Publ. Math. Debrecen
86/1-2 (2015), 1–17
DOI: 10.5486/PMD.2015.5911

Asymptotic behavior of solutions to neutral functional differential equations with infinite delay

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Abstract. In this paper, we first introduce some classes of μ-pseudo almost automorphic type functions and establish some properties of such functions. And then, we apply the obtained results to investigate the existence of μ-pseudo almost automorphic solutions to a first-order partial neutral functional differential equation with infinite delay.

1. Introduction

The concept of almost automorphy is an important generalization of the classical almost periodicity. It was introduced by Bochner [6], [7], for more details about this topic we refer the reader to [15], [16], [23], [24] and the references therein. Since then, almost automorphy has become one of the most attractive topics in the qualitative theory of evolution equations, and there have been several interesting, natural and powerful generalizations of the classical almost automorphic functions. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [22]. Liang, Xiao and Zhang in [20], [27] presented the concept of pseudo almost automorphy. In [25], N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to investigate the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation. Blot et al. introduced the

Mathematics Subject Classification: 45N05, 43A60, 34G20, 26A33.
Key words and phrases: μ-pseudo almost automorphic function of class p, μ-pseudo almost automorphic function of class infinity, functional differential equation with infinite delay.
This paper is supported by NSF of China (11361032), NCET-10-0022, and Program for Longyuan Youth Innovative Talents of Gansu Province of China (2014-4-80).

notion of weighted pseudo almost automorphic functions with values in a Banach space in [4], which generalizes that of pseudo-almost automorphic functions. Xia and Fan presented the notation of Stepanov-like (or S^p-) weighted pseudo almost automorphy in [26]. Zhang, Chang and N'Guérékata further investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [29], [31]. Recently, Blot, Cieutat and Ezzinbi in [5] applied the measure theory to define an ergodic function and they presented the concept of μ-pseudo almost automorphic functions, and thus the classical theory of pseudo almost automorphy becomes a particular case of their approach.

In recent years, the existence of pseudo almost periodic type or almost automorphic type solutions to some neutral differential equations has been considered in many publications such as [1], [2], [3], [8], [9], [10], [11], [12], [13], [14], [30]. Motivated by above mentioned works, the aim of this work is to introduce the notion of μ-pseudo almost automorphic functions of class p and the notion of μ-pseudo almost automorphic functions of class infinity, we first establish some basic results not only on the completeness of the space that consists of μ-pseudo almost automorphic functions of class p but also on the composition theorems of such functions. And finally, the previous results, are, subsequently utilized to investigate existence results of μ-pseudo almost automorphic solutions to the following first-order neutral functional-differential equations with infinite delay:

$$\frac{d}{dt}[u(t) + f(t, u_t)] = Au(t) + g(t, u_t), \tag{1.1}$$

where $(\mathbb{X}, \|\cdot\|)$ is a Banach space, $A : D(A) \subseteq \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of a uniformly exponentially stable semigroup of linear operators on $\mathbb{X}$, the history $u_t : (-\infty, 0] \to \mathbb{X}$ given by $u_t(\theta) := u(t + \theta)$, belongs to an abstract phase space $\mathcal{B}$ defined axiomatically, and $\mathcal{B}$ is a fading memory space, f, $g : \mathbb{R} \times \mathcal{B} \to \mathbb{X}$ are some suitable functions.

The rest of this paper is organized as follows. In Section 2, we introduce some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we first establish some composition theorems of μ-pseudo almost automorphic function of class p, and then we further investigate the existence of μ-pseudo almost automorphic mild solutions to the neutral functional differential equation (1.1).

2. Preliminaries

This section is devoted to some preliminary results needed in the sequel. In particular, to deal with infinite delay, we need to introduce some new classes of μ-pseudo almost automorphic function. Throughout the paper, the notation $(\mathbb{X}, \|\cdot\|)$ is a Banach space and $BC(\mathbb{R}, \mathbb{X})$ denotes the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$. From now on, $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ denotes the infinitesimal generator of an uniformly asymptotically stable semigroup of linear operators $(T(t))_{t\geq 0}$ on $\mathbb{X}$ and $\overline{M}$, ω are positive constants such that

$$\|T(t)\| \leq \overline{M} e^{-\omega t}$$

for each $t \geq 0$.

We denote by $\mathfrak{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measure μ on $\mathfrak{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbb{R} (a < b)$. For $\mu \in \mathcal{M}$ and $\tau \in \mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R}, \mathfrak{B})$ defined by

$$\mu_\tau(\mathcal{A}) = \mu(a + \tau : a \in \mathcal{A}) \quad \text{for } \mathcal{A} \in \mathfrak{B}.$$

From $\mu \in \mathcal{M}$, we suppose the following hypothesis ([5]) holds throughout this paper.

(H0) For all $\tau \in \mathbb{R}$, there exist $\gamma > 0$ and a bounded interval I such that

$$\mu_\tau(\mathcal{A}) \leq \gamma \mu(\mathcal{A}),$$

when $\mathcal{A} \in \mathfrak{B}$ satisfies $\mathcal{A} \bigcap I = \emptyset$.

Definition 2.1 ([24]). A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{R}, \mathbb{X})$.

Definition 2.2 ([20], [24]). A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in \mathbb{B}$, where $\mathbb{B}$ is any bounded subset of $\mathbb{X}$. The collection of all such functions will be denoted by $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

The notation $PAA_0(\mathbb{X})$ stands for the space of functions

$$PAA_0(\mathbb{R},\mathbb{X}) = \left\{\phi \in BC(\mathbb{R},\mathbb{X}) : \lim_{m\to\infty}\frac{1}{2m}\int_{-m}^{m}\|\phi(t)\|dt = 0\right\}.$$

To study issues related to delay we need to introduce the new space of functions defined for each $p>0$ by

$$PAA_0(\mathbb{R},\mathbb{X},p) := \left\{\phi \in BC(\mathbb{R},\mathbb{X}) : \lim_{m\to\infty}\frac{1}{2m}\int_{-m}^{m}\left(\sup_{\theta\in[t-p,t]}\|\phi(\theta)\|\right)dt = 0\right\}.$$

In addition to the above-mentioned spaces, the present setting requires the introduction of the following function spaces

$$PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X}) = \left\{\phi \in BC(\mathbb{R}\times\mathbb{X},\mathbb{X}) : \lim_{m\to\infty}\frac{1}{2m}\int_{-m}^{m}\|\phi(t,x)\|dt = 0\right\}$$

and

$$PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},p) := \left\{\phi \in BC(\mathbb{R}\times\mathbb{X},\mathbb{X}) : \lim_{m\to\infty}\frac{1}{2m}\times\int_{-m}^{m}\left(\sup_{\theta\in[t-p,t]}\|\phi(\theta,x)\|\right)dt = 0\right\}.$$

Definition 2.3 ([21], [28]). A continuous function $f:\mathbb{R}\to\mathbb{X}$ (respectively $\mathbb{R}\times\mathbb{X}\to\mathbb{X}$) is called pseudo-almost automorphic if it can be decomposed as $f=g+\phi$, where $g\in AA(\mathbb{R},\mathbb{X})$(respectively $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) and $\phi\in PAA_0(\mathbb{R},\mathbb{X})$(respectively $PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$). Denote by $PAA(\mathbb{R},\mathbb{X})$ (respectively $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) the set of all such functions.

Definition 2.4 ([5]). Let $\mu\in\mathcal{M}$. A bounded continuous function $f:\mathbb{R}\to\mathbb{X}$ is said to be μ-ergodic if

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|f(t)\|d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R},\mathbb{X},\mu)$.

Similarly, we denote ergodic functions related to delay by

$$\varepsilon(\mathbb{R},\mathbb{X},\mu,p) := \left\{f\in BC(\mathbb{R},\mathbb{X}) : \lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\times\int_{[-r,r]}\left(\sup_{\theta\in[t-p,t]}\|f(\theta)\|\right)d\mu(t) = 0\right\};$$

$$\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p) := \Big\{ f\in BC(\mathbb{R}\times\mathbb{X},\mathbb{X}) : \lim_{r\to+\infty}\frac{1}{\mu([-r,r])} \times \int_{[-r,r]}\Big(\sup_{\theta\in[t-p,t]}\|f(\theta,z)\|\Big)d\mu(t)=0\Big\}.$$

In view of the previous definitions, it is clear that $\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$ and $\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$ are continuously embedded into $\varepsilon(\mathbb{R},\mathbb{X},\mu)$ and $\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$, respectively. Furthermore, it is not hard to see that $\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$ and $\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$ are closed in $\varepsilon(\mathbb{R},\mathbb{X},\mu)$ and $\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$. Consequently, by $PAA(\mathbb{R},\mathbb{X},\mu)$ respectively, $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)) \subset BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{X},\mathbb{X})$), one obtains the following result from [5, Proposition 2.13.]:

Lemma 2.1. *Let $\mu\in\mathcal{M}$, then the spaces $\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$ and $\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$ endowed with the uniform convergence topology are Banach spaces.*

Definition 2.5 ([5]). Let $\mu\in\mathcal{M}$. A continuous function $f:\mathbb{R}\to\mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form: $f=g+\phi$, where $g\in AA(\mathbb{R},\mathbb{X})$ and $\phi\in\varepsilon(\mathbb{R},\mathbb{X},\mu)$. We denote the space of all such functions by $PAA(\mathbb{R},\mathbb{X},\mu)$.

Obviously, we have $AA(\mathbb{R},\mathbb{X})\subseteq PAA(\mathbb{R},\mathbb{X},\mu)\subseteq BC(\mathbb{R},\mathbb{X})$.

We now introduce the following new classes of μ-pseudo almost automorphic functions.

Definition 2.6. Let $\mu\in\mathcal{M}$. A continuous function $f:\mathbb{R}\to\mathbb{X}$ is said to be μ-pseudo almost automorphic of class p if f is written in the form: $f=g+\phi$, where $g\in AA(\mathbb{R},\mathbb{X})$ and $\phi\in\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$. We denote the space of all such functions by $PAA(\mathbb{R},\mathbb{X},\mu,p)$.

Definition 2.7. Let $\mu\in\mathcal{M}$. A continuous function $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is said to be μ-pseudo almost automorphic of class p if f is written in the form: $f=g+\phi$, where $g\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $\phi\in\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$. We denote the space of all such functions by $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$.

To deal with infinite delays, we need to introduce the following new spaces of functions:

$$\varepsilon(\mathbb{R},\mathbb{X},\mu,\infty) := \bigcap_{p\geq 0}\varepsilon(\mathbb{R},\mathbb{X},\mu,p),$$

$$\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,\infty) := \bigcap_{p\geq 0}\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p).$$

Obviously, $\varepsilon(\mathbb{R},\mathbb{X},\mu,\infty)$ and $\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,\infty)$ are respectively closed subspaces of $\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$ and $\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$, and hence both are Banach spaces.

Definition 2.8. Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic of class infinity if f is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, \infty)$. We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$.

Definition 2.9. Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic of class infinity if f is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu, \infty)$. We denote the space of all such functions by $PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu, \infty)$.

In this work we will defined the phase space $\mathcal{B}$ axiomatically, using ideas and notation developed in [19](see also [18]). More precisely, $\mathcal{B}$ will denote the vector space of functions defined from $(-\infty, 0]$ into $\mathbb{X}$ endowed with a seminorm denoted $\|\cdot\|_{\mathcal{B}}$ and such that the following axioms hold:

A. If $x : (-\infty, \sigma + b) \to \mathbb{X}$ with $b > 0$, is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathcal{B}$, then for each $t \in [\sigma, \sigma + b)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\|x(t)\| \leq \mathcal{H}\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - s)\|x_\sigma\|_{\mathcal{B}}$,

where $\mathcal{H} > 0$ is a constant, and K, $M : [0, \infty) \to [1, \infty)$ are functions such that $K(\cdot)$ and $M(\cdot)$ are respectively continuous and locally bounded, and K, $\mathcal{H}$, M are independent of $x(\cdot)$.

A1. If $x(\cdot)$ is a function as in (A), then x_t is a $\mathcal{B}$-valued continuous on $[\sigma, \sigma + b)$.

B. The space $\mathcal{B}$ is complete.

C2. If $(\varphi^n)_{n \in \mathbb{N}}$ is a sequence of continuous functions with compact support defined from $(-\infty, 0]$ into $\mathbb{X}$, which converges to φ uniformly on compact subsets of $(-\infty, 0]$, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Remark 2.1 ([18]). Throughout the rest of the paper, $\mathfrak{L}$ denotes a constant such that $\|\varphi\|_{\mathcal{B}} \leq \mathfrak{L} \cdot \sup_{\theta \leq 0} \|\varphi(\theta)\|$ for every $\varphi \in BC((-\infty, 0]; \mathbb{X})$, see [19, Proposition 7.1.1].

Definition 2.10 ([13]). Let $S(t) : \mathcal{B} \mapsto \mathcal{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0)$ and $S(t)\varphi(\theta) = \varphi(t + \theta)$ on $(-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory if $\|S(t)\varphi\|_{\mathcal{B}} \to 0$ as $t \to \infty$ for each $\varphi \in \mathcal{B}$ with $\varphi(0) = 0$.

Remark 2.2 ([18]). In this paper we suppose that there exists a constant $\mathcal{K} > 0$ such that $\max\{K(t), M(t)\} \leq \mathcal{K}$ for each $t \geq 0$. Observe that this condition is verified, for example, if $\mathcal{B}$ is a fading memory, see, e.g., [19, Proposition 7.1.5].

We give the following basic assumptions:
(H1) The function $s \mapsto T(s)x$ belongs to $C([0,\infty),\mathbb{X})$ for each $x \in \mathbb{X}$. Moreover, the function $s \to AT(s)$ defined from $(0,\infty)$ into $\mathcal{L}(\mathbb{R},\mathbb{X})$ is strongly measurable, and there exists a non-decreasing function $H : [0,\infty) \mapsto [0,\infty)$ and $\delta > 0$ such that $e^{-\delta s}H(s) \in L^1([0,\infty))$ with $\|AT(s)\|_{\mathcal{L}(\mathbb{R},\mathbb{X})} \le e^{-\delta s}H(s)$ for each $s > 0$.
(H2) $\mathcal{B}$ is a uniform fading memory space, $f, g \in PAA(\mathbb{R}\times\mathcal{B},\mathbb{X},\mu,\infty)$ and there are continuous and bounded functions L_f, $L_g : \mathbb{R} \mapsto [0,\infty)$ such that

$$\|f(t,u) - f(t,v)\| \le L_f(t)\|u - v\|_{\mathcal{B}},$$

and

$$\|g(t,u) - g(t,v)\| \le L_g(t)\|u - v\|_{\mathcal{B}},$$

for all u, $v \in \mathcal{B}$ and $t \in \mathbb{R}$.

3. Main results

In this section, we first prove some composition theorems for μ-pseudo almost automorphic functions of class p, and then apply these theorems to investigate some existence results for the problem (1.1).

3.1. Composition theorems for μ-pseudo almost automorphic functions of class p.

Theorem 3.1 (). *Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$ with $g \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $h \in \varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$. Assume that the following condition* (i) *and* (ii) *are satisfied:*

(i) *$f(t,x)$ satisfies a Lipschitz condition in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$, that is, there exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le L\|x - y\|$$

for all x, $y \in \mathbb{X}$ and $t \in \mathbb{R}$.

(ii) *$g(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

If $u = u_1 + u_2 \in PAA(\mathbb{R},\mathbb{X},\mu,p)$ with $u_1 \in AA(\mathbb{R},\mathbb{X})$, $u_2 \in \varepsilon(\mathbb{R},\mathbb{X},\mu,p)$. Then the function $f(\cdot,u(\cdot))$ belongs to $PAA(\mathbb{R},\mathbb{X},\mu,p)$.

Proof. Since $f \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$ and $u \in PAA(\mathbb{R},\mathbb{X},\mu,p)$, we have by definition that $f = g + h$ and $u = u_1 + u_2$, where $g \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $h \in$

$\varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$, $u_1\in AA(\mathbb{R},\mathbb{X})$ and $u_2\in\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$. The function f can be decomposed as

$$\begin{aligned} f(t,u(t)) &= g(t,u_1(t)) + f(t,u(t)) - g(t,u_1(t)) \\ &= g(t,u_1(t)) + f(t,u(t)) - f(t,u_1(t)) + h(t,u_1(t)). \end{aligned}$$

Define

$$G(t)=g(t,u_1(t)),\ F(t)=f(t,u(t))-f(t,u_1(t)),\ H(t)=h(t,u_1(t)).$$

Then $f(t,u(t))=G(t)+F(t)+H(t)$. Since the function g satisfies the condition (ii), it follows [20, Lemma 2.2] that the function $g(\cdot,u_1(\cdot))\in AA(\mathbb{R},\mathbb{X})$. To show that $f(\cdot,u(\cdot))\in PAA(\mathbb{R},\mathbb{X},\mu,p)$, it is sufficient to show that $F+H\in\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$.

Initially, we prove that $F\in\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$. Clearly, $f(t,u(t))-f(t,u_1(t))$ is bounded and continuous. Now, by (i), we have

$$\|f(t,u(t))-f(t,u_1(t))\|\le L\|u(t)-u_1(t)\|\le L\|u_2(t)\|.$$

Hence, by the fact that $u_2\in\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$, we obtain

$$\begin{aligned} &\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-p,t]}\|F(\theta)\|\right)d\mu(t) \\ &\qquad\le \lim_{r\to+\infty}\frac{L}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-p,t]}\|u_2(\theta)\|\right)d\mu(t)=0, \end{aligned}$$

which shows that $F(\cdot)\in\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$.

Next, we show that $H\in\varepsilon(\mathbb{R},\mathbb{X},\mu,p)$. Since $u(t)$, $u_1(t)$ are bounded, we can choose a bounded subset $B\in\mathbb{X}$ such that $u(\mathbb{R})$, $u_1(\mathbb{R})\subseteq B$. Since g satisfies the condition (ii), then for any $\epsilon>0$, there exits a $\delta>0$ such that x, $y\in B$ and $\|x-y\|\le\delta$ imply that $\|g(t,x)-g(t,y)\|\le\epsilon$ for all $t\in\mathbb{R}$.

Put $\delta_0=\min\{\epsilon,\delta\}$, then

$$\|h(t,x)-h(t,y)\|\le\|f(t,x)-f(t,y)\|+\|g(t,x)-g(t,y)\|\le(L+1)\epsilon.$$

for all x, $y\in B$ with $\|x-y\|\le\delta_0$.

Set $I=u_1([-r,r])$. Then I is compact in $\mathbb{R}$ since the image of a compact set under a continuous mapping is compact, and so one can find finite open balls O_k, $(k=1,2,\ldots,m)$ with center $x_k\in I$ and radius δ small enough such that $I\subseteq\bigcup_{k=1}^m O_k$ and

$$\|h(t,u_1(t))-h(t,x_k)\|\le(L+1)\epsilon,\quad u_1(t)\in O_k,\ t\in[-r,r].$$

Suppose $\|h(t,x_q)\| \leq \max_{1\leq k\leq m}\|h(t,x_k)\|$, where q is an index number among $\{1,2,\dots,m\}$. The set $B_k = \{t \in [-r,r] : u_1(t) \in O_k\}$ is open in $[-r,r]$ and $[-r,r] = \bigcup_{k=1}^m B_k$. Let

$$E_1 = B_1, \quad E_k = B_k \setminus \bigcup_{j=1}^{k-1} B_j \qquad (2 \leq k \leq m).$$

Then $E_i \cap E_j = \emptyset$ when $i \neq j$, $1 \leq i,j \leq m$. Observe that

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-p,t]}\|h(\theta,u_1(\theta))\|\right)d\mu(t)\\
&= \frac{1}{\mu([-r,r])}\int_{\bigcup_{k=1}^m E_k}\left(\sup_{\theta\in[t-p,t]}\|h(\theta,u_1(\theta))\|\right)d\mu(t)\\
&\leq \frac{1}{\mu([-r,r])}\sum_{k=1}^m\int_{E_k}\left(\sup_{\theta\in[t-p,t]}(\|h(\theta,u_1(\theta))-h(\theta,x_k)\|+\|h(\theta,x_k)\|)\right)d\mu(t)\\
&\leq \frac{1}{\mu([-r,r])}\sum_{k=1}^m\int_{E_k}(L+1)\epsilon d\mu(t) + \frac{1}{\mu([-r,r])}\sum_{k=1}^m\int_{E_k}\left(\sup_{\theta\in[t-p,t]}\|h(\theta,x_k)\|\right)d\mu(t)\\
&\leq (L+1)\epsilon + \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-p,t]}\|h(\theta,x_q)\|\right)d\mu(t).
\end{aligned}$$

Using the same arguments as above, we obtain

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-p,t]}\|h(\theta,u_1(\theta))\|\right)d\mu(t) = 0.$$

That is, $h(t,u_1(t)) \in \varepsilon(\mathbb{R},\mathbb{X},\mu,p)$. Hence $f(t,u(t)) \in PAA(\mathbb{R},\mathbb{X},\mu,p)$, which ends the proof. $\square$

Theorem 3.2. *Let $\mu \in \mathcal{M}$, let $F \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu,p)$ and $h \in PAA(\mathbb{R},\mathbb{X},\mu,p)$. Assume that there exists a function $L_F : \mathbb{R} \mapsto [0,\infty)$ satisfying*

$$\|F(t,x_1) - F(t,x_2)\|_{\mathbb{X}} \leq L_F(t)\|x_1 - x_2\|_{\mathbb{X}}, \quad \forall\, t \in \mathbb{R},\ \forall\, x_1,\ x_2 \in \mathbb{X}. \tag{3.1}$$

If

$$\limsup_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-p,t]}L_F(\theta)\right)d\mu(t) < \infty \tag{3.2}$$

and

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{\theta\in[t-p,t]}L_F(\theta)\right)\xi(t)d\mu(t) = 0 \tag{3.3}$$

for each $\xi \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$, then the function $t \mapsto F(t,h(t)) \in PAA(\mathbb{R},\mathbb{X},\mu,p)$.

Proof. Assume that $F = F_1 + \varphi$, $h = h_1 + h_2$, where $F_1 \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\varphi \in \varepsilon(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu, p)$, $h_1 \in AA(\mathbb{R}, \mathbb{X})$ and $h_2 \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, p)$. Consider the decomposition

$$F(t, h(t)) = F_1(t, h_1(t)) + [F(t, h(t)) - F(t, h_1(t))] + \varphi(t, h_1(t)),$$

In view of [20, Lemma 2.2], $F_1(t, h_1(t)) \in AA(\mathbb{R}, \mathbb{X})$, it remains to prove that both $[F(t, h(t)) - F(t, h_1(t))]$ and $\varphi(t, h_1(t))$ belong to $\varepsilon(\mathbb{R}, \mathbb{X}, \mu, p)$. Indeed, by (3.1) it follows that

$$\begin{aligned}
&\frac{1}{\mu([-r, r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-p,t]} \|F(\theta, h(\theta)) - F(\theta, h_1(\theta))\| \right) d\mu(t) \\
&\quad \leq \frac{1}{\mu([-r, r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-p,t]} L_F(\theta) \|h_2(\theta)\| \right) d\mu(t) \\
&\quad \leq \frac{1}{\mu([-r, r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-p,t]} L_F(\theta) \right) \cdot \left(\sup_{\theta \in [t-p,t]} \|h_2(\theta)\| \right) d\mu(t),
\end{aligned}$$

which implies that $[F(t, h(t)) - F(t, h_1(t))] \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, p)$ from (3.3).

Since $h_1(\mathbb{R})$ is relatively compact in $\mathbb{X}$ and F_1 is uniformly continuous on sets of the form $\mathbb{R} \times K$ where $K \subseteq \mathbb{X}$ is compact subset, for $\epsilon > 0$, there exists $\delta \in (0, \epsilon)$ such that

$$\|F_1(t, x) - F_1(t, \overline{x})\| \leq \epsilon, \quad x, \ \overline{x} \in h_1(\mathbb{R}),$$

with $\|x - \overline{x}\| < \delta$.

Now, fix $x_1, \ldots, x_n \in h_1(\mathbb{R})$ such that $h_1(\mathbb{R}) \subseteq \bigcup_{i=1}^n B_\delta(x_i, \mathbb{X})$. Obviously, the sets $E_i = h_1^{-1}(B_\delta(x_i))$ form an open covering of $\mathbb{R}$, and therefore using the sets $B_1 = E_1$, $B_2 = E_2 \setminus E_1$ and $B_i = E_i \setminus \bigcup_{j=1}^{i-1} E_j$, one obtains a coverage of $\mathbb{R}$ by disjoint open sets.

For $t \in B_i$, $h_1(t) \in B_\delta(x_i)$,

$$\begin{aligned}
\|\varphi(t, h_1(t))\| &\leq \|F(t, h_1(t)) - F(t, x_i)\| + \| - F(t, h_1(t)) + F_1(t, x_i)\| + \|\varphi(t, x_i)\| \\
&\leq L_F(t)\|h_1(t) - x_i\| + \epsilon + \|\varphi(t, x_i)\| \leq L_F(t)\epsilon + \epsilon + \|\varphi(t, x_i)\|.
\end{aligned}$$

Now using the previous inequality it follows that

$$\frac{1}{\mu([-r, r])} \int\limits_{[-r,r]} \left(\sup_{\theta \in [t-p,t]} \|\varphi(t, h_1(t))\| \right) d\mu(t)$$

$$\leq \frac{1}{\mu([-r,r])} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{\theta \in [t-p,t]} \|\varphi(\theta, h_1(\theta))\| \right) d\mu(t)$$

$$\leq \frac{1}{\mu([-r,r])} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{j=1,\dots,n} \left[\sup_{\theta \in [t-p,t] \cap B_j} \|\varphi(\theta, h_1(\theta))\| \right] \right) d\mu(t)$$

$$\leq \frac{1}{\mu([-r,r])} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{j=1,\dots,n} \left[\sup_{\theta \in [t-p,t] \cap B_j} \|F(\theta, h_1(\theta)) - F(\theta, x_j)\| \right] \right) d\mu(t)$$

$$+ \frac{1}{\mu([-r,r])} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{j=1,\dots,n} \left[\sup_{\theta \in [t-p,t] \cap B_j} \|F_1(\theta, h_1(\theta)) - F_1(\theta, x_j)\| \right] \right) d\mu(t)$$

$$+ \frac{1}{\mu([-r,r])} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{j=1,\dots,n} \left[\sup_{\theta \in [t-p,t] \cap B_j} \|\varphi(\theta, x_j)\| \right] \right) d\mu(t)$$

$$\leq \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-p,t]} L_F(\theta)\epsilon + \epsilon \right) d\mu(t)$$

$$+ \sum_{i=1}^{n} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-p,t]} \|\varphi(\theta, x_j)\| \right) d\mu(t).$$

In view of the above it is clear that $\varphi(t, h_1(t))$ belongs to $\varepsilon(\mathbb{R}, \mathbb{X}, \mu, p)$. Hence, $F(t, h(t)) \in PAA(\mathbb{R}, \mathbb{X}, \mu, p)$. This completes the proof. □

Lemma 3.1. *Let $\mu \in \mathcal{M}$, let $u \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$. Assume that $\mathcal{B}$ is a uniform fading memory space. Then the function $s \mapsto u_s$ belongs to $PAA(\mathbb{R}, \mathcal{B}, \mu, \infty)$.*

Proof. Assume that $u = \phi + \varphi$ where $\phi \in AA(\mathbb{R}, \mathbb{X})$ and $\varphi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, \infty)$. Clearly, $u_s = \phi_s + \varphi_s$.

First, we need to prove that $\phi_s \in AA(\mathcal{B})$. For a given sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$ and a function $v \in BC(\mathbb{R}, \mathbb{X})$ such that $u(s + s_n) \to v(s)$ for each $s \in \mathbb{R}$. Since $\mathcal{B}$ satisfies axiom C2, from [19, Proposition 7.1.1], we infer that $u_{s+s_n} \to v_s$ in $\mathcal{B}$ for each $s \in \mathbb{R}$. Let $L > 0$, for $\epsilon > 0$, fix $N_{\epsilon,L} \in \mathbb{N}$ such that

$$\|u(s + s_n) - v(s)\| \leq \epsilon, \qquad \|u_{-L+s_n} - v_{-L}\| \leq \epsilon,$$

whenever $n \geq N_{\epsilon,L}$. In view of the above, for $t \in \mathbb{R}$ and $n \geq N_{\epsilon,L}$ we get

$$\|u_{t+s_n} - v_t\|_{\mathcal{B}} \leq M(L+t)\|u_{-L+s_n} - v_{-L}\|_{\mathcal{B}} + K(L+t) \cdot \sup_{\theta \in [-L,L]} \|u(\theta + s_n) - v(\theta)\| \leq 2\mathcal{K}\epsilon,$$

where $\mathcal{K}$ is the constant appearing in Remark 2.2.
In view of the above, u_{t+s_n} converges to v_t for each $t \in \mathbb{R}$. Similarly, one can prove that v_{t-s_n} converges to u_t for each $t \in \mathbb{R}$. Thus, $\phi_s \in AA(\mathcal{B})$.

Now, we shall prove that $\varphi_s \in \varepsilon(\mathbb{R}, \mathcal{B}, \mu, \infty)$. Let $p > 0$ and $\epsilon > 0$, since $\mathcal{B}$ is a uniform fading memory space, from [10, Remark 2.14], we know that there is $\sigma_\epsilon > p$ such that $M(\sigma) < \epsilon$ for every $\sigma < \sigma_\epsilon$. Under these conditions, for $r > 0$ and $\sigma < \sigma_\epsilon$ we find that

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]} \sup_{\theta\in[t-p,t]} \|\varphi_\theta\|_{\mathcal{B}} d\mu(t) \\
&\leq \frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\sup_{\theta\in[t-p,t]} M(\sigma)\|\varphi_{\theta-\sigma}\|_{\mathcal{B}} + K(\sigma) \sup_{s\in[\theta-\sigma,\theta]} \|\varphi(s)\| \right) d\mu(t) \\
&\leq \mathfrak{L}\cdot \|\varphi\|_\infty \cdot \epsilon + \frac{\mathcal{K}}{\mu([-r,r])}\int_{[-r,r]} \sup_{s\in[\theta-\sigma,\theta]} \|\varphi(s)\| d\mu(t),
\end{aligned}$$

which enables to complete the proof as ϵ is arbitrary and $\varphi \in \varepsilon(\mathbb{R}, \mathcal{B}, \mu, \sigma)$. □

Corollary 3.1. *Let $\mu \in \mathcal{M}$, $f \in PAA(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu, \infty)$ and $u \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$. Assume that the conditions of Theorem 3.1 are satisfied for every $p > 0$, then the function $t \mapsto f(t, u(t))$ belongs to $PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$.*

3.2. Existence of μ-pseudo almost automorphic solutions to (1.1).

Definition 3.1. A continuous function $u : [\sigma, \sigma + a) \to \mathbb{X}$, $a > 0$ is called a mild solution for the neutral system (1.1) on $[\sigma, \sigma + a)$, if $u_s \in \mathcal{B}$ for every $s \in \mathbb{R}$, the function $s \to AT(t-s)f(s, u_s)$ is integrable on $[\sigma, t)$ for every $\sigma < t < \sigma + a$, and

$$\begin{aligned}
u(t) = T(t-\sigma)(\varphi(0) + f(\sigma, \varphi)) - f(t, u_t) - \int_\sigma^t AT(t-s)f(s, u_s)ds \\
+ \int_\sigma^t T(t-s)g(s, u_s)ds
\end{aligned}$$

Lemma 3.2. *Let $\mu \in \mathcal{M}$, let $u \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$. Under assumptions* (H1)*, if w is the function defined by $w(t) := \int_{-\infty}^t AT(t-s)u(s)ds$, $\forall t \in \mathbb{R}$, then $w \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$.*

Proof. Since $u \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$, we have $u = u_1 + u_2$ with $u_1 \in AA(\mathbb{R}, \mathbb{X})$, $u_2 \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, \infty)$ such that

$$w(t) = \int_{-\infty}^t AT(t-s)u_1(s)ds + \int_{-\infty}^t AT(t-s)u_2(s)ds.$$

Denote $\Phi(t) = \int_{-\infty}^{t} AT(t-s)u_1(s)ds$, $\Psi(t) = \int_{-\infty}^{t} AT(t-s)u_2(s)ds$ for each $t \in \mathbb{R}$. In order to prove $w \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$, we only need to verify $\Phi(t) \in AA(\mathbb{R}, \mathbb{X})$ and $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, \infty)$.

First, let us prove that $\Phi(t) \in AA(\mathbb{R}, \mathbb{X})$. Since $u_1(s) \in AA(\mathbb{R}, \mathbb{X})$, for a given sequence $(\sigma_n)_{n\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and a continuous function $v \in BC(\mathbb{R}, \mathbb{X})$ such that $u_1(t+s_n)$ converges to $v(t)$ in $\mathbb{X}$, and $v(t-s_n)$ converges to $u_1(t)$ in $\mathbb{X}$ for each $t \in \mathbb{R}$. From the Bochner's criterion related to integrable functions and the estimation

$$\|AT(t-s)u_1(s)\|_{\mathcal{L}(\mathbb{R},\mathbb{X})} = \|AT(t-s)\|_{\mathcal{L}(\mathbb{R},\mathbb{X})}\|u_1(s)\|_{\mathbb{X}} \ \le e^{-\delta(t-s)}H(t-s)\|u_1(s)\|_{\mathbb{X}}$$

it follows that the function $s \mapsto AT(t-s)u_1(s)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$. Furthermore, since

$$\begin{aligned} w(t+s_n) &= \int_{-\infty}^{t+s_n} AT(t+s_n-s)u_1(s)ds \\ &= \int_{-\infty}^{t} AT(t-s)u_1(t+s_n)ds, \quad t \in \mathbb{R},\ n \in \mathbb{N}. \end{aligned}$$

Using the above estimation and the Lebesgue dominated convergence theorem, it follows that $w(t+s_n)$ converges to $z(t) = \int_{-\infty}^{t} AT(t-s)v(s)ds$ for each $t \in \mathbb{R}$. Proceeding as previously, one can similarly prove that $z(t-s_n)$ converges to $w(t)$ for each $t \in \mathbb{R}$.

Next, we shall prove that $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu, \infty)$. By using the notation $H_p = \sup_{s\ge p} H(s)$, for positive numbers p, r we find that

$$\begin{aligned} &\frac{1}{\mu([-r,r])}\int_{[-r,r]} \sup_{\theta\in[t-p,t]} \|\Psi(\theta)\| d\mu(t) \\ &\le \frac{1}{\mu([-r,r])}\int_{[-r,r]} \sup_{\theta\in[t-p,t]} \int_{-\infty}^{\theta} \|AT(\theta-s)u_2(s)\| ds d\mu(t) \\ &\le \frac{1}{\mu([-r,r])}\int_{[-r,r]} \sup_{\theta\in[t-p,t]} \int_{-\infty}^{\theta} e^{-\delta(\theta-s)}H(\theta-s)\|u_2(s)\| ds d\mu(t) \\ &\le \frac{1}{\mu([-r,r])}\int_{[-r,r]} \sup_{\theta\in[t-p,t]} \int_{-\infty}^{\theta-p} e^{-\delta(\theta-s)}H(\theta-s)\|u_2(s)\| ds d\mu(t) \\ &\quad + \frac{1}{\mu([-r,r])}\int_{[-r,r]} \sup_{\theta\in[t-p,t]} \int_{\theta-p}^{\theta} e^{-\delta(\theta-s)}H(\theta-s)\|u_2(s)\| ds d\mu(t) \\ &\le \frac{H_p}{\mu([-r,r])}\int_{[-r,r]} \sup_{\theta\in[t-p,t]} \int_{-\infty}^{\theta-p} e^{-\delta(\theta-s)}\|u_2(s)\| ds d\mu(t) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-p,t]}\int_{\theta-p}^{\theta}H(\theta-s)\sup_{s\in[\theta-p,\theta]}\|u_2(s)\|dsd\mu(t)\\
&\le\frac{H_pe^{\delta p}}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-p,t]}\int_{-\infty}^{\theta-p}e^{-\delta(t-s)}\|u_2(s)\|dsd\mu(t)\\
&+\int_0^p H(s)ds\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-p,t]}\|u_2(\theta)\|d\mu(t)\\
&\le\frac{H_pe^{\delta p}}{\mu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t}e^{-\delta(t-s)}\|u_2(s)\|dsd\mu(t)\\
&+\int_0^p H(s)ds\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-p,t]}\|u_2(\theta)\|d\mu(t)\\
&\le\frac{H_pe^{\delta p}\|u_2\|_\infty}{\delta}+\int_0^p H(s)ds\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sup_{\theta\in[t-p,t]}\|u_2(\theta)\|d\mu(t),
\end{aligned}
$$

which enables to complete the proof as $u_2\in\varepsilon(\mathbb{R},\mathbb{X},\mu,\infty)$. □

Using the similar steps as in the proof of Lemma 3.2, one obtains the following result:

Lemma 3.3. *Let $\mu\in\mathcal{M}$, and let $u\in PAA(\mathbb{R},\mathbb{X},\mu,\infty)$, if w is the function defined by $w(t):=\int_{-\infty}^{t}T(t-s)u(s)ds$, $\forall t\in\mathbb{R}$, then $w\in PAA(\mathbb{R},\mathbb{X},\mu,\infty)$.*

Theorem 3.3. *Let $\mu\in\mathcal{M}$. Under assumption (H1)-(H2), there exists a unique μ-pseudo almost automorphic mild solution to* (1.1) *whenever*

$$
\overline{\theta}:=\left(L_f+\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\delta(t-s)}H(t-s)L_f(s)ds+\overline{M}\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\omega(t-s)}L_g(s)ds\right)\mathfrak{L}<1,
$$

where $L_f=\sup_{t\in\mathbb{R}}L_f(t)$ and $\mathfrak{L}$ is the constant appearing in Remark 2.1.

Proof. Let $\Gamma : PAA(\mathbb{R},\mathbb{X},\mu,\infty)\to BC(\mathbb{R},\mathbb{X})$ be the nonlinear operator defined by

$$
\Gamma u(t):=-f(t,u_t)-\int_{-\infty}^{t}AT(t-s)f(s,u_s)ds+\int_{-\infty}^{t}T(t-s)g(s,u_s)ds,\quad t\in\mathbb{R}.
$$

It is easy to see that Γu ia well defined and continuous. Moreover, from Theorem 3.1, Lemma 3.1 and Corollary 3.1 we obtain $f(t,u_t)\in PAA(\mathbb{R},\mathbb{X},\mu,\infty)$. Furthermore, from Lemma 3.2 and Lemma 3.3, we can infer that $\int_{-\infty}^{t}AT(t-s)f(s,u_s)ds$, $\int_{-\infty}^{t}T(t-s)g(s,u_s)ds\in PAA(\mathbb{R},\mathbb{X},\mu,\infty)$. That is Γ maps $PAA(\mathbb{R},\mathbb{X},\mu,\infty)$ into $PAA(\mathbb{R},\mathbb{X},\mu,\infty)$.

On the other hand, for u, $v \in PAA(\mathbb{R}, \mathbb{X}, \mu, \infty)$ we get

$$\begin{aligned}\|\Gamma u(t) - \Gamma v(t)\| &\leq L_f(t)\|u_t - v_t\|_{\mathcal{B}} + \int_{-\infty}^{t} L_f(s)e^{-\delta(t-s)}H(t-s)\|u_t - v_t\|_{\mathcal{B}}ds \\ &\quad + M\int_{-\infty}^{t} e^{-\omega(t-s)}L_g(s)\|u_t - v_t\|_{\mathcal{B}}ds \\ &\leq \left(L_f + \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\delta(t-s)}H(t-s)L_f(s)ds\right)\cdot \mathfrak{L}\cdot \|u_t - v_t\|_{\infty} \\ &\quad + \left(\overline{M}\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\omega(t-s)}L_g(s)ds\right)\cdot \mathfrak{L}\cdot \|u_t - v_t\|_{\infty} \\ &\leq \overline{\theta}\cdot \|u_t - v_t\|_{\infty}.\end{aligned}$$

The assertion is now a consequence of the classical Banach contraction mapping principle. □

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(Received June 26, 2013; revised July 14, 2014)