1)^n$ elements.

2. Proofs

We begin by establishing some auxiliary results.

Interval filling sequences play an important role in establishing the existence of various kinds of representations of real numbers; see, e.g., Daróczy, Járai and Kátai [1], Daróczy and Kátai [2]. We also need such a result here: a variant of a classical theorem of Kakeya [5], [6] (see also [9], Part 1, Exercise 131):

Proposition 3. *Let $\sum_{k=1}^{\infty} r_k$ be a convergent series of positive numbers, satisfying the inequalities*

$$r_n \leq 2m \sum_{k=n+1}^{\infty} r_k \tag{5}$$

for all $n = 1, 2, \dots$. Then the sums

$$\sum_{k=1}^{\infty} c_k r_k, \quad (c_k) \in A^{\infty} \tag{6}$$

fill the interval

$$\left[-m \sum_{k=1}^{\infty} r_k, m \sum_{k=1}^{\infty} r_k \right]. \tag{7}$$

Proof. It is clear that all sums (6) belong to the interval (7). Conversely, for each given x in this interval we define a sequence $(c_k) \in A^{\infty}$ by the following greedy algorithm. If $c_1, \dots, c_{n-1}$ are already defined (no assumption if $n = 1$), then let c_n be the largest element of A such that

$$\left(\sum_{k=1}^{n} c_k r_k \right) - m \left(\sum_{k=n+1}^{\infty} r_k \right) \leq x.$$

Letting $n \to \infty$ it follows that $\sum_{k=1}^{\infty} c_k r_k \leq x$. It remains to prove the converse inequality. This is obvious if $c_k = m$ for all $k \in \mathbb{N}$ because then

$$\sum_{k=1}^{\infty} c_k r_k = m \sum_{k=1}^{\infty} r_k \geq x$$

by the choice of x.

If $c_n < m$ for infinitely many indices, then

$$\left(\sum_{k=1}^{n-1} c_k r_k \right) + m r_n - m \left(\sum_{k=n+1}^{\infty} r_k \right) > x$$

for all such indices, and letting $n \to \infty$ we conclude that $\sum_{k=1}^{\infty} c_k r_k \geq x$.

The proof will be complete if we show that (c_k) cannot have a last term $c_n < m$, i.e., an index n such that $c_n = j < m$, and $c_k = m$ for all $k > n$. Assume on the contrary that there exists such an index n. Then we have

$$\left(\sum_{k=1}^{n-1} c_k r_k \right) + j r_n + m \left(\sum_{k=n+1}^{\infty} r_k \right) \leq x$$

and

$$\left(\sum_{k=1}^{n-1} c_k r_k\right) + (j+1) r_n - m\left(\sum_{k=n+1}^{\infty} r_k\right) > x$$

by construction. Hence

$$r_n > 2m \sum_{k=n+1}^{\infty} r_k,$$

contradicting (5). □

We also need two technical lemmas.

Lemma 4. *If $1 < q < (1+\sqrt{8m+1})/2$ and $p > q$, then the sequence $(r_k) := (q^{-i} - p^{-i})_{i \in \mathbb{N} \setminus n\mathbb{N}}$ satisfies (5) for all sufficiently large integers n.*

Proof. Fix a sufficiently large integer n such that

$$\frac{1}{2m} < \frac{1}{q(q-1)} - \frac{1}{q(q^n - 1)}.$$

This is possible by our assumption on q, because we have the following equivalences for $m > 0$ and $q > 1$:

$$\begin{aligned} \frac{1}{2m} < \frac{1}{q(q-1)} &\iff 4q(q-1) < 8m \\ &\iff (2q-1)^2 < 8m+1 \\ &\iff 2q - 1 < \sqrt{8m+1} \\ &\iff q < \left(1 + \sqrt{8m+1}\right)/2. \end{aligned}$$

Now, if

$$r_{h'} = q^{-h} - p^{-h} = q^{-h}\left(1 - (q/p)^h\right)$$

for some $h' \geq 1$, then

$$\sum_{k=h'+1}^{\infty} r_k = \sum_{i \in \mathbb{N} \setminus n\mathbb{N}, i > h} (q^{-i} - p^{-i}) = \sum_{i \in \mathbb{N} \setminus n\mathbb{N}, i > h} q^{-i}\left(1 - (q/p)^i\right).$$

Since $(1 - (q/p)^i) > (1 - (q/p)^h)$ for all $i > h$, it follows that (we use the choice of n in the last step)

$$\frac{\sum_{k=h'+1}^{\infty} r_k}{r_{h'}} \geq \sum_{i \in \mathbb{N} \setminus n\mathbb{N}, i > h} q^{h-i} = \left(\sum_{i=1}^{\infty} q^{-i}\right) - \left(\sum_{i > \frac{h}{n}} q^{h-in}\right)$$

$$\geq \left(\sum_{i=1}^{\infty} q^{-i}\right) - \left(\sum_{i=0}^{\infty} q^{-1-in}\right) = \left(\sum_{i=2}^{\infty} q^{-i}\right) - \left(\sum_{i=1}^{\infty} q^{-1-in}\right)$$
$$= \frac{1}{q(q-1)} - \frac{1}{q(q^n-1)} > \frac{1}{2m}. \qquad \square$$

Lemma 5. *Let $p > q > 1$. The sequence*

$$\left(\frac{\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})}{q^{-n}-p^{-n}}\right)_{n=1}^{\infty}$$

is strictly decreasing, and tends to $1/(q-1)$.

Proof. Since $1 > (q/p)^n \searrow 0$, the results follow from the identity

$$\frac{\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})}{q^{-n}-p^{-n}} = \frac{1}{q-1} + \frac{p-q}{(q-1)(p-1)}\frac{(q/p)^n}{1-(q/p)^n}. \tag{8}$$

Setting $x = q/p$ for brevity, the identity is proved as follows:

$$\begin{aligned}\frac{\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})}{q^{-n}-p^{-n}} &= \frac{q^{-n}}{(q-1)(q^{-n}-p^{-n})} - \frac{p^{-n}}{(p-1)(q^{-n}-p^{-n})}\\ &= \frac{1}{(q-1)(1-x^n)} - \frac{x^n}{(p-1)(1-x^n)}\\ &= \frac{1-x^n+x^n}{(q-1)(1-x^n)} - \frac{x^n}{(p-1)(1-x^n)}\\ &= \frac{1}{q-1} + \frac{x^n}{1-x^n}\left(\frac{1}{q-1}-\frac{1}{p-1}\right)\\ &= \frac{1}{q-1} + \frac{p-q}{(q-1)(p-1)}\frac{x^n}{1-x^n}. \qquad \square\end{aligned}$$

Remark 6. Let us note for further reference the following equivalent form of (8), obtained during the proof:

$$\frac{\sum_{i=n+1}^{\infty}(q^{-i}-p^{-i})}{q^{-n}-p^{-n}} = \frac{1}{q-1} + \left(\frac{1}{q-1}-\frac{1}{p-1}\right)\frac{q^n}{p^n-q^n}. \tag{9}$$

Now we are ready to prove our theorem.

Proof of Theorem 1 (i). We adapt the proof of Theorem 3 in [3], which states that if $1 < q < (1+\sqrt{5})/2$, then every $q < x < 1/(q-1)$ has a continuum of expansions in base q with digits 0 or 1.

Applying Lemma 4 we fix a large positive integer n such that the sequence $(r_k) := (q^{-i} - p^{-i})_{i \in \mathbb{N} \setminus n\mathbb{N}}$ satisfies (5). Next we fix a large positive integer N such that

$$\Big[-m \sum_{i=N}^{\infty} (q^{-in} - p^{-in}), m \sum_{i=N}^{\infty} (q^{-in} - p^{-in}) \Big] \subset \Big[-m \sum_{i \in \mathbb{N} \setminus n\mathbb{N}} (q^{-in} - p^{-in}), m \sum_{i \in \mathbb{N} \setminus n\mathbb{N}} (q^{-in} - p^{-in}) \Big]. \tag{10}$$

This is possible because the right side interval contains 0 in its interior. The sets

$$B := \mathbb{N} \setminus n\mathbb{N},$$
$$C := \{in : i = N, N+1, \ldots\},$$
$$D := \{in : i = 1, \ldots, N-1\}$$

form a partition of $\mathbb{N}$.

Choose an arbitrary sequence $(c_i)_{i \in C} \in A^C$; there is a continuum of such sequences because C is an infinite set. Since

$$-\sum_{i \in C} c_i (q^{-i} - p^{-i})$$

belongs to the left side interval in (10), applying Proposition 3 there exists a sequence $(c_i)_{i \in B} \in A^B$ such that

$$\sum_{i \in B \cup C} c_i (q^{-i} - p^{-i}) = 0.$$

Setting $c_i = 0$ for $i \in D$ we obtain a sequence $(c_i)_{i \in \mathbb{N}} \in C(p,q)$. □

Proof of Theorem 1 (ii). We show that for each positive integer n there exists a sequence $(c_i) \in C(p,q)$, beginning with $c_1 = \cdots = c_{n-1} = 0$ and $c_n = -1$. Indeed, since $q \le m+1$, by Lemma 5 we have

$$0 < q^{-n} - p^{-n} < (q-1) \sum_{i=n+1}^{\infty} (q^{-i} - p^{-i}) \le m \sum_{i=n+1}^{\infty} (q^{-i} - p^{-i}).$$

Since $q \le 2m+1$, Lemma 5 also shows that the condition (5) of Proposition 3 is satisfied for the alphabet $A = \{-m, \cdots, m\}$ and the sequence $r_k := q^{-k-n} - p^{-k-n}$, $k = 1, 2, \ldots$. Hence there exists a sequence $(c_i)_{i=n+1}^{\infty} \in A^{\infty}$ satisfying

$$q^{-n} - p^{-n} = \sum_{i=n+1}^{\infty} c_i (q^{-i} - p^{-i});$$

setting $c_1 = \cdots = c_{n-1} = 0$ and $c_n = -1$ this yields (2). □

Proof of Theorem 1 (iii) (a). We show that there is a sequence $(c_i) \in C(p,q)$, beginning with $c_1 = -1$. Since $q \le 2m+1$, by Proposition 3 and Lemma 5 it is sufficient to show that

$$(0<) q^{-1} - p^{-1} \le m \sum_{i=2}^{\infty} (q^{-i} - p^{-i}).$$

By (8) this is equivalent to the inequality

$$\frac{1}{m} \le \frac{1}{q-1} + \frac{p-q}{(p-1)(q-1)} \frac{\frac{q}{p}}{1-\frac{q}{p}} = \frac{1}{q-1} + \frac{q}{(p-1)(q-1)},$$

i.e., to $p \le (m+1)(q-1)/(q-m-1)$. Indeed, since $m>0$, $q>1$ and $p>m+1$, we have

$$\begin{aligned} \frac{1}{m} \le \frac{1}{q-1} + \frac{q}{(p-1)(q-1)} &\iff (p-1)(q-1) \le m(p-1) + mq \\ &\iff p(q-m-1) \le (m+1)(q-1) \\ &\iff p \le \frac{(m+1)(q-1)}{q-m-1}. \end{aligned}$$

□

Remark 7. Now we prove our statement in Remark 2 (i). If $m+1 < q \le 2m+1$ and $p > q$ is closer to q so that

$$(0<) q^{-n} - p^{-n} \le m \sum_{i=n+1}^{\infty} (q^{-i} - p^{-i})$$

or equivalently (see (9))

$$\frac{1}{m} \le \frac{1}{q-1} + \left(\frac{1}{q-1} - \frac{1}{p-1}\right) \frac{q^n}{p^n - q^n}$$

for some positive integer n, then the adaptation of the preceding proof shows that for each $k = 1, \dots, n$ there exists a sequence $(c_i) \in C(p,q)$, beginning with $c_1 = \dots = c_{k-1} = 0$ and $c_k = -1$.

The right side of the above inequality is a decreasing function of p because the function

$$f(p) := \left(\frac{1}{q-1} - \frac{1}{p-1}\right) \frac{1}{p^n - q^n}$$

has a negative derivative for all $p > q$.

Indeed, we have

$$f'(p) = \frac{1}{(p-1)^2(p^n-q^n)} - \left(\frac{1}{q-1} - \frac{1}{p-1}\right)\frac{np^{n-1}}{(p^n-q^n)^2},$$

whence

$$\frac{(p-1)^2(p^n-q^n)^2}{p-q} f'(p) = \frac{p^n-q^n}{p-q} - np^{n-1}\frac{p-1}{q-1} < \frac{p^n-q^n}{p-q} - np^{n-1}.$$

We conclude by noticing that $\frac{p^n-q^n}{p-q} = nr^{n-1}$ by the Lagrange mean value theorem for some $q < r < p$ and therefore

$$\frac{p^n-q^n}{p-q} - np^{n-1} = n\left(r^{n-1} - p^{n-1}\right) \le 0.$$

Proof of Theorem 1 (iv) (a). Since $1/(q-1) < 1/2m$, by Lemma 5 we have

$$q^{-n} - p^{-n} > 2m \sum_{i=n+1}^{\infty} (q^{-i} - p^{-i})$$

for all sufficiently large integers n, say for all $n > N$.[1] This implies that if two different sequences $(c_i), (c_i') \in A^\infty$ satisfy $c_i = c_i'$ for $i = 1, \ldots, N$, then

$$\sum_{i=1}^{\infty} c_i(q^{-i} - p^{-i}) \ne \sum_{i=1}^{\infty} c_i'(q^{-i} - p^{-i}).$$

Indeed, if n is the first index for which $c_n \ne c_n'$, then $n > N$, and therefore

$$\begin{aligned}\left|\sum_{i=1}^{\infty}(c_i - c_i')(q^{-i} - p^{-i})\right| &\ge |c_n - c_n'|\,(q^{-n} - p^{-n}) - \sum_{i=n+1}^{\infty} |c_i - c_i'|\,(q^{-i} - p^{-i})\\ &\ge (q^{-n} - p^{-n}) - 2m\sum_{i=n+1}^{\infty}(q^{-i} - p^{-i}) > 0.\end{aligned}$$

It follows that if two different sequences $(c_i), (c_i') \in A^\infty$ satisfy

$$\sum_{i=1}^{\infty}\frac{c_i}{q^i} - \sum_{i=1}^{\infty}\frac{c_i}{p^i} = \sum_{i=1}^{\infty}\frac{c_i'}{q^i} - \sum_{i=1}^{\infty}\frac{c_i'}{p^i} = 0,$$

then already their beginning words $c_1 \ldots c_N$ and $c_1' \ldots c_N'$ must differ. We conclude that there are at most $(2m+1)^N$ sequences $(c_i) \in A^\infty$ satisfying (2). □

[1] If this inequality holds for some n, then it also holds for all larger integers by the monotonicity property of Lemma 5.

Proof of Theorem 1 (iv) (b). Thanks to (a) it is sufficient to exhibit a continuum of sequences $(c_i) \in A^\infty$ such that each sequence satisfies (2) for at least one base $p > q$.

Our assumption $q < m+1+\sqrt{m(m+1)}$ implies the inequality

$$\frac{1}{q^2} < m \sum_{i=2}^{\infty} \frac{i}{q^{i+1}}. \tag{11}$$

Indeed, differentiating the identity

$$\sum_{i=1}^{\infty} \frac{1}{q^i} = \frac{1}{q-1}$$

we get

$$\sum_{i=1}^{\infty} \frac{i}{q^{i+1}} = \frac{1}{(q-1)^2},$$

so that, since $m > 0$ and $q > 1$, (11) is equivalent to

$$\frac{m+1}{q^2} < \frac{m}{(q-1)^2}.$$

This inequality can be rewritten as

$$q^2 - 2q(m+1) + m + 1 < 0. \tag{12}$$

The polynomial $x^2 - 2x(m+1) + m + 1$ has exactly one root, which is larger than one, namely $x = m+1+\sqrt{m(m+1)}$. Thus (11) holds if and only if $q < m+1+\sqrt{m(m+1)}$.

In view of (11) we may choose a sufficiently large positive integer N such that

$$\frac{1}{q^2} < m \sum_{i=2}^{N} \frac{i}{q^{i+1}}. \tag{13}$$

Now fix an arbitrary sequence $(c_i) \in A^\infty$ satisfying

$$c_1 = -1, \quad c_2 = \cdots = c_N = m \quad \text{and} \quad c_i \ge 0 \text{ for all } i > N. \tag{14}$$

(There is a continuum of such sequences.) We are going to prove that (2) holds for at least one base $p > q$.

It is sufficient to show that

$$\sum_{i=1}^{\infty} c_i (q^{-i} - p^{-i}) < 0$$

if $p > q$ is large enough, and

$$\sum_{i=1}^{\infty} c_i(q^{-i} - p^{-i}) > 0$$

if $p > q$ is close enough to q. Indeed, then we will have equality for some intermediate value of p by continuity.

The first property will follow from the stronger relation

$$\lim_{p\to\infty} \sum_{i=1}^{\infty} c_i(q^{-i} - p^{-i}) < 0, \quad \text{i.e.,} \quad \sum_{i=1}^{\infty} \frac{c_i}{q^i} < 0.$$

The proof is straightforward: since $c_1 = -1$ and $q > m + 1$, we have

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} \le \frac{-1}{q} + \sum_{i=2}^{\infty} \frac{m}{q^i} = \frac{-1}{q} + \frac{m}{q(q-1)} < \frac{-1}{q} + \frac{1}{q} = 0.$$

Since $c_i \ge 0$ for all $i > N$, the second property is weaker than the inequality

$$\sum_{i=1}^{N} c_i(q^{-i} - p^{-i}) > 0$$

for all p with $p > q$ that are close enough, and this is weaker than the relation

$$\lim_{p\to q} \frac{1}{p-q} \sum_{i=1}^{N} c_i(q^{-i} - p^{-i}) > 0.$$

The last property follows by using (13) and (14):

$$\lim_{p\to q} \frac{1}{p-q} \sum_{i=1}^{N} c_i(q^{-i} - p^{-i}) = \sum_{i=1}^{N} \frac{ic_i}{q^{i+1}} = -\frac{1}{q^2} + m\sum_{i=2}^{N} \frac{i}{q^{i+1}} > 0. \qquad \square$$

Proof of Theorem 1 (iii) (b), (iv) (c) and (v). If $p > q > m + 1$ satisfy (4), then the proof of (iii) (a) shows that

$$q^{-1} - p^{-1} > m\sum_{i=2}^{\infty}(q^{-i} - p^{-i}).$$

Then by Lemma 5 we also have, more generally,

$$q^{-n} - p^{-n} > m \sum_{i=n+1}^{\infty} (q^{-i} - p^{-i})$$

for all positive integers n.

Now if a sequence $(c_i) \in A^\infty$ has a first nonzero term c_n, then

$$\left|\sum_{i=n+1}^{\infty} c_i(q^{-i}-p^{-i})\right| \le \sum_{i=n+1}^{\infty} m(q^{-i}-p^{-i}) < q^{-n}-p^{-n} \le \left|c_n(q^{-n}-p^{-n})\right|,$$

so that (2) cannot hold. This completes the proof of (iii) (b) and (iv) (c).

For the proof of (v) it remains to check that in case $q \ge m+1+\sqrt{m(m+1)}$ the condition (4) holds for all $p > q$. This is equivalent to

$$q \ge \frac{(m+1)(q-1)}{q-m-1},$$

which can be rewritten as

$$q^2 - 2q(m+1) + m + 1 \ge 0.$$

By our observation after (12) this inequality holds if and only if $q \ge m+1+\sqrt{m(m+1)}$. □

3. Open questions

(1) Find the optimal conditions on p and q in Theorem 1. In particular,
   (a) Can $C(p,q)$ be infinite for some $p > q > m+1$?
   (b) In case $2m+1 < q < m+1+\sqrt{m(m+1)}$ is $C(p,q)$ nontrivial for *all* $p > q$ sufficiently close to q?

(2) Construct an alphabet and three (or more) different bases such that a continuum of (or infinitely many) real numbers have identical expansions in all three bases.

(3) Given two bases $p > q > 1$ investigate the set of points of the form

$$\sum_{i=1}^{\infty} c_i(p^{-i}-q^{-i}), \quad (c_i) \in A^\infty.$$

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