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On group algebras with unit groups of derived length at most four

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Abstract. In this paper, we examine the relation between a finite group G and the units U in the group algebra of G over a field K of positive characteristic. By imposing certain natural conditions on the derived subgroups of U so that it has solvable length at most four, we show that the group G must be commutative.

1. Introduction

Let KG be the group algebra of a group G over a field K of positive characteristic p and let $U(KG) = U$ denote its multiplicative group of units. For subsets X, Y of a group G, we denote by (X, Y) the subgroup of G generated by all commutators $(x, y) = x^{-1}y^{-1}xy$ with $x \in X$ and $y \in Y$. The derived subgroups of G are defined as $G^{(0)} = G$, $G^{(1)} = G' = (G, G)$, and $G^{(i)} = (G^{(i-1)}, G^{(i-1)})$ for all $i > 0$. The descending normal series $G^{(0)} \rhd G^{(1)} \rhd \dots G^{(i)}$ is called the derived series of G. If the series terminates, i.e., if $G^{(n)} = 1$ for some integer n, then G is said to be solvable and the smallest such integer is called the derived length of G.

The investigation of necessary and sufficient conditions for the solvability of $U(KG)$ dates back to the 1970s with the works of Bateman and Passman [1], [2]. A lot of work has been done on this context with a complete solution of the problem being given by Bovdi [3]. However, computation of the derived length of U and the converse problem of finding the nature of G or its commutator subgroup G' for a fixed derived length of U still remain open. Various results on the derived length of U have been obtained (for example, [4]–[6]). Shalev [7] has classified

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group algebras of finite groups over fields of odd characteristic whose unit group is metabelian, and Kurdics [8] has done the same for even characteristic. Further, a necessary and sufficient condition for U to be centrally metabelian, that is, $(U^{(2)}, U) = 1$, is given by Sahai [9]. Also, characterisation of group algebras over fields of odd characteristic such that $(U^{(2)}, U') = 1$ has been given by Sahai [10] and the same with U satisfying $(U^{(2)}, U^{(2)}) = 1$ has been investigated by Chandra and Sahai [11], [12].

In this article we consider group algebras with unit group U which satisfies $(U^{(3)}, U') = 1$. Such U is obviously of derived length at most four, i.e., $U^{(4)} = 1$. Lie algebraic properties of KG play an important role in our investigation. For $X, Y \subseteq KG$, we denote by $[X, Y]$ the additive subgroup generated by all Lie commutators $[x, y] = xy - yx$, where $x \in X$ and $y \in Y$. Also, $O_p(G)$ stands for the maximal normal p-subgroup of G, and $\Delta(G)$ denotes the augmentation ideal of the group algebra KG. For any two elements $x, h \in G$, x^h denotes the conjugation of x by h, that is, $h^{-1}xh$. We denote the Frattini subgroup of a group G by $\Phi(G)$, which is the intersection of all maximal subgroups of G. It is well-known that $\Phi(G)$ is a characteristic subgroup of G. By a p'-element or a p'-automorphism of a group G we mean an element or an automorphism of G whose order is not divisible by p. All groups considered are finite. Our main result is as follows:

Theorem 1.1. *Let K be a field of characteristic $p \geq 17$ and let G be a group of odd order. Then G is abelian if and only if U satisfies $(U^{(3)}, U') = 1$.*

2. Background

In this section we discuss a few important known results which provide useful tools for the proof of our theorem. We first state the complete set of necessary and sufficient conditions for U to be solvable, given by Bovdi [3].

Theorem 2.1. *Let K be a field of finite characteristic p, and $O_p(G)$ a maximal normal p-subgroup of the finite group G. Then the group $U(KG)$ is solvable if and only if one of the following statements holds:*

(i) *G is abelian.*

(ii) *$G/O_p(G)$ is abelian and F is a field of characteristic p.*

(iii) *$|K| = 2$ and $G/O_p(G)$ is an extension of an elementary abelian 3-group A by a group $\langle b \rangle$ of order 2, and $bab = a^{-1}$ for all $a \in A$.*

(iv) $|K| = 3$ *and* $G/O_p(G)$ *is an extension of an elementary abelian* 2*-group* A *by a group* $\langle b\rangle$ *of order* 2.

(v) $|K| = 3$ *and* G *is an extension of an abelian group* A *of exponent* 4 *by a group* $\langle b\rangle$ *of order* 2 *and* $bab^{-1} = a^{-1}$ *for all* $a \in A$.

(vi) $|K| = 3$ *and* $G/O_p(G)$ *is an extension of an abelian group* A *of exponent* 8 *by a group* $\langle b\rangle$ *of order* 2 *and* $bab = a^3$ *for all* $a \in A$.

The next result can be found in ([13], Theorem 3.5)

Theorem 2.2. *Let* G *be a group of order* $p^a b$ *and* $(p, b) = 1$ *and let* K *be a field of characteristic* p*. Assume that* G *has a normal Sylow* p*-subgroup* P*. Then the Jacobson radical* $J = J(KG)$ *of* KG *is* $J = \Delta(P)KG$*.*

The following result can be found in ([14], Chapter 9.6). Recall that the nilpotency index of a nilpotent ideal I is the least positive integer n such that $I^n = 0$.

Theorem 2.3. *Let* G *be a finite* p*-group of order* p^m*. Then the nilpotency index of* $\Delta(G)$ *over a field of characteristic* p *is* p^m *if and only if* G *is cyclic. Further, if* $G = P_1 \times P_2 \times \cdots \times P_k$*, where each* P_i *is a cyclic subgroup of order* p^{t_i}*,* $t_i \in \mathbb{Z}$*, for* $i = 1, 2, \ldots, k$*, then the nilpotency index of* $\Delta(G)$ *is* $(p^{t_1} + p^{t_2} + \cdots + p^{t_k} - k + 1)$*.*

Next, we state a theorem by Burnside which can be found in ([15], Theorem 5.1.4).

Theorem 2.4. *Let* ψ *be a* p'*-automorphism of a* p*-group* P*, which induces the identity on* $P/\Phi(P)$*. Then* ψ *is the identity automorphism on* P*.*

The following can be found in ([15], Theorem 5.3.6).

Theorem 2.5. *If* A *is a* p'*-group of automorphisms of the* p*-group* P*, then* $(P, A, A) = (P, A)$*. In particular, if* $(P, A, A) = 1$*, then* $A = 1$*.*

If r is a real number, then by $\lceil r\rceil$ we denote the minimal integer not smaller than r. The next result can be found in [5].

Result 2.6. If KG is a non-commutative group algebra of a torsion nilpotent group G over a field K of positive characteristic p such that U is solvable, then the derived length of U is at least $\lceil \log_2(p + 1)\rceil$.

Let $\mathfrak{J}(G')$ denote the ideal $KG\Delta(G')$. Let $x, y \in U$ be such that $x-1 \in \mathfrak{J}(G')^i$ and $y - 1 \in \mathfrak{J}(G')^j$ for some $i, j > 0$. Then we have

$$(x, y) \equiv 1 + [x, y] \pmod{\mathfrak{J}(G')^{i+j+1}}. \tag{2.1}$$

Another important identity is, for any two elements x, y in KG, we have:

$$xy - 1 = (x-1)(y-1) + (x-1) + (y-1) \tag{2.2}$$

3. Proof of Theorem 1.1

- **Necessary conditions:**
  Let G be a group of odd order. When G is abelian, the result follows trivially.
- **Sufficient conditions:**
  Let KG be the group algebra of a finite group G over a field K of characteristic $p \geq 5$, such that, the unit group $U = U(KG)$ satisfies the condition $(U^{(3)}, U') = 1$. Then U is solvable and according to Theorem 2.1, $G/O_p(G)$ is abelian.

If $G/O_p(G)$ is abelian then $O_p(G)$ is a Sylow p-subgroup of G. Let $P = O_p(G)$. Now, $|P|$ and $[G : P]$ are relatively prime, hence by Schur–Zassenhaus Theorem ([15], Theorem 6.2.1), we have $G = P \rtimes H$, where H is a p'-prime subgroup of G. Also, by the above conditions, H is abelian.

Lemma 3.1. *Let* $\operatorname{Char} K = p \geq 11$. *Let G be a group of odd order. Suppose that U satisfies $(U^{(3)}, U') = 1$. Then $G = P \times H$, where P is a p-group and H is an abelian p'-group, where p' is odd.*

Proof. We know from above that $G = P \rtimes H$, where P is a p-group and H is an abelian p'-group. Also $P \trianglelefteq G$. Since G is of odd order, we have p' is odd. We need to show that $(P, H) = 1$. We will show that if $(P, H) \neq 1$, then we can construct nontrivial element in $(U^{(3)}, U')$. $\square$

We first assume that P is elementary abelian. Suppose, $(P, h) \neq 1$ for some $h \in H$. Then, $h^2 \neq 1$, and $(P, h) \leq P$ (as $P \trianglelefteq G$) and hence (P, h) is a p-group. Since h induces a p'-automorphism on P, by Theorem 2.5, $(P, h, h) = (P, h)$. Let $L = \langle (P, h), h \rangle$. Then $L' = (P, h, h) = (P, h)$ and $(P, h) \trianglelefteq L$. So on replacing G with L if necessary, we may assume that $P = G' = (P, h)$. By Theorem 2.2, the Jacobson radical $J = J(KG) = \Delta(P)KG$. Now, since $(P, h) \neq 1$, we can find $x \in P$ such that $(x, h) \neq 1$. Put $\alpha = x - 1$. Then $u = 1 + h\alpha$ is a unit in KG. Also, $(x, h), (x, h)^h, (x, h^2) \in P$. By forming commutators of suitable elements in U, we obtain elements in $U', U^{(2)}$ and then in $U^{(3)}$. Now consider $u_1 = (u, h) \in U'$ and $v_1 = (u, x) \in U'$. We have

$$\begin{aligned} u_1 = (u, h) &= 1 + u^{-1}h^{-1}[u, h] \\ &\equiv 1 + (1 - h\alpha)(\alpha h - h\alpha) \pmod{J^2} \end{aligned}$$

$$\begin{aligned}
&\equiv 1 + \alpha h - h\alpha \pmod{J^2}\\
&= 1 + hx((x,h)-1) \pmod{J^2}\\
&\equiv 1 + h((x,h)-1) \pmod{J^2} \text{ (as } x \equiv 1 \pmod{J}. \qquad (3.1)
\end{aligned}$$

As $G' = P$, we use identity (2.1) to obtain the following:

$$\begin{aligned}
v_1 = (u,x) &\equiv 1 + [u,x] \pmod{J^3}\\
&= 1 + (x + h\alpha x - x - xh\alpha) \pmod{J^3}\\
&= 1 + \{h(x-1)x - xh(x-1)\} \pmod{J^3}\\
&= 1 + (hx - xh)(x-1) \pmod{J^3}\\
&= 1 + hx(1-(x,h))(x-1) \pmod{J^3}\\
&\equiv 1 - h((x,h)-1)(x-1) \pmod{J^3} \text{ (as } x \equiv 1 \pmod{J}. \qquad (3.2)
\end{aligned}$$

Next we consider $u_2 = (u_1, x)$ and $v_2 = (v_1, x)$. As $x \in P = G' \subset U'$, u_2 and v_2 are in $U^{(2)}$.

$$\begin{aligned}
u_2 = (u_1,x) &\equiv 1 + [u_1,x] \pmod{J^3}\\
&= 1 + \{x + h((x,h)-1)x - x - xh((x,h)-1)\} \pmod{J^3}\\
&= 1 + (hx - xh)((x,h)-1) \pmod{J^3}\\
&= 1 + hx(1-(x,h))((x,h)-1) \pmod{J^3}\\
&\equiv 1 - h((x,h)-1)^2 \pmod{J^3} \text{ (as } x \equiv 1 \pmod{J}) \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
v_2 = (v_1,x) &\equiv 1 + [v_1,x] \pmod{J^4}\\
&= 1 + \{x - h((x,h)-1)(x-1)x - x + xh((x,h)-1)(x-1)\} \pmod{J^4}\\
&= 1 + (xh - hx)((x,h)-1)(x-1) \pmod{J^4}\\
&= 1 + hx((x,h)-1)^2(x-1) \pmod{J^4}\\
&\equiv 1 + h((x,h)-1)^2(x-1) \pmod{J^4} \text{ (as } x \equiv 1 \pmod{J}). \qquad (3.4)
\end{aligned}$$

Finally we obtain an element $w = (u_2, v_2)$ in $U^{(3)}$. We have

$$\begin{aligned}
w = (u_2,v_2) &\equiv 1 + [u_2,v_2] \pmod{J^6}\\
&= 1 + [u_2 - 1, v_2 - 1] \pmod{J^6}\\
&= 1 + \{-h((x,h)-1)^2 h((x,h)-1)^2(x-1)\\
&\qquad + h((x,h)-1)^2(x-1)h((x,h)-1)^2\} \pmod{J^6}
\end{aligned}$$

$$
\begin{aligned}
&= 1 + h((x,h)-1)^2\{(x-1)h - h(x-1)\}((x,h)-1)^2 \pmod{J^6} \\
&= 1 + h((x,h)-1)^2 hx((x,h)-1)^3 \pmod{J^6} \\
&\equiv 1 + h((x,h)-1)^2 h((x,h)-1)^3 \pmod{J^6} \text{ (as } x \equiv 1 \pmod{J}) \\
&= 1 + h^2((x,h)^h-1)^2((x,h)-1)^3 \pmod{J^6}. \qquad (3.5)
\end{aligned}
$$

We now show that the element (w,x) in $(U^{(3)}, U')$ is nontrivial. It suffices to show that $V_1 = (w,x)$ is nontrivial modulo J^7. Now,

$$
\begin{aligned}
V_1 = (w,x) &\equiv 1 + [w,x] \pmod{J^7} \\
&= 1 + [w-1,x] \pmod{J^7} \\
&= 1 + \{h^2((x,h)^h-1)^2((x,h)-1)^3x - xh^2((x,h)^h-1)^2((x,h)-1)^3\} \pmod{J^7} \\
&= 1 + (h^2x - xh^2)((x,h)^h-1)^2((x,h)-1)^3 \pmod{J^7} \\
&= 1 + h^2x(1-(x,h^2))((x,h)^h-1)^2((x,h)-1)^3 \pmod{J^7} \\
&\equiv 1 - h^2((x,h^2)-1)((x,h)^h-1)^2((x,h)-1)^3 \pmod{J^7} \\
&\qquad \text{(as } x \equiv 1 \pmod{J}). \qquad (3.6)
\end{aligned}
$$

Since P is elementary abelian, let $P = P_1 \times P_2 \times \cdots \times P_k$, where $k \geq 1$, each $P_i \cong C_p$, $i = 1, 2, \ldots k$ and C_p is a cyclic group of order p. Let x_i be the generator of P_i, for $i = 1, 2, \ldots, k$. Let

$$
\begin{aligned}
(x,h^2) &= {x_1}^{a_1}{x_2}^{a_2}\ldots{x_k}^{a_k} \\
(x,h)^h &= {x_1}^{b_1}{x_2}^{b_2}\ldots{x_k}^{b_k} \\
(x,h) &= {x_1}^{c_1}{x_2}^{c_2}\ldots{x_k}^{c_k}
\end{aligned}
$$

where a_i's, b_i's and c_i's are integers such that $1 \leq a_i, b_i, c_i \leq p$ for every $i = 1, 2, \ldots, k$, with at least one element in every set of a_i's, b_i's and c_i's being greater than or equal to one but strictly less than p. Then equation (3.6) can be written as:

$$
\begin{aligned}
V_1 = (w,x) \equiv 1 - h^2({x_1}^{a_1}{x_2}^{a_2}\ldots{x_k}^{a_k}-1)({x_1}^{b_1}{x_2}^{b_2}\ldots{x_k}^{b_k}-1)^2 \\
({x_1}^{c_1}{x_2}^{c_2}\ldots{x_k}^{c_k}-1)^3 \pmod{J^7}. \qquad (3.7)
\end{aligned}
$$

Now by repeated use of identity (2.2) and working modulo J^7, equation (3.7) becomes

$$
\begin{aligned}
V_1 \equiv 1 - h^2&\{({x_1}^{a_1}-1) + ({x_2}^{a_2}-1) + \cdots + ({x_k}^{a_k}-1)\} \\
&\times \{({x_1}^{b_1}-1) + ({x_2}^{b_2}-1) + \cdots + ({x_k}^{b_k}-1)\}^2 \\
&\times \{({x_1}^{c_1}-1) + ({x_2}^{c_2}-1) + \cdots + ({x_k}^{c_k}-1)\}^3. \qquad (3.8)
\end{aligned}
$$

Now, whenever we have $d \in \mathbb{N}$, we can write:

$$\begin{aligned}\therefore\ x^d - 1 &= (x-1)(1 + x + \cdots + x^{d-1})\\ &= (x-1)\{1 + ((x-1)+1) + \cdots + ((x^{d-1}-1)+1)\}\\ &= (x-1)(d + D_1) \text{ where } D_1 \in \Delta(P) \subseteq J = (x-1)D. \qquad (3.9)\end{aligned}$$

If $p \nmid d$, then $D = (d + D_1)$ is a unit in KG. With the help of this technique, the second term in RHS of equation (3.8) can be written as:

$$\begin{aligned}M = &\{(x_1-1)A_1 + (x_2-1)A_2 + \cdots + (x_k-1)A_k\}\\ &\times \{(x_1-1)B_1 + (x_2-1)B_2 + \cdots + (x_k-1)B_k\}^2\\ &\times \{(x_1-1)C_1 + (x_2-1)C_2 + \cdots + (x_k-1)C_k\}^3 \qquad (3.10)\end{aligned}$$

$$\begin{aligned}\text{i.e., } M = &\big\{(x_1-1)A_1 + (x_2-1)A_2 + \cdots + (x_k-1)A_k\big\}\\ &\times \Bigg\{\sum_{i=1}^{k}(x_i-1)^2{B_i}^2 + 2\sum_{\substack{i,j=1\\ i\neq j}}^{k}(x_i-1)(x_j-1)B_iB_j\Bigg\}\\ &\times \Bigg\{\sum_{i=1}^{k}(x_i-1)^3{C_i}^3 + 3\sum_{\substack{i,j=1\\ i\neq j}}^{k}(x_i-1)^2(x_j-1){C_i}^2C_j\\ &\qquad + 6\sum_{\substack{i,j,l=1\\ i\neq j\neq l}}^{k}(x_i-1)(x_j-1)(x_l-1)C_iC_jC_l\Bigg\} \qquad (3.11)\end{aligned}$$

where all the A_i's, B_i's, C_i's, for $i = 1, 2, \ldots, k$, belong to KG with at least one element in every set of A_i's, B_i's, C_i's, for $i = 1, 2, \ldots, k$, is a unit in KG. Now, by Theorem 2.3, nilpotency index of $\Delta(P)$ as well as J in this case is $(kp-k+1)$. Let, if possible, $V_1 \equiv 1 \pmod{J^7}$, that is , let $M \in J^7$.

Let $I = \{1, 2, \ldots, k\}$. Let $I_A = \{t \in I \mid A_t \text{ is a unit}\}$, $I_B = \{t \in I \mid B_t \text{ is a unit}\}$ and $I_C = \{t \in I \mid C_t \text{ is a unit}\}$. Whenever there is an element i in the intersection of any of these sets, we apply a trick by adjusting the powers of the element $(x_i - 1)$ to get an element contradicting the nilpotency index of $\Delta(P)$. Now the following mutually exclusive cases may arise:

Case (I) $I_C \cap I_B \neq \emptyset$. We consider the following mutually exclusive subcases:

(a) $I_C \cap I_B \cap I_A \neq \emptyset$. Let $r \in I_C \cap I_B \cap I_A$. We examine the term

$$\Bigg\{(x_r-1)^{p-7}\prod_{\substack{i=1\\ i\neq r}}^{k}(x_i-1)^{p-1}\Bigg\}M,$$

and find that $M \in J^7$ would imply

$$A_r B_r^2 C_r^3\,(x_1-1)^{p-1}(x_2-1)^{p-1}\dots(x_k-1)^{p-1} \in J^{kp-k-6}\,J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_r B_r^2 C_r^3$ is a unit in KG.

(b) $I_C \cap I_B \cap I_A = \emptyset$. Pick $m \in I_C \cap I_B$ and $m_A \in I_A$, so $m \notin I_A$. We examine the term

$$\left\{(x_m-1)^{p-6}(x_{m_A}-1)^{p-2}\prod_{\substack{i=1\\ i\neq m,m_A}}^{k}(x_i-1)^{p-1}\right\}M$$

and find that $M \in J^7$ would imply

$$A_{m_A} B_m^2 C_m^3\,(x_1-1)^{p-1}(x_2-1)^{p-1}\dots(x_k-1)^{p-1} \in J^{kp-k-6}\,J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_{m_A} B_m^2 C_m^3$ is a unit in KG.

Case (II) $I_C \cap I_B = \emptyset$. Again, this has the following subcases:

(a) $I_C \cap I_A \neq \emptyset$. Pick $l \in I_C \cap I_A$ and $l_B \in I_B$, so $l \notin I_B$ and $l_B \notin I_C$. We examine the term

$$\left\{(x_l-1)^{p-5}(x_{l_B}-1)^{p-3}\prod_{\substack{i=1\\ i\neq l,l_B}}^{k}(x_i-1)^{p-1}\right\}M$$

and find that $M \in J^7$ would imply

$$A_l B_{l_B}^2 C_l^3(x_1-1)^{p-1}(x_2-1)^{p-1}\dots(x_k-1)^{p-1} \in J^{kp-k-6}\,J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_l B_{l_B}^2 C_l^3$ is a unit in KG.

(b) $I_C \cap I_A = \emptyset$. This has the following subcases.

(i) $I_B \cap I_A \neq \emptyset$. Let $n \in I_B \cap I_A$, and $n_C \in I_C$, so $n \notin I_C$ and $n_C \notin I_A \cup I_B$. We examine the term

$$\left\{(x_n-1)^{p-4}(x_{n_C}-1)^{p-4}\prod_{\substack{i=1\\ i\neq n,n_C}}^{k}(x_i-1)^{p-1}\right\}M$$

and find that $M \in J^7$ would imply

$$A_n B_n^2 C_{n_C}^3(x_1-1)^{p-1}(x_2-1)^{p-1}\dots(x_k-1)^{p-1} \in J^{kp-k-6}\,J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_n B_n^2 C_{n_C}^3$ is a unit in KG.

(ii) $I_B \cap I_A = \emptyset$, so that I_A, I_B and I_C are pairwise disjoint. Let $d \in I_A$, $e \in I_B$ and $f \in I_C$. Examining the term

$$\left\{(x_d-1)^{p-2}(x_e-1)^{p-3}(x_f-1)^{p-4}\prod_{\substack{i=1\\ i\neq d,e,f}}^{k}(x_i-1)^{p-1}\right\}M$$

and find that $M \in J^7$ would imply

$$A_dB_e^2C_f^3\,(x_1-1)^{p-1}(x_2-1)^{p-1}\ldots(x_k-1)^{p-1} \in J^{kp-k-6}\,J^7=(0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_dB_e^2C_f^3$ is a unit in KG.

Therefore, $V_1 \not\equiv 1 \pmod{J^7}$, which implies that V_1 is a nontrivial element in $(U^{(3)}, U') = 1$, a contradiction to our given condition. So, when P is elementary abelian, we get that $G = P \times H$.

Now, let P be any p-group. Assume $(P, h) \neq 1$ for some $h \in H$. As the Frattini subgroup $\Phi(P)$ is a characteristic subgroup of P, we have $h\Phi(P) = \Phi(P)$ and hence h induces an automorphism on $P/\Phi(P)$. Now $P/\Phi(P)$ is elementary abelian (by [15], Theorem 5.1.3). Now, we have already proved that h induces the identity automorphism on the elementary abelian group $P/\Phi(P)$. Hence by Theorem 2.4, h induces the identity automorphism on P as well. Hence we get, $G = P \times H$.

Proposition 3.2. *Let* $\operatorname{Char} K = p \geq 17$ *and let* G *be a finite* p*-group such that* U *satisfies* $(U^{(3)}, U') = 1$, *then* G *is abelian.*

Proof. A finite p-group is nilpotent and torsion. If G is non-abelian then by Result 2.6, the derived length of U for $p \geq 17$ is $\lceil \log_2(p+1) \rceil \geq \lceil \log_2(17+1) \rceil \approx \lceil 4.16 \rceil = 5$. Thus U can only satisfy $(U^{(3)}, U') = 1$, if G is abelian. □

- **Conclusion:**
  Combining lemma 3.1 and Proposition 3.2, we find that when $\operatorname{Char} K \geq 17$ and G is a group of odd order such that U satisfies $(U^{(3)}, U') = 1$, then G is abelian. Hence, Theorem 1.1 is proved.

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