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On univalence, starlikeness and convexity of certain analytic functions

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Abstract. Let $t=(t_0,t_1,\dots,t_k)\in(-\infty\ \infty)^{k+1}$. By $\boldsymbol{R}_t$ we denote the class of functions $f(z)=z+a_2z^2+\dots$ that are analytic in the unit disk $U=\{z:|z|<1\}$ and satisfy the condition

$$\operatorname{Re}\left[t_2\frac{f(z)}{z}+t_1f^{(1)}(z)+\dots+t_kf^{(k)}(z)-t_0-t_1\right]>0\,.$$

It is known that (i) if $f,g\in\boldsymbol{R}=\boldsymbol{R}(0,1,1)$ then $f*g\in\boldsymbol{R}$, $\operatorname{Re}\left[\frac{f(z)}{z}\right]>\frac{1}{2}$, $\operatorname{Re}f(z)>-1+2\log 2$, and $f*g$ is convex, (ii) if $f\in\boldsymbol{R}\left(0,\frac{4}{3},\frac{4}{3}\right)$ then f is startlike.

In the present paper we solve problems similar to the above in a large number of classes $\boldsymbol{R}_t$. Some known inequalities are improved and some known results are proved under weaker conditions.

Introduction

Let $\boldsymbol{H}(\boldsymbol{U})$ be the class of holomorphic functions on the open unit disc $\boldsymbol{U}=\{z:|z|<1\}$ and $\boldsymbol{A}$ be the class of functions $f\in\boldsymbol{H}(\boldsymbol{U})$ such that $f(0)=f'(0)-1=0$. The class $\boldsymbol{H}(\boldsymbol{U})$ is a locally convex linear topological space under the topology of uniform convergence on the compact subsets of $\boldsymbol{U}$.

We denote by $\boldsymbol{S}$ the subclass of $\boldsymbol{A}$ consisting of univalent functions and by $\boldsymbol{K},\boldsymbol{S}_t$ the usual subclasses of $\boldsymbol{S}$ whose members are convex, starlike (w.r.t. the origin).

Let $\boldsymbol{P}$ be the class of functions $f\in\boldsymbol{H}(\boldsymbol{U})$ for which $f(0)=1$ and $\operatorname{Re}f>0$ and $t=(t_0,t_1,\dots,t_k)\in(-\infty,+\infty)^{k+1}$. We define $\boldsymbol{R}_t$ to be the

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class of functions $f \in \boldsymbol{A}$ for which

$$t_0 \frac{f(z)}{z} + t_1 f'(z) + \cdots + t_k z^{k-1} f^{(k)}(z) + 1 - t_0 - t_1 \in \boldsymbol{P}\,.$$

We mention certain basic results from the paper [3].

i) If $f \in \boldsymbol{R} = \boldsymbol{R}(0,1,1)$ then $\operatorname{Re} f'(z) > -1 + 2\log 2$, $\operatorname{Re}\left[\frac{f(z)}{z}\right] > \frac{1}{2}$
ii) If $f \in \boldsymbol{A}$ and $\operatorname{Re}[f'(z) + zf''(z)] > -\frac{1}{4}$ then $f \in \boldsymbol{S}_t$
iii) If $f \in \boldsymbol{R}$, $g \in \boldsymbol{R}$ then $f * g \in \boldsymbol{K}$.

In the present paper we deal with problems similar to those of [3] in a large number of classes $\boldsymbol{R}_t$. We improve certain estimations that are made in [1] and certain results of that paper are proved with weaker assumptions.

2. Preliminaries

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \boldsymbol{H}(\boldsymbol{U}), \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \in \boldsymbol{H}(\boldsymbol{U})$$

then so does their Hadamard product

$$f * g(z) = \sum_{n=1}^{\infty} a_n b_n z^n\,.$$

Lemma 1. *The transformation*

$$* : \boldsymbol{H}(\boldsymbol{U}) \times \boldsymbol{H}(\boldsymbol{U}) \to \boldsymbol{H}(\boldsymbol{U})$$

is continuous (see [2], p. 27).

Lemma 2. *Let $\boldsymbol{Y}$ be a locally convex linear topological space, $\boldsymbol{N}_1 \subset \boldsymbol{Y}$, $\boldsymbol{N}_2 \subset \boldsymbol{Y}$ be convex and compact and $\boldsymbol{EN}_1$, $\boldsymbol{EN}_2$ be the corresponding subsets of its extremal points. Then we have*

(a) *If $\boldsymbol{L}_1$ is a linear continuous functional on the space Y then*

$$\min\{\operatorname{Re} \boldsymbol{L}_1(x) : x \in \boldsymbol{N}_1\} = \min\{\operatorname{Re} \boldsymbol{L}_1(x) : x \in \boldsymbol{EN}_1\}\ .$$

(b) *If $\boldsymbol{L}_2$ linear continuous functional on the space $\boldsymbol{Y} \times \boldsymbol{Y}$ then*

$$\min\{\operatorname{Re} \boldsymbol{L}_2(x_1,x_2) : (x_1,x_2) \in \boldsymbol{N}_1 \times \boldsymbol{N}_2\} = \\ = \min\{\operatorname{Re} L_1(x_1,x_2) : (x_1,x_2) \in \boldsymbol{EN}_1 \times \boldsymbol{EN}_2\}$$

(see 2, p.173).

Lemma 3. *The class* $\boldsymbol{P}$ *is a convex and compact subset of the space* $\boldsymbol{H}(\boldsymbol{U})$ *and,*

$$\boldsymbol{EP} = \left\{1 + 2\sum_{n=1}^{\infty}(\eta z)^n : |\eta| = 1\right\} \qquad \text{(see [1])}.$$

Proposition 1. *Let* $t = (t_0, t_1, \dots, t_k) \in (-\infty, \infty)^{k+1}$ *such that the polynomial*

$$\underline{t}(x) = t_0 + t_1(x+1) + \cdots + t_k(x+1)x\dots(x-k+2)$$

does not have any natural number as a root. Then we have

(a) $\boldsymbol{R}_t$ *is a convex and compact subset of* $\boldsymbol{H}(\boldsymbol{U})$
(b) $\boldsymbol{ER}_t = \{zt^*(\eta z) : |\eta| = 1\}$ *where*

$$t^*(z) = 1 + 2\sum_{n=1}^{\infty}\frac{z^n}{\underline{t}(n)}.$$

Proof. It is obvious that

$$z + \sum_{n=2}^{\infty} a_n z^n \in \boldsymbol{R}_t \quad \text{if and only if} \quad f(z) = 1 + \sum_{n=1}^{\infty} a_{n+1}\underline{t}(n) z^n \in \boldsymbol{P}$$

or

$$1 + \sum_{n=1}^{\infty}\beta_n z^n \in \boldsymbol{P} \quad \text{if and only if} \quad z\left[1 + \sum_{n=1}^{\infty}\frac{\beta_n}{\underline{t}(n)} z^n\right] \in \boldsymbol{R}_t.$$

If

$$f_0(z) = 1 + \sum_{n=1}^{\infty}\frac{1}{\underline{t}(n)} z^n,$$

we consider the operator $\boldsymbol{T}$ with

$$\boldsymbol{T} : \boldsymbol{H}(\boldsymbol{U}) \to \boldsymbol{H}(\boldsymbol{U}) \quad \text{and} \quad \boldsymbol{T}(f)(z) = z f_0 * f(z).$$

From Lemma 1 we get that $\boldsymbol{T}$ is continuous, therefore $\boldsymbol{R}_t = \boldsymbol{T}(\boldsymbol{P})$ is compact. Since, furthermore, $\boldsymbol{T}$ is linear and injective we have

$$\boldsymbol{ER}_t = \boldsymbol{ET}(\boldsymbol{P}) = \boldsymbol{T}(\boldsymbol{EP}).$$

From Lemma 3 we get (b).

Lemma 4. *Let* $p(x) = (x - x_1)(x - x_2)\dots(x - x_k)$, $q(x) = (x - \varrho_1)(x - \varrho_2)\dots(x - \varrho_m)$, $k \le m$, $x_i \le \varrho_i$ *for* $1 \le i \le k$, $\varrho_1 < \varrho_2 < \dots \varrho_m < 1$ *and*

$$\frac{p(x)}{q(x)} = \delta_{k,m} + \frac{A_1}{x - \varrho_1} + \frac{A_2}{x - \varrho_2} + \dots + \frac{A_m}{x - \varrho_m}\,.$$

Then we have

(a) $$A_1\omega^{-\varrho_1} + \dots + A_m\omega^{-\varrho_m} \ge 0 \quad \forall\, \omega \in (0,1]\,.$$

(b) $$\operatorname{Inf}\left\{\operatorname{Re}\left[\sum_{n=1}^{\infty}\frac{p(n)}{q(n)}z^n\right] : z \in U\right\} = \lim_{r\to 1}\sum_{n=1}^{\infty}\frac{p(n)}{q(n)}(-r)^n = \sum_{n=1}^{\infty}\left[\frac{p(n)}{q(n)} - \delta_{k,m}\right](-1)^n - \frac{\delta_{k,m}}{2}\,.$$

Proof. We first prove (a) for $p(x) = 1$. Its obvious if $m = 1$. Suppose it is true for $m = n - 1$. If

$$\frac{1}{(x - \varrho_1)(x - \varrho_2)\dots(x - \varrho_{n-1})} = \frac{B_1}{x - \varrho_1} + \dots + \frac{B_{n-1}}{x - \varrho_{n-1}}$$

and

$$\frac{1}{(x - \varrho_1)(x - \varrho_2)\dots(x - \varrho_n)} = \frac{A_1}{x - \varrho_1} + \frac{A_2}{x - \varrho_2} + \dots + \frac{A_n}{x - \varrho_n}$$

then

(3) $$A_1(\varrho_n - \varrho_1) = -B_1, \dots, A_{n-1}(\varrho_n - \varrho_{n-1}) = -B_{n-1}\,.$$

From (3) we have

$$\frac{d}{d\omega}\left(A_1\omega^{\varrho_n-\varrho_1} + A_2\omega^{\varrho_n-\varrho_2} + \dots + A_n\right) = = -\omega^{\varrho_n-1}\left(B_1\omega^{-\varrho_1} + \dots + B_{n-1}\omega^{\varrho_{n-1}}\right) \le 0$$

and therefore we have a min in $(0,1]$ when $\omega = 1$.
From the equality

$$A_1 + A_2 + \dots + A_n = 0$$

we get the required result for $p(x) = 1$.

From the equality

$$\frac{(x - x_1)}{q(x)} = \frac{(\varrho_1 - x_1)}{q(x)} + \frac{1}{(x - \varrho_2)\dots(x - \varrho_m)}$$

we get the required result for $p(x) = x - x_1$. Continuing in this way we get (a).

(b) From the equality

$$\sum_{n=1}^{\infty} \frac{z^n}{n-\varrho} = \sum_{n=1}^{\infty} z^n \left[\int_0^1 \omega^{n-\varrho-1} d\omega\right] =$$

$$= \int_0^1 \left[\sum_{n=1}^{\infty} z^n \omega^{n-\varrho-1}\right] d\omega = \int_0^1 \frac{\omega^{-\varrho} z}{1-\omega z} d\omega$$

we have

$$\sum_{n=1}^{\infty} \frac{p(n)}{q(n)} z^n = \int_0^1 \frac{(A_1\omega^{-\varrho_1} + A_2\omega^{-\varrho_2} + \dots)z}{1-\omega z} d\omega + \delta_{m,k} \frac{z}{1-z}\,. \tag{4}$$

From (4), (a) and by the equality

$$\min_{|z|=r} \operatorname{Re}\left[\frac{z}{1-z}\right] = \frac{-r}{1+r}$$

we obtain

$$\min\left\{\operatorname{Re}\left[\sum_{n=1}^{\infty} \frac{p(n)}{q(n)} z^n\right], |z| = r\right\} = \sum_{n=1}^{\infty} \frac{p(n)}{q(n)} (-r)^n\,. \tag{5}$$

If we use the minimum principle for harmonic functions, we get from (5) the first part of equation (b). The second part is an immediate corollary of the first if we use the relation

$$\lim_{r\to 1-} \sum_{n=1}^{\infty} \frac{(-r)^n}{n-\varrho} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n-\varrho}\,.$$

Notes. The conclusion (b) of the Lemma holds also in the case that $q(x)$ has multiple roots. Indeed if we let $\varrho_2 = \varrho_1 + \varepsilon$ we have a relation of the form $\operatorname{Re} f_\varepsilon(z) \geq f_\varepsilon(-r)$. For $\varepsilon \to 0$ we get the conclusion for roots of multiplicity two. Continuing in this manner, we can generalize the result.

We prove now the following proposition

Proposition 2. *Let $\boldsymbol{E}_m$ be the set of $t=(t_0, t_1, \dots, t_k) \in (-\infty, \infty)^{k+1}$ that satisfy the condition* i) $t_k > 0$, (ii) *if* $\underline{t}(\varrho) = 0$ *then* $\varrho < 1$, (iii) *the number of roots of* $\underline{t}(x)$ *that belong to* $[-1, 1]$ *is at least* m. *Then*

$$\text{(a)} \quad \inf\left\{\operatorname{Re}\left[\frac{f(z)}{z}\right] : z \in \boldsymbol{U},\ f \in \boldsymbol{R}_t\right\} = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{\underline{t}(n)}, \quad \forall\, t \in \boldsymbol{E}_0$$

(a′) $\inf\left\{\mathrm{Re}\left[\frac{f(z)}{z}\right] : z \in \boldsymbol{U},\ f \in \boldsymbol{R}_t * \boldsymbol{R}_t\right\} =$

$$= 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{\underline{t}^2(n)}, \quad \forall\, t \in \boldsymbol{E}_0$$

(b) $\inf\{\mathrm{Re}\,[f'(z)] : z \in \boldsymbol{U},\ f \in \boldsymbol{R}_t\} =$

$$= 1 + 2\sum_{n=1}^{\infty}(-1)^n\left[\frac{n+1}{\underline{t}(n)} - \frac{\delta_{1,k}}{t_k}\right] - \frac{\delta_{1,k}}{t_k}, \quad \forall\, t \in \boldsymbol{E}_1$$

(b′) $\inf\{\mathrm{Re}[f'(z)] : z \in \boldsymbol{U}\ f \in \boldsymbol{R}_t * \boldsymbol{R}_t\} = 1 + 4\sum_{n=1}^{\infty} \frac{(n+1)}{\underline{t}^2(n)}, \quad \forall\, t \in \boldsymbol{E}_1$

(c) $\inf\{\mathrm{Re}[f'(z) + zf''(z)] : z \in \boldsymbol{U},\ f \in \boldsymbol{R}_t\} =$

$$= 1 + 2\sum_{n=1}^{\infty}(-1)^n\left[\frac{(n+1)^2}{\underline{t}(n)}\right] - \frac{\delta_{2,k}}{t_k}, \quad \forall\, t \in \boldsymbol{E}_2$$

(c′) $\inf\{\mathrm{Re}[f'(z) + zf''(z) : z \in \boldsymbol{U},\ f \in \boldsymbol{R}_t * \boldsymbol{R}_t\} =$

$$= 1 + 4\sum_{n=1}^{\infty}(-1)^n\left[\frac{(n+1)^2}{t^2(n)} - \frac{\delta_{1k}}{t_k^2}\right] - \frac{\delta_{1,k}}{2t_k^2}, \quad \forall\, t \in \boldsymbol{E}_1$$

(d) $\inf\left\{\mathrm{Re}[f'(z) + 3zf''(z) + z^2 f'''(z)] : z \in \boldsymbol{U},\ f \in \boldsymbol{R}_t\right\} =$

$$= 1 + 2\sum_{n=1}^{\infty}(-1)^n\left[\frac{(n+1)^3}{\underline{t}(n)} - \frac{\delta_{3,k}}{t_k}\right] - \frac{\delta_{3,k}}{t_k}, \quad \forall\, t \in \boldsymbol{E}_3$$

(d′) $\inf\left\{\mathrm{Re}[f'(z) + 3zf''(z) + z^2 f'''(z)] : z \in \boldsymbol{U},\ f \in \boldsymbol{R}_t * \boldsymbol{R}_t\right\} =$

$$= 1 + 4\sum_{n=1}^{\infty}(-1)^n \frac{(n+1)^3}{\underline{t}^2(n)}, \quad \forall\, t \in \boldsymbol{E}_2\,.$$

Proof. (a) If $z \neq 0$ we consider the linear functional

$$\boldsymbol{L}_z : \boldsymbol{L}_z(f) = \frac{f(z)}{z}\,.$$

Then $\boldsymbol{L}_z$ is continuous on $\boldsymbol{H}(\boldsymbol{U})$. From Lemma 3a and Proposition 1 we have

$$\min\left\{\operatorname{Re}\left[\frac{f(z)}{z}\right] : f \in \boldsymbol{R}_t\right\} = \min\left\{\operatorname{Re}\left[\frac{f(z)}{z}\right] : f \in \boldsymbol{ER}_t\right\} =$$

$$\min\left\{\operatorname{Re}\left[1 + 2\sum_{n=1}^{\infty}\frac{(\eta z)^n}{\underline{\mathrm{t}}(n)}\right] : |\eta| = 1\right\}. \tag{6}$$

From (6) and Lemma 4b we get the required result.

(a′) If on the space $\boldsymbol{H}(\boldsymbol{U}) \times \boldsymbol{H}(\boldsymbol{U})$ we consider the bilinear mapping

$$\boldsymbol{L}_z^* : \boldsymbol{L}_z^*(f,g) = \frac{1}{2} f * g(z)$$

then $\boldsymbol{L}_z^*$ is continuous. From Lemma 3b and Prop. 1 we have

$$\begin{aligned}
&\min\left\{\operatorname{Re}\left[\frac{1}{2} f * g(z)\right] : f \in \boldsymbol{R}_t,\ g \in \boldsymbol{R}_t\right\} = \\
&= \min\left\{\operatorname{Re}\left[\frac{1}{2} f * g(z)\right] : f \in \boldsymbol{ER}_t,\ g \in \boldsymbol{ER}_t\right\} = \\
&= \min\left\{\operatorname{Re}\left[1 + 4\sum_{n=1}^{\infty}\frac{(\eta z)^n}{\underline{\mathrm{t}}^2(n)}\right] : |\eta| = 1\right\}.
\end{aligned} \tag{7}$$

From (7) and Lemma 4b we obtain the required result.

The ramaining conclusions are proved by a similar process.

3. Univalence, starlikeness and convexity criteria

Theorem 1. *Let $\boldsymbol{R}_t$ be as in Lemma 4, $a < 1$, and*

$$\boldsymbol{Q}_a = \left\{f \in \boldsymbol{A} :\ \operatorname{Re}\left[\frac{f(z)}{z}\right] > a\right\}.$$

Then we have

(a) $\boldsymbol{R}_t * \boldsymbol{Q}_a = \boldsymbol{R}_t$ *if and only if* $a = \frac{1}{2}$,
(b) $\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{R}_t$ *if and only if* $\boldsymbol{R}_t \subset \boldsymbol{Q}_{1/2}$.

Proof. (a) It is obvious that $f \in \boldsymbol{R}_t$ if and only if

$$\operatorname{Re}\left[\frac{1}{2} h * f(z)\right] > 0 \quad \text{where} \quad h(z) = z + \sum_{n=1}^{\infty}\underline{\mathrm{t}}(n) z^n .$$

Also

$$\boldsymbol{Q}_a = \{az + (1-a)zf(z) : f \in \boldsymbol{P}\}\,,$$

$$\boldsymbol{EQ}_a = \left\{ z\left[1 + 2(1-a)\sum_{n=1}^{\infty}(\eta z)^n\right] : |\eta| = 1 \right\}.$$

For $z \neq 0$, consider on the space $\boldsymbol{H}(\boldsymbol{U}) \times \boldsymbol{H}(\boldsymbol{U})$ the continuous bilinear function $\boldsymbol{L}_z$ which is defined by the formula

$$\boldsymbol{L}_z(f,g) = \frac{1}{2} h * f * g(z)\,.$$

From (1) and Lemma 2(b) we have

$$\text{(2)}\quad \min\left\{ \operatorname{Re}\left[\frac{1}{2}h * f * g(z)\right] : f \in \boldsymbol{R}_t,\ g \in \boldsymbol{Q}_a \right\} =$$

$$= \min\left\{ \operatorname{Re}\left[1 + 4(1-a)\sum_{n=1}^{\infty}(\eta z)^n : |\eta| = 1\right]\right\}.$$

The right hand side of (2) is positive for every $z \in \boldsymbol{U}$ if and only if $a \geq \frac{1}{2}$. Also if $f \in \boldsymbol{R}_t$ then $f = f * g_0$ where $g_0(z) = z + \sum\limits_{n=1}^{\infty} z^{n+1} \in \boldsymbol{Q}_{1/2}$.

(b) By arguments similar to those of (a) we have the equalities

$$\min\left\{ \operatorname{Re}\left[\frac{1}{2}h * f * g(z)\right] : f \in \boldsymbol{R}_t,\ g \in \boldsymbol{R}_t \right\} =$$

$$= \min\left\{ \operatorname{Re}\left[1 + 4\sum_{n=1}^{\infty}\frac{(\eta z)^n}{\underline{t}(n)}\right] : |\eta| = 1 \right\} =$$

$$= -1 + 2\min\left\{ \operatorname{Re}\left[\frac{f(z)}{z}\right] : f \in \boldsymbol{ER}_t \right\} =$$

$$= -1 + 2\min\left\{ \operatorname{Re}\left[\frac{f(z)}{z}\right] : f \in \boldsymbol{R}_t \right\}.$$

Corollary 1. *If* $z + \sum\limits_{n=1}^{\infty} a_n z^{n+1} \in \boldsymbol{R}_t$ *then* $z + \sum\limits_{n=1}^{\infty} a_{nk} z^{nk+1} \in \boldsymbol{R}_t$.

Theorem 2. a) *If* $t \in \boldsymbol{E}_0$ *such that*

$$1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\underline{t}(n)} \geq \frac{1}{2}$$

then $\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{R}_t$.

b) *If* $t \in \boldsymbol{E}_1$ *such that*

$$1 + 2\sum_{n=1}^{\infty}(-1)^n \left[\frac{n+1}{\underline{t}(n)} - \frac{\delta_{1,k}}{t_k}\right] - \frac{\delta_{1,k}}{t_k} \geq 0$$

then $\boldsymbol{R}_t \subset \boldsymbol{S}$.

c) *If* $t \in \boldsymbol{E}_1$ *such that*

$$1 + 4\sum_{n=1}^{\infty}\frac{(n+1)}{\underline{t}^2(n)} \geq 0$$

then $\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{S}$.

PROOF. Conclusion (a) follow by combining Prop. 2 (a) and Th. 1. Conclusions 2(b) and 2(b′) follow analogously combining Prop. 2(b), 2(b′) and Th. 1.

Theorem 2′**.** *If* $a < 1,\ t = (0, \frac{1}{1-a}, \frac{1}{1-a})$ *or*

$$\boldsymbol{R}_t \equiv \left\{f \in \boldsymbol{A} : \operatorname{Re}\left[f'(z) + zf''(z)\right] > a\right\}$$

then we have

(a) $$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1 + 2(1-a)\left(\frac{\pi^2}{12} - 1\right) \quad \forall f \in \boldsymbol{R}_t$$

and

$$\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{R}_t \quad \text{if} \;\; a \geq 1 + \frac{1}{4}\left[\frac{\pi^2}{12} - 1\right]^{-1}$$

(a′) $$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1 + 4\left(\frac{7}{720}\pi^4 - 1\right)(1-a)^2 \quad \forall\, f \in \boldsymbol{R}_t \times \boldsymbol{R}_t$$

(b) $$\operatorname{Re} f'(z) > 1 + 2(1-a)(\log 2 - 1) \quad \forall\, f \in \boldsymbol{R}_t$$

and

$$\boldsymbol{R}_t \subset \boldsymbol{S} \quad \text{if} \;\; a \geq 1 - 2^{-1}(\log 2 - 1)^{-1}$$

(b′) $$\operatorname{Re} f'(z) > 1 + 4(\gamma - 1)(1-a)^2 \quad \forall\, f \in \boldsymbol{R}_t * \boldsymbol{R}_t$$

and

$$\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{S} \quad \text{if} \;\; a \geq 1 - 2^{-1}(1-\gamma)^{-\frac{1}{2}}$$

where $\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = 0,90152508\ldots$.

The constants in the right-hand sides of the above inequalities cannot be replaced by smaller ones.

PROOF. The proof is an immediate consequence of Prop. 2 if we assume that the relations below are known

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2} = \frac{\pi^2}{12} - 1, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} = \log 2 - 1,$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^3} = -1 + \frac{7}{720}\pi^4$$

Remark. Th. 2 and Corollary 1, in the special case $\boldsymbol{R}_t = \boldsymbol{R}$ yield Th. 3, Th. 3′ and Corollary 3 of paper [3].

Part (b) of Th. 2, in the special case $\boldsymbol{R}_t = \boldsymbol{R}$ yields part (a) of Th. 1 of paper [3].

Part (a) of Th. 2, in the special case $\boldsymbol{R}_t = \boldsymbol{R}$ yields

$$\mathrm{Re}\left[\frac{f(z)}{z}\right] > -1 + \frac{\pi^2}{6}$$

instead of

$$\mathrm{Re}\left[\frac{f(z)}{z}\right] > \frac{1}{2}$$

found in paper [3].

The number $-1 + \frac{\pi^2}{6}$ cannot be replaced by a smaller one.

Theorem 3. *If* $f \in \boldsymbol{A}$ *and* $\mathrm{Re}[f'(z) + zf''(z)] > \lambda$ *where* $\lambda = (6-\pi^2)(24-\pi^2)^{-1} = -0,27\ldots$ *then* $f_t \in \boldsymbol{S}_t$.

PROOF. The inequality $\mathrm{Re}[f'(z) + zf''(z)] > a$ yields

$$\mathrm{Re}\left[\frac{f(z)}{z}\right] > 1 + 2(1-a)\left(\frac{\pi^2}{12} - 1\right) \geq 0 \quad \text{when} \quad a \geq \frac{6-\pi^2}{12-\pi^2},$$
$$\mathrm{Re}[f'(z)] > 2\log 2 - 1 + a(2 - \log 2) \geq 0 \quad \text{when} \quad a \geq \frac{1 - 2\log 2}{2 - 2\log 2}.$$
$$\text{If } \frac{zf'(z)}{f(z)} = \frac{1+w(z)}{1-w(z)} \quad \text{we require} \quad |w(z)| < 1 \; \forall\, z \in \boldsymbol{U}.$$

If this last requirement is not true, then there exists $z_0 \in \boldsymbol{U},\ k \geq 1$ such that $w(z_0) = e^{i\theta}$ and $w(z_0) = kz_0 w'(z_0)$ (see [3], Lemma 2) and therefore

$$\operatorname{Re}[f'(z_0) + z_0 f''(z_0)] = \operatorname{Re}\left\{\left[\frac{f(z_0)}{z_0}\right]\left[\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2 - 2k\frac{e^{i\theta}}{(1-e^{i\theta})^2}\right]\right\}$$

$$\leq -\frac{1}{2}\operatorname{Re}\left[\frac{f(z_0}{z_0}\right] < -\frac{1}{2}\left[\frac{\pi^2}{6} - 1 + a\left(2 - \frac{\pi^2}{6}\right)\right] \leq a \quad \text{when} \quad a \leq \frac{6-\pi^2}{24-\pi^2}\,.$$

Corollary 3. *If* $f \in \boldsymbol{A}$ *and* $\operatorname{Re}\left[f'(z) + 3zf''(z) + z^2 f''(z)\right] > \lambda$ *then* $f \in \boldsymbol{K}$.

Theorem 4. (a) *If* $t \in \boldsymbol{E}_2$ *and*

$$1 + 2\sum_{n=1}^{\infty}(-1)^n\left[\frac{(n+1)^2}{\underline{t}(n)} - \frac{\delta_{2k}}{t_k}\right] - \frac{\delta_{2k}}{t_k} \geq \lambda$$

then $\boldsymbol{R}_t \subset \boldsymbol{S}_t$.

(a′) *If* $t \in \boldsymbol{E}_2$ *and*

$$1 + 4\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)^2}{\underline{t}^2(n)} \geq \lambda$$

then $\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{S}_t$.

(b) *If* $t \in \boldsymbol{E}_3$ *and*

$$1 + 2\sum_{n=1}^{\infty}(-1)^n\left[\frac{(n+1)^3}{\underline{t}(n)} - \frac{\delta_{3,k}}{t_k}\right] - \frac{\delta_{3,k}}{t_k} \geq \lambda$$

then $\boldsymbol{R}_t \subset \boldsymbol{K}$.

(b′) *If* $t \in \boldsymbol{E}_3$ *and*

$$1 + 4\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)^3}{\underline{t}^2(n)} > \lambda$$

then $\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{K}$.

PROOF. Conclusions (a) and (a′) follow by combining Prop. 2 and Th. 3 and 5. Conclusions (b) and (b′) follow analogously combining Theorem 5 and Corollary 3.

Corollary 4. *If* $a < 1,\ t = (0, \frac{1}{1-a}, \frac{1}{1-a})$ *or* $R_t = \{f \in A : \operatorname{Re}[f'(z) + zf'(z)] > a\}$ *then*

(a) *If* $a \geq 1 - 3[6(12-\pi^2)^{-1}(24-\pi^2)^{-1}]^{1/2}$ *then* $\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{S}_t$,

(b) *if* $a \geq 1 - 3[2(1-\log 2)(24-\pi^2)]^{-1/2}$ *then* $\boldsymbol{R}_t * \boldsymbol{R}_t \subset \boldsymbol{K}$.

Remark. Theorem 3 provides a stronger starlikeness criterion than Theorem 2 in paper [3], since the hypothesis in [3] is

$$\mathrm{Re}[f'(z) + zf''(z)] > -1/4\,.$$

Similarly, Corollary 3 in this paper forms a stronger convexity criterion than Corollary 2 in [3].

Part (b) of Corollary 4 provides a stronger convexity criterion than Theorem 4 in [3], since the hypothesis in paper [3] is

$$\mathrm{Re}[f'(z) + zf''(z)] > 0\,.$$

References

[1] F. Holland, The extreme points of a class of functions with positive real part, *Math. Ann.* **209** (1973), 85–87.

[2] G. Schober, Univalent function selected topics, Lecture. Notes in Math., vol. 478, *Springer–Verlag*, *New York*, 1975.

[3] R. Singh and S. Singh, Convolution properties of a class of strarlike functions, *Proc. Amer. Math. Soc.* **1** (106) (1989), 145–153.

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