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Characterization of p-groups by sum of the element orders

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Abstract. Let G be a finite group. Then we denote $\psi(G) = \sum_{x\in G} o(x)$ where $o(x)$ is the order of the element x in G. In this paper we characterize some finite p-groups (p a prime) by ψ and their orders.

1. Introduction and main results

In what follows all groups are finite and p is a prime.

Given a finite group G, let $\psi(H) = \sum_{x\in H} o(x)$ for $H \subseteq G$, where as usual, $o(x)$ is the order of the element x. In this note, we ask what information about some classes of p-groups G can be recovered if we know both $\psi(G)$ and $|G|$. The starting point for the function ψ is given by the paper [1] which investigates the maximum of ψ among all groups of the same order. In [2] the authors determined the structure of the groups which have the minimum sum of the element orders on all groups of the same order.

Let CP_2 be the class of finite groups G such that $o(xy) \leq \max\{o(x), o(y)\}$ for all $x, y \in G$. We denote $\Omega_i(G) = \langle\{x \in G \mid x^{p^i} = 1\}\rangle$ for all $i \in \mathbb{N}$. Now we state the first main result as follows.

Theorem 1.1. *Suppose that P and Q are contained in CP_2 of the same order p^n. Then the following statements are equivalent:*

(1) $\psi(P) = \psi(Q)$.

(2) $|\Omega_i(P)| = |\Omega_i(Q)|$ *for all* $i \in \mathbb{N}$.

(3) $\psi(\Omega_i(P)) = \psi(\Omega_i(Q))$ *for all* $i \in \mathbb{N}$.

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Note that the class CP_2 of p-groups is more large than the class of abelian p-groups, regular p-groups (see Theorem 3.14 of [6], II, p. 47) and p-groups whose subgroup lattices are modular (see Lemma 2.3.5 of [5]). Moreover by the main theorem in [7], we infer that powerful p-groups for p odd also belong to CP_2.

The following is the second main result.

Theorem 1.2. *Let P and Q be two finite p-groups of the same order and $\Omega_{m-1}(P) \neq P$, where $\exp(P) = p^m$. If $\exp(P) > \exp(Q)$, then $\psi(P) > \psi(Q)$.*

In general, it is not true that if P and Q are p-groups of the same order such that $\exp(P) > \exp(Q)$, then $\psi(P) > \psi(Q)$. For example consider $Q = (C_4)^4$ and $P = D_{16} \times (C_2)^4$. The authors would like to thank Prof. E. Khukhru for giving this example.

But if $\exp(P) = \exp(Q)$, then we have the following.

Theorem 1.3. *Let P and Q belong to CP_2 of the same order and the same exponent p^m. Also suppose that $|\Omega_{m-i}(P)| = |\Omega_{m-i}(Q)|$ for $i = 1, 2, \ldots, t$. If $|\Omega_{m-t-1}(P)| < |\Omega_{m-t-1}(Q)|$, then $\psi(P) > \psi(Q)$.*

As an application of Theorems 1.1 and 1.2 we have the following.

Theorem 1.4. *Let P and Q belong to CP_2 of the same order p^n. Then $\psi(P) = \psi(Q)$ if and only if there is a bijection $f : P \to Q$ such that $o(f(x)) = o(x)$ for all $x \in P$.*

2. Proof of the main results

Lemma 2.1. *Let P be a finite p-group, $\exp(P) = p^m$ and $M = \Omega_{m-1}(P) \neq P$. Then $\psi(P) = \psi(M) + |M|p^m\big(\frac{|P|}{|M|} - 1\big)$.*

Proof. Suppose that X is a left transversal to M in P containing identity element. For all $y \in M$ and $1 \neq x \in X$, we have $o(xy) = p^m$. Therefore $\psi(xM) = |M|p^m$ for all $1 \neq x \in X$. This completes the proof. □

Theorem 2.2. *Let P and Q be two finite p-groups of order p^n and $\exp(P) = p^m$. If $\Omega_{m-1}(P) \neq P$ and $\exp(P) > \exp(Q)$, then $\psi(P) > \psi(Q)$.*

Proof. Let $M = \Omega_{m-1}(P)$. Then

$$\begin{aligned}\psi(P) &= \psi(M) + |M|p^m\left(\frac{|P|}{|M|} - 1\right) > |M|p^m\left(\frac{|P|}{|M|} - 1\right)\\ &= p^m(|P| - |M|) = p^m(p^n - |M|) \geq p^m(p^n - p^{n-1}) \geq p^n p^{m-1}.\end{aligned}$$

Since $\exp(Q) \leq p^{m-1}$, we have $\psi(Q) < p^n p^{m-1} < \psi(P)$. □

We observe that if a finite group G belongs to CP_2, then for every $x, y \in G$ satisfying $o(x) \neq o(y)$ we have $o(xy) = \max\{o(x), o(y)\}$.

We shall need the following theorem about the groups belonging to CP_2.

Theorem 2.3 (See Theorem D in [4]). *A finite group G is contained in CP_2 if and only if one of the following statements holds:*

(1) *G is a p-group and $\Omega_n(G) = \{x \in G \mid x^{p^n} = 1\}$.*

(2) *G is a Frobenius group of order $p^\alpha q^\beta$, $p < q$, with kernel $F(G)$ of order p^α and cyclic complement.*

In the sequel assume that P and Q are p-groups belonging to CP_2.

Lemma 2.4. *If $|\Omega_1(P)| = p^r$, then $\psi(P) = 1 - p + p^{r+1}\psi\big(\frac{P}{\Omega_1(P)}\big)$.*

Proof. Suppose that $\Omega_1(P) = N$. Then we have $\langle x\rangle \cap N \neq 1$ for all $1 \neq x \in P$, Since $\big\langle x^{\frac{o(x)}{p}} \big\rangle$ is a subgroup of $\langle x\rangle \cap N$. Let X be a left transversal to N in P such that $1 \in X$. Suppose that $1 \neq x \in X$. Then $o(x) \geq p^2$ since N does not contain x. If $y \in N$, then by Theorem 2.3 part one $\exp(N) = p$ and so we have $o(xy) = o(x)$. This implies that

$$\psi(P) = \sum_{x\in X} \psi(xN) = \psi(N) + |N| \sum_{1\neq x\in X} o(x)$$

If $1 \neq x \in X$, then $\langle x\rangle \cap N \neq 1$ which follows that $o(x) = po(xN)$. Hence

$$\begin{aligned}\psi(P) &= \psi(N) + |N| \sum_{1\neq x\in X} o(x) = \psi(N) + |N|p \sum_{1\neq x\in X} o(xN)\\ &= \psi(N) + |N|p\left(\psi\left(\frac{P}{N}\right) - 1\right).\end{aligned}$$

Since $|N| = p^r$, we have $\psi(N) = p^{r+1} - p + 1$, which completes the proof. □

Lemma 2.5. *If $\psi(P) = \psi(Q)$, then $|\Omega_1(P)| = |\Omega_1(Q)|$.*

Proof. Suppose that $\Omega_1(P) = N$ and $\Omega_1(Q) = M$. If $|N| = p^r$ and $|M| = p^t$, then it follows from previous lemma that $p^{r+1}\psi\big(\frac{P}{N}\big) = p^{t+1}\psi\big(\frac{Q}{M}\big)$. If $r+1 < t+1$, then $p^{r+1}\psi\big(\frac{P}{N}\big) \equiv 0 \pmod{p^{t+1}}$. Since $\frac{P}{N}$ is a p-group, we have $\psi\big(\frac{P}{N}\big) = 1 + kp$ and so $\psi\big(\frac{P}{N}\big) = 1 + kp \equiv 1 \equiv 0 \pmod{p}$, a contradiction. Thus $r = t$. □

Lemma 2.6. *We have $\Omega_i\big(\frac{P}{\Omega_1(P)}\big) = \frac{\Omega_{i+1}(P)}{\Omega_1(P)}$ for all $i \in \mathbb{N}$.*

Proof. Since $\frac{\Omega_{i+1}(P)}{\Omega_1(P)} \leq \Omega_i\big(\frac{P}{\Omega_1(P)}\big)$, it is enough to show that $\Omega_i\big(\frac{P}{\Omega_1(P)}\big) \leq \frac{\Omega_{i+1}(P)}{\Omega_1(P)}$. Suppose that $t\Omega_1(P) \in \Omega_i\big(\frac{P}{\Omega_1(P)}\big)$. Then $t^{p^i} \in \Omega_1(P)$ and since $\exp(\Omega_1(P)) = p$, we have $t^{p^{i+1}} = 1$. Therefore $t \in \Omega_{i+1}(P)$ and so $t\Omega_1(P) \in \frac{\Omega_{i+1}(P)}{\Omega_1(P)}$. □

Corollary 2.7. $\frac{P}{\Omega_1(P)}$ *belongs to* CP_2.

Proof. It follows from previous lemma that $\Omega_i\big(\frac{P}{\Omega_1(P)}\big) = \frac{\Omega_{i+1}(P)}{\Omega_1(P)}$, for all $i \in \mathbb{N}$. Since P belongs to CP_2, we have $\Omega_{i+1}(P) = \{x \in G \mid x^{p^{i+1}} = 1\}$ for all $i \in \mathbb{N}$. Since $x^{p^i} \in \Omega_1(P)$, we see $\frac{\Omega_{i+1}(P)}{\Omega_1(P)} = \{x\Omega_1(P) \in \frac{P}{\Omega_1(P)} \mid x^{p^i}\Omega_1(P) = \Omega_1(P)\}$ for all $i \in \mathbb{N}$ by Theorem 2.3 and so $\frac{P}{\Omega_1(P)}$ is contained in CP_2. □

Theorem 2.8. *Let P and Q have the same order p^n and the same exponent p^m and suppose that $|\Omega_{m-i}(P)| = |\Omega_{m-i}(Q)|$ for $i = 0, 1, 2, \ldots, t$. If $|\Omega_{m-t-1}(P)| < |\Omega_{m-t-1}(Q)|$, then $\psi(P) > \psi(Q)$.*

Proof. If $P \in CP_2$ and $\exp(P) = p^m$, then for all $i < m$, $\Omega_i(P) \neq P$ and

$$1 < \Omega_1(P) < \Omega_2(P) < \cdots < \Omega_m(P) = P.$$

Note that for all $i \leq j$, we have $\Omega_i(\Omega_j(P)) = \Omega_i(P)$. Using Lemma 2.1 we can get

$$\psi(P) = \psi(\Omega_{m-t}(P)) + \sum_{i=1}^{t} |\Omega_{m-i}(P)| p^{m-i+1} \left(\frac{|\Omega_{m-i+1}(P)|}{|\Omega_{m-i}(P)|} - 1 \right)$$

and

$$\psi(Q) = \psi(\Omega_{m-t}(Q)) + \sum_{i=1}^{t} |\Omega_{m-i}(Q)| p^{m-i+1} \left(\frac{|\Omega_{m-i+1}(Q)|}{|\Omega_{m-i}(Q)|} - 1 \right).$$

Since

$$\sum_{i=1}^{t} |\Omega_{m-i}(P)| p^{m-i+1} \left(\frac{|\Omega_{m-i+1}(P)|}{|\Omega_{m-i}(P)|} - 1 \right) = \sum_{i=1}^{t} |\Omega_{m-i}(Q)| p^{m-i+1} \left(\frac{|\Omega_{m-i+1}(P)|}{|\Omega_{m-i}(Q)|} - 1 \right),$$

it is enough to prove that $\psi(\Omega_{m-t}(P)) > \psi(\Omega_{m-t}(Q))$. Suppose that $|\Omega_{m-t-1}(Q)| = p^a |\Omega_{m-t-1}(P)|$, where $a \geq 1$. By Lemma 2.1, we have

$$\begin{aligned} \psi(\Omega_{m-t}(P)) - \psi(\Omega_{m-t}(Q)) &= \psi(\Omega_{m-t-1}(P)) - \psi(\Omega_{m-t-1}(Q)) \\ &\quad + p^{m-t}(|\Omega_{m-t-1}(Q)| - |\Omega_{m-t-1}(P)|) \\ &> p^{m-t}(|\Omega_{m-t-1}(Q)| - |\Omega_{m-t-1}(P)|) - \psi(\Omega_{m-t-1}(Q)) \end{aligned}$$

$$= p^{m-t-a}|\Omega_{m-t-1}(Q)|(p^a - 1) - \psi(\Omega_{m-t-1}(Q))$$
$$\geq p^{m-t-1}(p-1)|\Omega_{m-t-1}(Q)| - \psi(\Omega_{m-t-1}(Q))$$
$$\geq \psi(\Omega_{m-t-1}(Q)) - \psi(\Omega_{m-t-1}(Q)) = 0.$$

This completes the proof. $\square$

Using Lemmas 2.4 and 2.5 we can propose another proof for Corollary 6 in [3].

Corollary 2.9. *Let P and Q be abelian p-groups of the same order. Then $\psi(P) = \psi(Q)$ if and only if $P \cong Q$.*

Proof. It is sufficient to show that if $\psi(P) = \psi(Q)$, then $P \cong Q$. We prove this by induction on $|P|$. Base step of induction is trivial. Let $|\Omega_1(P)| = p^t$ and $|\Omega_1(Q)| = p^r$. It follows from Lemma 2.4 that $\psi(P) = 1 - p + p^{r+1}\psi\big(\frac{P}{\Omega_1(P)}\big)$ and $\psi(Q) = 1 - p + p^{t+1}\psi\big(\frac{Q}{\Omega_1(Q)}\big)$. We have $r = t$ by Lemma 2.5. Therefore $\psi\big(\frac{P}{\Omega_1(P)}\big) = \psi\big(\frac{Q}{\Omega_1(Q)}\big)$. So we have $\frac{P}{\Omega_1(P)} \cong \frac{Q}{\Omega_1(Q)}$ by induction hypothesis which implies that $P \cong Q$. $\square$

The above result is not true for regular p-groups or p-groups of nilpotent class 2. For example there exists a regular 3-group P such that $|P| = 27$ and $\exp(P) = 3$, but P is not abelian, so $\psi(P) = 79 = \psi((C_3)^3)$ but P is not isomorphic to $(C_3)^3$.

Now we are ready to prove Theorem 1.1.

Theorem 2.10. *Suppose that P and Q have the same order. Then the following statements are equivalent:*

(1) $\psi(P) = \psi(Q)$.

(2) $|\Omega_i(P)| = |\Omega_i(Q)|$ *for all* $i \in \mathbb{N}$.

(3) $\psi(\Omega_i(P)) = \psi(\Omega_i(Q))$ *for all* $i \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2). We prove by induction on $|P|$. Suppose that P and Q are contained in CP_2 and $\psi(P) = \psi(Q)$. It follows from Lemma 2.4 that $\psi(P) = 1 - p + p^{r+1}\psi\big(\frac{P}{\Omega_1(P)}\big)$ and $\psi(Q) = 1 - p + p^{t+1}\psi\big(\frac{Q}{\Omega_1(Q)}\big)$ where $|\Omega_1(P)| = p^r$ and $|\Omega_1(Q)| = p^t$. Since $\psi(P) = \psi(Q)$, we obtain $r = t$ by Lemma 2.5 and so $\psi\big(\frac{P}{\Omega_1(P)}\big) = \psi\big(\frac{Q}{\Omega_1(Q)}\big)$. By corollary 2.7 we have $\frac{P}{\Omega_1(P)}$ and $\frac{Q}{\Omega_1(Q)}$ are in CP_2. Since $\big|\frac{P}{\Omega_1(P)}\big| = \big|\frac{Q}{\Omega_1(Q)}\big|$, the induction assumption yields that $\big|\Omega_i\big(\frac{P}{\Omega_1(P)}\big)\big| = \big|\Omega_i\big(\frac{Q}{\Omega_1(Q)}\big)\big|$ for all $i \in \mathbb{N}$. Therefore $|\Omega_i(P)| = |\Omega_i(Q)|$ by Lemmas 2.3 and 2.4.

(2) $\Rightarrow$ (1). Let $\exp(P) = p^m$. By Theorem 2.2, we have $\exp(Q) = p^m$. Since P and Q are contained in CP_2, we have $\exp(\Omega_j(P)) = \exp(\Omega_j(Q)) = p^j$ for all $j \in \mathbb{N}$. But

$$\psi(P) = 1+\sum_{j=1}^{m}(|\Omega_j(P)|-|\Omega_{j-1}(P))|p^j = 1+\sum_{j=1}^{m}(|\Omega_j(Q)|-|\Omega_{j-1}(Q)|)p^j = \psi(Q),$$

where the second equality holds by the hypothesis (2).

(2) $\Rightarrow$ (3). Since $|\Omega_i(P)| = |\Omega_i(Q)|$ for all $i \in \mathbb{N}$, we have $\exp(P) = \exp(Q)$. Let $\exp(P) = p^m$. Since P and Q are contained in CP_2, we have $\exp(\Omega_j(P)) = \exp(\Omega_j(Q)) = p^j$ for all $j \in \mathbb{N}$. So

$$\begin{aligned}\psi(\Omega_i(P)) &= 1 + \sum_{j=1}^{i}(|\Omega_j(P)| - |\Omega_{j-1}(P))|p^j \\ &= 1 + \sum_{j=1}^{i}(|\Omega_j(Q)| - |\Omega_{j-1}(Q)|)p^j = \psi(\Omega_i(Q)).\end{aligned}$$

(3) $\Rightarrow$ (2). Since $\psi(\Omega_i(P)) = \psi(\Omega_i(Q))$ for all $i \in \mathbb{N}$, we have $\exp(P) = \exp(Q) = p^m$. Let $M = \Omega_{m-1}(P)$ and $N = \Omega_{m-1}(Q)$. By Lemma 2.1 we have

$$\psi(P) = \psi(M) + |M|p^m\left(\frac{|P|}{|M|} - 1\right) = \psi(N) + |N|p^m\left(\frac{|Q|}{|N|} - 1\right) = \psi(Q).$$

Since $\psi(M) = \psi(N)$, we obtain that $|N| = |M|$. By repeated use of this technique we shall reach the claimed. This completes the proof. $\square$

Finally we prove the last main result.

Theorem 2.11. *Let P and Q have the same order p^n. Then $\psi(P) = \psi(Q)$ if and only if there is a bijection $f : P \to Q$ such that $o(f(x)) = o(x)$ for all $x \in P$.*

Proof. It is clear that if there is a bijection $f : P \to Q$ such that $o(f(x)) = o(x)$ for all $x \in P$, then $\psi(P) = \psi(Q)$. Conversely suppose that $\psi(P) = \psi(Q)$. We proceed by induction on n. Base step is trivial. By Theorem 2.2 we have $\exp(P) = \exp(Q) = p^m$. It follows from Theorem 2.10 that $\psi(\Omega_{m-1}(P)) = \psi(\Omega_{m-1}(Q))$ and so by inductive hypothesis there is a bijection $f : \Omega_{m-1}(P) \to \Omega_{m-1}(Q)$ such that $o(f(x)) = o(x)$ for all $x \in \Omega_{m-1}(P)$. Theorem 2.10 follows that $|\Omega_m(P)| - |\Omega_{m-1}(P)| = |\Omega_m(Q)| - |\Omega_{m-1}(Q)|$ and hence there is a bijection g from $\Omega_m(P) - \Omega_{m-1}(P)$ to $\Omega_m(Q) - \Omega_{m-1}(Q)$. Define h from P to Q by

$$h(x) = \begin{cases} f(x) & x \in \Omega_{m-1}(P), \\ g(x) & \text{otherwise.} \end{cases}$$

It is easily seen that h is a bijection from P to Q such that $o(h(x)) = o(x)$ for all $x \in P$, as wanted. □

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