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Peripherally multiplicative maps between Figà–Talamanca–Herz algebras

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Abstract. The main purpose of this paper is to characterize, not necessarily linear, generalized (weakly) peripherally multiplicative maps between Figà–Talamanca–Herz algebras. Let G_1 and G_2 be locally compact Hausdorff groups, Γ and Ω be arbitrary nonempty sets, and $1 < p < \infty$. We characterize surjections $S_1 : \Gamma \longrightarrow A_p(G_1)$, $S_2 : \Omega \longrightarrow A_p(G_1)$, $T_1 : \Gamma \longrightarrow A_p(G_2)$ and $T_2 : \Omega \longrightarrow A_p(G_2)$ satisfying $\|T_1(\gamma)T_2(\omega)\|_\infty = \|S_1(\gamma)S_2(\omega)\|_\infty$ for all $\gamma \in \Gamma$, $\omega \in \Omega$. We apply this to get a description of certain peripherally multiplicative maps. In particular, it is shown that if surjections $T_1, T_2 : A_p(G_1) \longrightarrow A_p(G_2)$ satisfy $R_\pi(T_1(f)T_2(g)) \subseteq R_\pi(fg)$ for all $f, g \in A_p(G_1)$, or $R_\pi(fg) \subseteq R_\pi(T_1(f)T_2(g))$ for all $f, g \in A_p(G_1)$, then T_1 and T_2 are weighted composition operators. For amenable groups G_1 and G_2, T_1 and T_2 are shown to be weighted isomorphisms which induce an algebra isomorphism between $A_p(G_1)$ and $A_p(G_2)$. Moreover, when one of G_1 or G_2 is first countable, precise characterizations of weakly peripherally multiplicative maps are obtained. Conditions are also given to guarantee that T_1 and T_2 are algebra isomorphisms.

1. Introduction

Finding the general form of algebra isomorphisms between various classes of Banach algebras has a long history. More generally, the problem of characterizing maps which preserve some properties or certain subsets of the spectrum of elements in the domain algebra (the so called spectral persevere problem) has recently gained a lot of interest [8].

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The Gleason–Kahan–Żelazko theorem [32] on the characterization of multiplicative linear functionals of Banach algebras is one of the famous results in this direction. Motivated by this theorem, Kowalski and Słodkowski in [19] obtained a similar result without the linearity assumption. On the other hand, Molnár in [25] described surjections $T : C(X) \longrightarrow C(X)$ satisfying $(T(f)T(g))(X) = (fg)(X)$ for all $f, g \in C(X)$ as weighted composition operators, where $C(X)$ is the Banach algebra of all continuous complex-valued functions on a first countable compact Hausdorff space X. If moreover, $T(1) = 1$, then T is an isometric algebra isomorphism. Generalizations of this result were given in [9], [10], [12], [27], [28] for some semisimple commutative Banach algebras, and in [13] for particular topological function algebras.

Replacing the range or the spectrum by a subset, this result is extended in [24] by Luttman and Tonev, who introduced the notions of *peripheral range* and *peripheral spectrum* of an algebra element. Let A be a Banach function algebra on a locally compact Hausdorff space X and $f \in A$. The peripheral range and the peripheral spectrum of f are defined, respectively by

$$R_\pi(f) = \{z \in f(X) : |z| = \|f\|_\infty\} \quad \text{and} \quad \sigma_\pi(f) = \{z \in \sigma(f) : |z| = r(f)\},$$

where $\sigma(f)$ and $r(f)$ denote the spectrum and the spectral radius of f. It should be noted that these two sets coincide for functions in a uniform algebra [24, Lemma 1]. A function $h \in A$ is called a peaking function, writing $h \in \mathcal{P}(A)$, if $R_\pi(h) = \{1\}$.

Let A and B be Banach function algebras on locally compact Hausdorff spaces X and Y, respectively. A map $T : A \longrightarrow B$ is peripherally multiplicative [24] if it preserves the peripheral spectrum of the products, that is, $R_\pi(T(f)T(g)) = R_\pi(fg)$ for all $f, g \in A$. The peripherally multiplicative surjections between unital and non-unital uniformly closed function algebras are known to be weighted composition operators in [24] and [6], respectively. In [16] the same is proved for maps between Lipschitz algebras. Generalizations of results between uniform algebras were obtained in [7], [5] by considering maps with $R_\pi(T^m(f)T^n(g)) \subseteq R_\pi(f^m g^n)$ $(m, n \in \mathbb{N})$. A map $T : A \longrightarrow B$ is weakly peripherally multiplicative if $R_\pi(T(f)T(g)) \cap R_\pi(fg) \neq \varnothing$ for all $f, g \in A$ [20].

In general it is interesting to find conditions on maps between algebras of functions to force them to be weighted composition operators. For weakly peripherally multiplicative maps, it is not known if such maps between general uniform algebras are weighted composition operators. The question has been answered in the affirmative, under certain additional assumptions for algebras, or for maps between uniformly closed function algebras [20], [30]. In [20] it is shown that if a weakly peripherally multiplicative surjection $T : A \longrightarrow B$ between uniform algebras A and B satisfies $T(\mathcal{P}(A)) = \mathcal{P}(B)$, then T is an isometric algebra isomorphism. Weakly peripherally multiplicative maps between Lipschitz algebras are characterized in [17]. In [21], LEE and LUTTMAN studied weakly peripherally multiplicative surjections $T : A \longrightarrow B$ between uniform algebras A and B, without the assumption that T preserves the peaking functions, for the case where underling spaces are first countable and showed them to be weighted composition operators. They also obtained a similar result for arbitrary pair of maps $T_1, T_2 : A \longrightarrow B$ satisfying $R_\pi(T_1(f)T_2(g)) \cap R_\pi(fg) \neq \varnothing$ for all $f, g \in A$. Then involving more than two maps between uniform algebras, extensions of some previous results are given in [29]. Results on weakly peripherally multiplicative maps under alternative conditions are obtained in [8], [18]. The most recent result is as follows. The pointed Lipschitz algebra $\mathrm{Lip}_0(X)$, on a pointed compact metric space X with distinguished base point e_X, is the Banach function algebra of all complex-valued Lipschitz functions f on X that $f(e_X) = 0$. It is shown in [15] that if surjections $S_1, S_2 : \mathrm{Lip}_0(X) \longrightarrow \mathrm{Lip}_0(X)$ and $T_1, T_2 : \mathrm{Lip}_0(X) \longrightarrow \mathrm{Lip}_0(Y)$ satisfy $R_\pi(T_1(f)T_2(g)) \cap R_\pi(S_1(f)S_2(g)) \neq \varnothing$ for all $f, g \in \mathrm{Lip}_0(X)$, then there exist continuous functions $h_1, h_2 : Y \longrightarrow \mathbb{C}$ and a base point preserving Lipschitz homeomorphism $\varphi : Y \longrightarrow X$ such that $h_1(y)h_2(y) = 1$ and $T_i(f)(y) = h_i(y)S_i(f)(\varphi(y))$, for all $f \in \mathrm{Lip}_0(X)$ and $y \in Y$ $(i = 1, 2)$.

The main purpose of this paper is to characterize certain peripheral preservers between Figà–Talamanca–Herz algebras. Given a locally compact group G, the maximal ideal space of the Figà–Talamanca–Herz algebra $A_p(G)$ is G, and so for each $f \in A_p(G)$, $\sigma_\pi(f) = R_\pi(f)$. Therefore the peripheral range preserving maps coincide with the peripheral spectrum preserving ones in this case. This follows from the observation that there are an abundance of peaking functions for these algebras.

This paper is organized as follows. In Section 2, we review the basic notions and preliminaries, used throughout the rest of the paper. Section 3 is devoted to the study of generalized norm-multiplicative maps between Figà–Talamanca–Herz algebras (compare with [15]). Characterizations of generalized peripherally multiplicative maps are given in Section 4. In Section 5, we study weakly peripherally multiplicative maps between Figà–Talamanca–Herz algebras. We give a complete description when one of the underling groups is first countable. Besides, sufficient conditions are given to ensure that such maps are algebra isomorphisms. Note that we consider the maps on different sets and obtain their descriptions independent of any structure of the index sets as previous results (see [7], [29]), in fact, because in most cases we can ignore the index sets and define induced maps on the algebras. We finally note that using the multiplicative Bishop's lemma in the context of uniformly closed function algebras, our proofs are also applicable to uniformly closed function algebras, extending some results of [7], [14], [18], [21].

2. Preliminaries

Throughout this section X is a locally compact Hausdorff space and G is a locally compact Hausdorff group with the left Haar measure λ. Let $C_0(X)$ denote the algebra of continuous complex-valued functions on X vanishing at infinity, with the supremum norm $\|\cdot\|_\infty$. A subalgebra A of $C_0(X)$ is called a *function algebra* on X if A strongly separates the points of X in the sense that for each $x, x' \in X$ with $x \neq x'$ there exists a function $f \in A$ with $f(x) \neq f(x')$ and for each $x \in X$ there exists a function $g \in A$ with $g(x) \neq 0$. A function algebra on X is called a *Banach function algebra* on X if it is a Banach algebra under a norm. In the case where X is compact, all function algebras on X are assumed to contain the constants and each uniformly closed function algebra on X is also called a *uniform algebra* on X.

Let A be a Banach function algebra on a locally compact Hausdorff space X. We denote the uniform norm on A by $\|.\|_\infty$ (some authors use the notation $\|.\|_X$). For $f \in A$ let $R_\pi(f)$ and $\sigma_\pi(f)$ be the peripheral range and the peripheral spectrum of f, respectively. These sets coincide if A is a uniform algebra [24, Lemma 1]. A function $f \in A$ is called a *peaking function* if $R_\pi(f) = \{1\}$. The set of all peaking functions in A is denoted by $\mathcal{P}(A)$; by $\mathcal{P}_x$ (or $\mathcal{P}_x(A)$) the set of all functions $f \in \mathcal{P}(A)$ with $f(x) = 1$; by $\mathcal{F}_x$ (or $\mathcal{F}_x(A)$) the set of all functions $f \in A$ such that $f(x) = 1 = \|f\|_\infty$. A point $x \in X$ is called a *strong boundary point* for A if for each neighborhood V of x there exists a function $f \in A$ with $\|f\|_X = f(x) = 1$ and $|f| < 1$ on $X \setminus V$. The *Choquet boundary* $\mathrm{Ch}(A)$ is the set of all points $x \in X$ for which δ_x, the evaluation functional at x, is an extreme point of the unit ball of the dual space of $(A, \|.\|_\infty)$. It is known that if A is a uniformly closed function algebra on a locally compact Hausdorff space X, then $x \in \mathrm{Ch}(A)$ if and only if x is a strong boundary point (see [22] and [28, Theorem 2.1]), but in general points in the Choquet boundary are not necessarily strong boundary points [2]. Given $f \in A$ we denote the maximum modulus set of f by $M_f = \{x \in X : |f(x)| = \|f\|_\infty\}$.

Let $1 < p < \infty$ and q be the conjugate to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Given a locally compact group G, the space $A_p(G)$ consists of all functions $f \in C_0(G)$ which can be represented as $f = \sum_{i=1}^{\infty} f_i * \check{g}_i$, where $f_i \in L_p(G)$, $g_i \in L_q(G)$, $\check{g}_i(x) = g_i(x^{-1})$, $(f_i * \check{g}_i)(x) = \int_G f_i(xy) g_i(y) d\lambda(y)$ for all $x \in G$, and $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$. The norm of $f \in A_p(G)$ is defined by

$$\|f\| = \inf \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q,$$

where the infimum is taken over all representations of f from the above. $A_p(G)$ is a Banach algebra, called the *Figà–Talamanca–Herz algebra*. For $p = 2$, $A_2(G)$ is the *Fourier algebra* of G introduced by Eymard [3]. If G is abelian with dual group $\hat{G}$, $A_2(G)$ is the set of Fourier transforms of all functions in $L_1(\hat{G})$. By [1, Theorem 4.5.31], $A_p(G)$ is indeed a regular Banach function algebra with maximal ideal space G and every point in G is a strong boundary point for $A_p(G)$, in particular, $\mathrm{Ch}(A_p(G)) = G$. Since $A_p(G)$ is self-adjoint, for each $x \in X$ and neighborhood V of x there is a peaking function $f \in A_p(G)$ with $0 \le f \le 1$, $f(x) = 1$ and $f = 0$ on $X \setminus V$.

For $f \in L_\infty(G)$, the left translation of f by $x \in G$ is $L_x f(y) = f(xy)$. A group G is called amenable if there exists a continuous linear functional $m \in L_\infty(G)^*$ such that $\|m\| = m(1) = 1$ and $m(L_x f) = m(f)$ for every $x \in G$, $f \in L_\infty(G)$. These include abelian and compact groups.

3. Jointly norm-multiplicative maps

In the rest of this paper G, G_1 and G_2 are locally compact groups, Γ and Ω are arbitrary nonempty sets and $1 < p < \infty$ is a fixed number with conjugate q. The identity element of a group G is denoted by e. Let $S_1 : \Gamma \longrightarrow A_p(G_1)$, $S_2 : \Omega \longrightarrow A_p(G_1)$, $T_1 : \Gamma \longrightarrow A_p(G_2)$ and $T_2 : \Omega \longrightarrow A_p(G_2)$ be surjective maps satisfying

$$\|T_1(\gamma) T_2(\omega)\|_\infty = \|S_1(\gamma) S_2(\omega)\|_\infty \quad (\gamma \in \Gamma,\ \omega \in \Omega).$$

These are the so called jointly norm-multiplicative maps. In this section, we give a complete characterization of such maps between Figà–Talamanca–Herz algebras. Similar results are obtained for uniform algebras in [21], [29] and for Lipschitz algebras in [15]. Note that if $T_1 = T_2$ and $S_1 = S_2 = \mathrm{id}$, the results are proved in [30] for arbitrary dense subalgebras of uniformly closed function algebras (where $A_p(G)$ is a particular cases).

The following lemma is a multiplicative version of the classical Bishop's lemma (cf. [8]) for Figà–Talamanca–Herz algebras. It should be mentioned that a similar generalization of the Bishop's lemma for uniformly closed function algebras is proved by TONEV in [30, Proposition 3.1], and we used the idea of his argument.

Lemma 3.1. *If $f \in A_p(G)$ and $x_0 \in G$ with $f(x_0) \neq 0$, then there exists a peaking function $u \in A_p(G)$ such that $u(x_0) = 1$ and $R_\pi(fu) = \{f(x_0)\}$.*

PROOF. Without loss of generality we may assume that $f(x_0) = 1$. For $n \in \mathbb{N}$, consider the open set

$$U_n = \left\{ x \in G : |f(x) - 1| < \frac{1}{2^{n+1}} \right\}.$$

Choose a compact symmetric neighborhood V_n of e such that $V_n V_n x_0 \subseteq U_n$. Since $\chi_{V_n} \in L_p(G)$ and $\chi_{x_0^{-1} V_n} \in L_q(G)$, the function $k_n = \lambda(V_n)^{-1} \chi_{V_n} * \chi_{V_n x_0}$ is in $A_p(G)$ with $k_n(x_0) = 1$, $k_n = 0$ on $G \setminus U_n$ and $1 \leq \|k_n\|_\infty \leq \lambda(V_n)^{-1} \|\chi_{V_n}\|_p \|\chi_{V_n x_0}\|_q = 1$. In particular, $\|k_n\| = 1$. Thus since $A_p(G)$ is self-adjoint, $u_n = k_n \overline{k_n}$ belongs to $A_p(G)$, where $\overline{k_n}$ is the complex conjugate of k_n. Hence one can see that u_n is a peaking function in $A_p(G)$ with $u_n(x_0) = 1 = \|u_n\|$ and $u_n = 0$ on $G \setminus U_n$. Set $u = \sum_{n=1}^{\infty} \frac{u_n}{2^n}$. Since $\|u_n\| = 1$ for each $n \in \mathbb{N}$, the series converges and $u \in A_p(G)$. Then u is a peaking function with $u(x_0) = 1 = \|u\|$ and $M_u \subseteq \bigcap_{n=1}^{\infty} M_{u_n} \subseteq \bigcap_{n=1}^{\infty} U_n = f^{-1}\{1\}$.

Next, observe that $R_\pi(fu) = \{1\}$. Clearly $(fu)(x_0) = 1$ and if $x \notin \bigcup_{n=1}^{\infty} U_n$, then $(fu)(x) = 0$. For $x \in \bigcap_{n=1}^{\infty} U_n$, $f(x) = 1$ and $|f(x)u(x)| \leq 1$ and moreover, it is apparent that $|f(x)u(x)| = 1$ if and only if $f(x)u(x) = 1$. Finally if x belongs to $U_1, \ldots, U_{n-1}$ but not to U_n, then

$$\begin{aligned} |(fu)(x)| \leq (1 + |f(x) - 1|)|u(x)| &\leq \left(1 + \frac{1}{2^n}\right) \sum_{i=1}^{n-1} \frac{1}{2^i} \\ &< \left(1 + \frac{1}{2^{n-1}}\right)\left(1 - \frac{1}{2^{n-1}}\right) < 1. \end{aligned}$$ □

Lemma 3.2. *For $f, g \in A_p(G)$, $|f| \leqslant |g|$ iff for each $r \geqslant 0$ and $h \in A_p(G)$, $|gh| \leqslant r$ implies $|fh| \leqslant r$.*

PROOF. We only need to prove the necessity. Assume on the contrary that there exists $x_0 \in G$ such that $|g(x_0)| < |f(x_0)|$. Set $t = \frac{1}{2}(|f(x_0)| + |g(x_0)|)$, then $|g(x_0)| < t < |f(x_0)|$. Choose a neighborhood V of x_0 such that $|g| < t$ on V. Since x_0 is a strong boundary point for $A_p(G)$, there is a peaking function $h \in \mathcal{P}_{x_0}$ such that $|h| < \frac{t}{\|g\|_\infty + 1}$ on $G \setminus V$. Thus $|(fh)(x_0)| > t$, while $|gh| \leq t$, which is a contradiction. □

The above lemma immediately implies the following.

Lemma 3.3. *For $\gamma, \gamma' \in \Gamma$, $|S_1(\gamma)| \leq |S_1(\gamma')|$ if and only if $|T_1(\gamma)| \leq |T_1(\gamma')|$. Similarly, given $\omega, \omega' \in \Omega$, $|S_2(\omega)| \leq |S_2(\omega')|$ if and only if $|T_2(\omega)| \leq |T_2(\omega')|$.*

Proof. Let $\gamma, \gamma' \in \Gamma$ and $|S_1(\gamma)| \leq |S_1(\gamma')|$. Assume that $r \geqslant 0$, $h \in A_p(G_1)$ and $|T_1(\gamma')h| \leqslant r$. Taking $\omega_0 \in \Omega$ such that $h = T_2(\omega_0)$ we have

$$\begin{aligned}\|T_1(\gamma)h\|_\infty &= \|T_1(\gamma)T_2(\omega_0)\|_\infty = \|S_1(\gamma)S_2(\omega_0)\|_\infty \leq \|S_1(\gamma')S_2(\omega_0)\|_\infty \\ &= \|T_1(\gamma')T_2(\omega_0)\|_\infty \leq r,\end{aligned}$$

in particular, $|T_1(\gamma)h| \leqslant r$. Consequently, $|T_1(\gamma)| \leq |T_1(\gamma')|$, by Lemma 3.2. The other cases could be concluded similarly. □

Lemma 3.4. *For each $x \in G_1$ there is $y \in G_2$ such that $y \in M_{T_1(\gamma)T_2(\omega)}$ for all $\gamma \in \Gamma$ and $\omega \in \Omega$ with $S_1(\gamma), S_2(\omega) \in \mathcal{F}_x$.*

Proof. Let $\gamma \in \Gamma$ and $\omega \in \Omega$. For each $S_1(\gamma), S_2(\omega) \in \mathcal{F}_x$, the maximizing set $M_{T_1(\gamma)T_2(\omega)}$ is a compact subset of the one point compactification $G_2 \cup \{\infty\}$ of G_2. It is enough to show that the family $\{M_{T_1(\gamma)T_2(\omega)} : S_1(\gamma), S_2(\omega) \in \mathcal{F}_x\}$ has the finite intersection property. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $\omega_1, \ldots, \omega_n \in \Omega$ be elements in $S_1^{-1}(\mathcal{F}_x)$ and $S_2^{-1}(\mathcal{F}_x)$, respectively. There exist $\gamma \in \Gamma$ and $\omega \in \Omega$ with $S_1(\gamma) = \prod_{i=1}^n S_1(\gamma_i)$ and $S_2(\omega) = \prod_{i=1}^n S_2(\omega_i)$. Clearly, $S_1(\gamma), S_2(\omega) \in \mathcal{F}_x$,$|S_1(\gamma)| \leq |S_1(\gamma_i)|$ and $|S_2(\omega)| \leq |S_2(\omega_i)|$ for all $i \in \{1, \ldots, n\}$. By Lemma 3.3,

$$|T_1(\gamma)| \leq |T_1(\gamma_i)|, |T_2(\omega)| \leq |T_2(\omega_i)| \quad (i = 1, \ldots, n).$$

Now, choose $y \in G_2$ such that $|T_1(\gamma)(y)T_2(\omega)(y)| = 1 = \|T_1(\gamma)T_2(\omega)\|_\infty$, then $y \in \bigcap_{i=1}^n M_{T_1(\gamma_i)T_2(\omega_i)}$. Therefore $\bigcap_{i=1}^n M_{T_1(\gamma_i)T_2(\omega_i)} \neq \emptyset$ as desired. □

For each $x \in G_1$ let $\mathcal{I}_x$ denote the nonempty set of those $y \in G_2$ satisfying the statement of Lemma 3.4.

Lemma 3.5. *Let $\gamma \in \Gamma$, $\omega \in \Omega$, $x \in G_1$ and $y \in \mathcal{I}_x$. Then $S_1(\gamma)S_2(\omega) \in \mathcal{F}_x$ if and only if $T_1(\gamma)T_2(\omega) \in \mathcal{F}_y$.*

Proof. Let $T_1(\gamma)T_2(\omega) \in \mathcal{F}_y$. We show that $|(S_1(\gamma)S_2(\omega))(x)| = 1$. If $(S_1(\gamma)S_2(\omega))(x) = 0$, we may assume that $S_1(\gamma)(x) = 0$. Since x is a strong boundary point for $A_p(G_1)$, there is an $f \in \mathcal{P}_x$ such that $\|S_1(\gamma)f\| < \frac{1}{\|S_2(\omega)\|_\infty}$. Let $\gamma_1 \in \Gamma$ and $\omega_1 \in \Omega$ satisfy $S_1(\gamma_1) = S_2(\omega_1) = f$. Then $|T_1(\gamma_1)(y)T_2(\omega_1)(y)| = 1$ and

$$\begin{aligned}1 &= \|T_1(\gamma)T_2(\omega)T_1(\gamma_1)T_2(\omega_1)\|_\infty \leq \|T_1(\gamma)T_2(\omega_1)\|_\infty \|T_1(\gamma_1)T_2(\omega)\|_\infty \\ &= \|S_1(\gamma)S_2(\omega_1)\|_\infty \|S_1(\gamma_1)S_2(\omega)\|_\infty < \frac{1}{\|S_2(\omega)\|_\infty}\|S_2(\omega)\|_\infty = 1,\end{aligned}$$

which is impossible. Therefore, $(S_1(\gamma)S_2(\omega))(x) \neq 0$. By Lemma 3.1 there exist peaking functions $f_1, f_2 \in \mathcal{F}_x$ such that $R_\pi(S_1(\gamma)f_1) = \{S_1(\gamma)(x)\}$ and $R_\pi(S_2(\omega)f_2) = \{S_2(\omega)(x)\}$. If we choose $\gamma' \in \Gamma$ and $\omega' \in \Omega$ such that $S_1(\gamma') = f_2$ and $S_2(\omega') = f_1$, then $T_1(\gamma')T_2(\omega') \in \mathcal{F}_y$. Consequently

$$\begin{aligned}|(S_1(\gamma)S_2(\omega))(x)| &= \|S_1(\gamma)f_1\|_\infty \|S_2(\omega)f_2\|_\infty = \|T_1(\gamma)T_2(\omega')\|_\infty \|T_1(\gamma')T_2(\omega)\|_\infty \\ &\geq \|T_1(\gamma)T_2(\omega')T_1(\gamma')T_2(\omega)\|_\infty = 1.\end{aligned}$$

Therefore $|(S_1(\gamma)S_2(\omega))(x)| = 1$, that is, $S_1(\gamma)S_2(\omega) \in \mathcal{F}_x$. This shows that if $T_1(\gamma)T_2(\omega) \in \mathcal{F}_y$ then $S_1(\gamma)S_2(\omega) \in \mathcal{F}_x$. Now using this conclusion the converse is obtained similarly. □

Lemma 3.6. *For each $x \in G_1$ there is a unique $y \in G_2$ such that $\mathcal{I}_x = \{y\}$.*

Proof. By Lemma 3.4, $\mathcal{I}_x \neq \emptyset$. Choose a point $y \in \mathcal{I}_x$. We claim that $\mathcal{I}_x = \{y\}$. If there exists $z \in \mathcal{I}_x \setminus \{y\}$, and V is a neighborhood of y not containing z, we may choose a peaking function $f \in A_p(G_2)$ with $f(y) = 1$ and $|f| < 1$ on $G_2 \setminus V$, in particular $|f(z)| < 1$. Let $\gamma \in \Gamma$ and $\omega \in \Omega$ be such that $T_1(\gamma) = T_2(\omega) = f$. Then $T_1(\gamma)T_2(\omega) \in \mathcal{F}_y$ and by Lemma 3.5, $S_1(\gamma)S_2(\omega) \in \mathcal{F}_x$. Again by Lemma 3.5, $|f^2(z)| = |(T_1(\gamma)T_2(\omega))(z)| = 1$, which is a contradiction. □

Now we may define the map $\psi : G_1 \longrightarrow G_2$ by $\psi(x) := y$, where y is the unique element of $\mathcal{I}_x$ obtained from the above lemma.

Lemma 3.7. *The map $\psi : G_1 \longrightarrow G_2$ is bijective.*

Proof. Let $x, x' \in G_1$ and $\psi(x) = \psi(x')$. If $x \neq x'$, consider a neighborhood U of x such that $x' \notin U$. Choose a peaking function $f \in A_p(G_1)$ with $f(x) = 1$ and $|f| < 1$ on $G_1 \setminus U$. Let $\gamma \in \Gamma$ and $\omega \in \Omega$ be such that $f = S_1(\gamma) = S_2(\omega)$. Then $S_1(\gamma)S_2(\omega) \in \mathcal{F}_x$ and by Lemma 3.5, $T_1(\gamma)T_2(\omega) \in F_{\psi(x)} = F_{\psi(x')}$. Again by Lemma 3.5, $f^2 = S_1(\gamma)S_2(\omega) \in F_{x'}$, which is a contradiction. Therefore ψ is injective.

To show surjectivity of ψ, let $y \in G_2$. By arguments as before, we may conclude that there is a unique point $x \in G_1$ such that $x \in M_{S_1(\gamma)S_2(\omega)}$ for all $T_1(p), T_2(q) \in \mathcal{F}_y$; and also for given $\gamma \in \Gamma$ and $\omega \in \Omega$, $S_1(\gamma)S_2(\omega) \in \mathcal{F}_x$ if and only if $T_1(\gamma)T_2(\omega) \in \mathcal{F}_y$. By Lemma 3.5, $S_1(\gamma)S_2(\omega) \in \mathcal{F}_x$ if and only if $T_1(\gamma)T_2(\omega) \in F_{\psi(x)}$, where $\gamma \in \Gamma, \omega \in \Omega$. Thus for each $\gamma \in \Gamma$ and $\omega \in \Omega$, $T_1(\gamma)T_2(\omega) \in F_{\psi(x)}$ if and only if $T_1(\gamma)T_2(\omega) \in \mathcal{F}_y$, so it follows easily that $\psi(x) = y$. □

Lemma 3.8. *For each $\gamma \in \Gamma$, $\omega \in \Omega$ and $x \in G_1$, $|(T_1(\gamma)T_2(\omega))(\psi(x))| = |(S_1(\gamma)S_2(\omega))(x)|$.*

Proof. Since $\|T_1(\gamma)T_2(\omega)\|_\infty = \|S_1(\gamma)S_2(\omega)\|_\infty$, we may assume that $T_1(\gamma)$, $T_2(\omega)$, $S_1(\gamma)$ and $S_2(\omega)$ are not identically zero. If $(S_1(\gamma)S_2(\omega))(x) = 0$, then we may assume, without loss of generality, that $S_1(\gamma)(x) = 0$. Given $\epsilon > 0$, choose a peaking function $f \in A_p(G_1)$ such that $f(x) = 1$ and $\|S_1(\gamma)f\|_\infty < \frac{\epsilon}{\|S_2(\omega)\|_\infty}$. Take $\gamma' \in \Gamma$ and $\omega' \in \Omega$ with $S_1(\gamma') = S_2(\omega') = f$, then $T_1(\gamma')T_2(\omega') \in F_{\psi(x)}$. Hence

$$\begin{aligned}|(T_1(\gamma)T_2(\omega))(\psi(x))| &\le \|T_1(\gamma)T_2(\omega)T_1(\gamma')T_2(\omega')\|_\infty\\ &\le \|T_1(\gamma)T_2(\omega')\|_\infty\|T_1(\gamma')T_2(\omega)\|_\infty\\ &= \|S_1(\gamma)S_2(\omega')\|_\infty\|S_1(\gamma')S_2(\omega)\|_\infty < \frac{\epsilon}{\|S_2(\omega)\|_\infty}\|S_2(\omega)\|_\infty = \epsilon.\end{aligned}$$

Since ϵ is arbitrary we conclude that $(T_1(\gamma)T_2(\omega))(\psi(x)) = 0$.

If $(S_1(\gamma)S_2(\omega))(x) \ne 0$, choose peaking functions $f_1, f_2 \in A_p(G_1)$ with $f_1(x) = 1 = f_2(x)$, $R_\pi(S_1(\gamma)f_1) = \{S_1(\gamma)(x)\}$ and $R_\pi(S_2(\omega)f_2) = \{S_2(\omega)(x)\}$, by Lemma 3.1. For $\gamma_1 \in \Gamma$ and $\omega_1 \in \Omega$ if $S_1(\gamma_1) = f_2$ and $S_2(\omega_1) = f_1$, then $S_1(\gamma_1), S_2(\omega) \in \mathcal{F}_x$, and hence by Lemma 3.5, $T_1(\gamma_1)T_2(\omega_1) \in F_{\psi(x)}$. Therefore

$$\begin{aligned}|(T_1(\gamma)T_2(\omega))(\psi(x))| &\le \|T_1(\gamma)T_2(\omega)T_1(\gamma_1)T_2(\omega_1)\|_\infty\\ &\le \|T_1(\gamma)T_2(\omega_1)\|_\infty\|T_1(\gamma_1)T_2(\omega)\|_\infty\\ &= \|S_1(\gamma)S_2(\omega_1)\|_\infty\|S_1(\gamma_1)S_2(\omega)\|_\infty = |S_1(\gamma)(x)||S_2(\omega)(x)|.\end{aligned}$$

Thus $|(T_1(\gamma)T_2(\omega))(\psi(x))| \le |(S_1(\gamma)S_2(\omega))(x)|$.

By the same argument as in the first part, for $(T_1(\gamma)T_2(\omega))(\psi(x))$ instead of $(S_1(\gamma)S_2(\omega))(x)$, we may assume that $(T_1(\gamma)T_2(\omega))(\psi(x)) \ne 0$ and conclude that $|(S_1(\gamma)S_2(\omega))(x)| \le |(T_1(\gamma)T_2(\omega))(\psi(x))|$. Therefore $|(T_1(\gamma)T_2(\omega))(\psi(x))| = |(S_1(\gamma)S_2(\omega))(x)|$. □

Lemma 3.9. *The map $\psi : G_1 \longrightarrow G_2$ is a homeomorphism.*

Proof. To show the continuity of ψ, fix $x_0 \in G_1$ and $y_0 \in G_2$ with $\psi(x_0) = y_0$. Let W be a neighborhood of y_0 in G_2. There exists a peaking function $g \in A_p(G_2)$ such that $g(y_0) = 1$ and $|g| < 1/2$ on $G_2 \setminus W$, since y is a strong boundary point for $A_p(G_2)$. For $\gamma \in \Gamma$ and $\omega \in \Omega$ with $T_1(\gamma) = T_2(\omega) = g$, set $V = \{x \in G_1 : |(S_1(\gamma)S_2(\omega))(x)| > 1/4\}$. By Lemma 3.8, V is a neighborhood of x_0. Moreover, if $x \in V$ then by the above lemma, $|(T_1(\gamma)T_2(\omega))(\psi(x))| = |(S_1(\gamma)S_2(\omega))(x)| > 1/4$ which implies that $\psi(V) \subseteq W$. Hence ψ is continuous. Similarly, the inverse of ψ is also continuous. □

Now we can conclude the main result of this section.

Theorem 3.10. *Let* $S_1 : \Gamma \longrightarrow A_p(G_1)$, $S_2 : \Omega \longrightarrow A_p(G_1)$, $T_1 : \Gamma \longrightarrow A_p(G_2)$ *and* $T_2 : \Omega \longrightarrow A_p(G_2)$ *be surjections satisfying*

$$\|T_1(\gamma)T_2(\omega)\|_\infty = \|S_1(\gamma)S_2(\omega)\|_\infty \quad (\gamma \in \Gamma,\ \omega \in \Omega).$$

Then there exists a homeomorphism $\psi : G_1 \longrightarrow G_2$ *such that*

$$|(T_1(\gamma)T_2(\omega))(\psi(x))| = |(S_1(\gamma)S_2(\omega))(x)|,$$

for all $\gamma \in \Gamma$, $\omega \in \Omega$ *and* $x \in G_1$.

Corollary 3.11.

(i) *Let* $T_1, T_2 : A_p(G_1) \longrightarrow A_p(G_2)$ *be surjections such that* $\|T_1(f)T_2(g)\|_\infty = \|fg\|_\infty$ *for all* $f, g \in A_p(G_1)$. *Then there exists a homeomorphism* $\varphi : G_2 \longrightarrow G_1$ *such that for each* $f, g \in A_p(G_1)$ *and* $y \in G_2$, $|(T_1(f)T_2(g))(y)| = |(fg)(\varphi(y))|$.

(ii) *Let* $T : A_p(G_1) \longrightarrow A_p(G_2)$ *be a norm-multiplicative surjection. Then there exists a homeomorphism* $\varphi : G_2 \longrightarrow G_1$ *such that for each* $f \in A_p(G_1)$ *and* $y \in G_2$, $|T(f)(y)| = |f(\varphi(y))|$.

4. Generalized peripherally multiplicative maps

In this section we characterize the generalized peripherally multiplicative maps between Figà–Talamanca–Herz algebras.

It should be noted that if $T_1 = T_2$ and $S_1 = S_2 = \mathrm{id}$, the result is proved in [18] for arbitrary uniformly closed function algebras.

Theorem 4.1. *Let* $S_1 : \Gamma \longrightarrow A_p(G_1)$, $S_2 : \Omega \longrightarrow A_p(G_1)$, $T_1 : \Gamma \longrightarrow A_p(G_2)$ *and* $T_2 : \Omega \longrightarrow A_p(G_2)$ *be surjections such that* $R_\pi(T_1(\gamma)T_2(\omega)) \subseteq R_\pi(S_1(\gamma)S_2(\omega))$ *for all* $\gamma \in \Gamma$ *and* $\omega \in \Omega$. *Then there exist a homeomorphism* $\varphi : G_2 \longrightarrow G_1$ *and continuous functions* $h_1, h_2 : G_2 \longrightarrow \mathbb{C} \setminus \{0\}$ *such that* $h_1(y)h_2(y) = 1$ *and*

$$T_1(\gamma)(y) = h_1(y)S_1(\gamma)(\varphi(y)), \quad T_2(\omega)(y) = h_2(y)S_2(\omega)(\varphi(y))$$

for all $\gamma \in \Gamma$, $\omega \in \Omega$ *and* $y \in G_2$.

Proof. By assumption $\|T_1(\gamma)T_2(\omega)\|_\infty = \|S_1(\gamma)S_2(\omega)\|_\infty$ holds for all $\gamma \in \Gamma$ and $\omega \in \Omega$. According to Theorem 3.10, there exists a homeomorphism φ :

$G_2 \longrightarrow G_1$ such that $|(T_1(\gamma)T_2(\omega))(y)| = |(S_1(\gamma)S_2(\omega))(\varphi(y))|$ for all $\gamma \in \Gamma$, $\omega \in \Omega$ and $y \in G_2$. Indeed, φ is the inverse of ψ given by Theorem 3.10.

Let $y \in G_2$, $\gamma \in \Gamma$, $\omega \in \Omega$ and $S_1(\gamma)$ and $S_2(\omega)$ be peaking functions in $\mathcal{P}_{\varphi(y)}$. We have $(T_1(\gamma)T_2(\omega))(y) = 1$, since $|(T_1(\gamma)T_2(\omega))(y)| = |(S_1(\gamma)S_2(\omega))(\varphi(y))| = 1$ and $R_\pi(T_1(\gamma)T_2(\omega)) \subseteq R_\pi(S_1(\gamma)S_2(\omega)) = \{1\}$.

Define the functions h_1 and h_2 on G_2 as follows. Given $y \in G_2$, choose peaking functions $S_1(\gamma)$ and $S_2(\omega)$ in $\mathcal{P}_{\varphi(y)}$, for some $\gamma \in \Gamma$ and $\omega \in \Omega$, and put $h_1(y) := T_1(\gamma)(y)$, $h_2(y) := T_2(\omega)(y)$. Let us show that h_1 (similarly h_2) is well-defined. Let $\gamma, \gamma' \in \Gamma$ and $S_1(\gamma), S_1(\gamma')$ be peaking functions in $\mathcal{P}_{\varphi(y)}$. From the previous paragraph for each $\omega_0 \in \Omega$ with $S_2(\omega_0) \in \mathcal{P}_{\varphi(y)}$ we have

$$T_1(\gamma)(y)T_2(\omega_0)(y) = 1 = T_1(\gamma')(y)T_2(\omega_0)(y),$$

thus $T_1(\gamma)(y) = T_1(\gamma')(y)$, as required. Moreover, note that $h_1(y)h_2(y) = 1$ for every $y \in G_2$. Let $y \in G_2$ and $\gamma \in \Gamma$. We show that $T_1(\gamma)(y) = h_1(y)S_1(\gamma)(\varphi(y))$. Put $x = \varphi(y)$ for simplicity. If $S_1(\gamma)(x) = 0$, choose $\omega \in \Omega$ such that $S_2(\omega) \in \mathcal{P}_x$ then $|(T_1(\gamma)T_2(\omega))(y)| = |(S_1(\gamma)S_2(\omega))(x)| = 0$. Therefore, $T_1(\gamma)(y) = 0 = h_1(y)S_1(\gamma)(x)$. Now assume that $S_1(\gamma)(x) \neq 0$ and choose a peaking function $u \in A_p(G_1)$ such that $u(x) = 1$ and $R_\pi(S_1(\gamma)u) = \{S_1(\gamma)(x)\}$, by Lemma 3.1. Choose $\omega' \in \Omega$ such that $S_2(\omega') = u$. Since $\|T_1(\gamma)T_2(\omega')\|_\infty = \|S_1(\gamma)S_2(\omega')\|_\infty = |S_1(\gamma)(x)| = |S_1(\gamma)(x)S_2(\omega')(x)| = |T_1(\gamma)(y)T_2(\omega')(y)|$, we have

$$(T_1(\gamma)T_2(\omega'))(y) \in R_\pi(T_1(\gamma)T_2(\omega')) \subseteq R_\pi(S_1(\gamma)S_2(\omega')) = \{S_1(\gamma)(x)\}.$$

Hence $T_1(\gamma)(y)T_2(\omega')(y) = S_1(\gamma)(x)$. But $h_1(y) = T_2(\omega')(y)^{-1}$, thus

$$T_1(\gamma)(y) = T_2(\omega')(y)^{-1}\, S_1(\gamma)(x) = h_1(y)S_1(\gamma)(x).$$

That is $T_1(\gamma)(y) = h_1(y)S_1(\gamma)(\varphi(y))$.

A similar argument shows that $T_2(\omega)(y) = h_2(y)S_2(\omega)(\varphi(y))$ for all $\omega \in \Omega$ and $y \in G_2$. We only need to show that h_1 (similarly h_2) is continuous. Let $y \in G_2$ and $(y_\alpha)_{\alpha \in I}$ be a net in G_2 converging to y. Choose $S_1(\gamma) \in A_p(G_1)$ with $S_1(\gamma)(\varphi(y)) \neq 0$. Since φ is continuous we may assume that $(S_1(\gamma) \circ \varphi)(y_\alpha) \neq 0$ for all $\alpha \in I$. Thus

$$h_1(y_\alpha) = \frac{T_1(\gamma)(y_\alpha)}{(S_1(\gamma) \circ \varphi)(y_\alpha)} \to \frac{T_1(\gamma)(y)}{(S_1(\gamma) \circ \varphi)(y)} = h_1(y). \qquad \square$$

Since Figà–Talamanca–Herz algebras are self-adjoint, the above theorem implies easily the following.

Corollary 4.2. *If $T_1, T_2 : A_p(G_1) \longrightarrow A_p(G_2)$ are surjections such that $R_\pi(T_1(f)T_2(g)) \subseteq R_\pi(fg)$ for all $f, g \in A_p(G_1)$, or, $R_\pi(fg) \subseteq R_\pi(T_1(f)T_2(g))$ for all $f, g \in A_p(G_1)$, then there exist a homeomorphism $\varphi : G_2 \longrightarrow G_1$ and continuous functions $h_1, h_2 : G_2 \longrightarrow \mathbb{C}$ such that $h_1 h_2 = 1$, $T_i(f)(y) = h_i(y)f(\varphi(y))$ for all $f \in A_p(G_1)$ and $y \in G_2$ $(i = 1, 2)$. In particular, when $T_2(f) = \overline{T_1(f)}$ for all $f \in A_p(G_1)$, we have $h_2 = \overline{h_1}$.*

Remark 4.3. (i) From the representation obtained in the above corollary it follows that T_1 and T_2 are linear, and since $\|f\|_\infty \le \|f\|$ for each f in $A_p(G)$, the Closed Graph theorem implies the continuity of T_1, T_2 under $\|\cdot\|$.

(ii) If G_1 and G_2 are amenable, T_1 and T_2 are weighted isomorphisms. Indeed, the weight functions h_1 and h_2 belong to the multiplier algebra $B_p(G_2)$ of $A_p(G_2)$, and the map $\mathcal{T}f = f \circ \varphi$ is an algebra isomorphism of $A_p(G_1)$ and $A_p(G_2)$ [26, Theorem 4.22].

5. Jointly weakly peripherally multiplicative maps

In this section we study jointly weakly peripherally multiplicative maps. Let $S_1 : \Gamma \longrightarrow A_p(G_1)$, $S_2 : \Omega \longrightarrow A_p(G_1)$, $T_1 : \Gamma \longrightarrow A_p(G_2)$ and $T_2 : \Omega \longrightarrow A_p(G_2)$ be surjective maps such that $R_\pi(T_1(\gamma)T_2(\omega)) \cap R_\pi(S_1(\gamma)S_2(\omega)) \neq \varnothing$, for all $\gamma \in \Gamma$ and $\omega \in \Omega$. Maps satisfying this condition are called jointly weakly peripherally multiplicative. We do not know if such maps are forced to be weighted composition operators, but we could characterize them under some additional conditions.

Lemma 5.1. *Let G be first countable and $x_0 \in G$. If $f \in A_p(G)$ and $f(x_0) \neq 0$, there exists a peaking function $u \in A_p(G)$ such that $M_u = M_{fu} = \{x_0\}$ and in particular, $R_\pi(fu) = \{f(x_0)\}$.*

Proof. Let $f \in A_p(G)$ and $f(x_0) \neq 0$. By Lemma 3.1, there exists a peaking function $u_0 \in A_p(G)$ such that $u_0(x_0) = 1$ and $R_\pi(fu_0) = \{f(x_0)\}$. Since G is first countable there is a sequence $\{U_n\}$ of neighborhoods of x_0 with $\bigcap_{n=1}^\infty U_n = \{x_0\}$. From the proof of Lemma 3.1 it follows that for each $n \in \mathbb{N}$, we may choose a peaking function u_n such that $u_n(x_0) = 1 = \|u_n\|$ and $|u| < 1$ off U_n. Then the peaking function $u = u_0 \sum_{n=1}^\infty \frac{u_n}{2^n}$ belongs to $A_p(G)$, and also $M_u = M_{fu} = \{x_0\}$. □

Since the maps S_1, S_2, T_1, T_2 satisfy the assumptions of Theorem 3.10, so in the sequel we assume ψ is the homeomorphism given by this theorem.

Lemma 5.2. *Let $\gamma \in \Gamma$, $\omega \in \Omega$ and $x \in G_1$. If $S_1(\gamma)$ and $S_2(\omega)$ are peaking functions with $M_{S_1(\gamma)} = M_{S_2(\omega)} = \{x\}$, then $(T_1(\gamma)T_2(\omega))(\psi(x)) = 1$. Similarly, if $T_1(\gamma)$ and $T_2(\omega)$ are peaking functions with $M_{T_1(\gamma)} = M_{T_2(\omega)} = \{\psi(x)\}$, then $S_1(\gamma)(x)S_2(\omega)(x) = 1$.*

Proof. Since $R_\pi(T_1(\gamma)T_2(\omega)) \cap R_\pi(S_1(\gamma)S_2(\omega)) \neq \varnothing$ and $R_\pi(S_1(\gamma)S_2(\omega)) = \{1\}$, there is $y' \in G_2$ with $T_1(\gamma)(y')T_2(\omega)(y') = 1$. Next $M_{S_1(\gamma)S_2(\omega)} = \{x\}$, and by Theorem 3.10, $|S_1(\gamma)(\varphi(y'))S_2(\omega)(\varphi(y'))| = 1$, hence $x = \varphi(y')$ and then $\psi(x) = y'$. Therefore, $T_1(\gamma)(\psi(x))T_2(\omega)(\psi(x)) = 1$. The second part is similar. □

Theorem 5.3. *Assume that G_1 or G_2 is first countable, and $S_1 : \Gamma \longrightarrow A_p(G_1)$, $S_2 : \Omega \longrightarrow A_p(G_1)$, $T_1 : \Gamma \longrightarrow A_p(G_2)$ and $T_2 : \Omega \longrightarrow A_p(G_2)$ are jointly weakly peripherally multiplicative surjections. Then there exist a homeomorphism $\varphi : G_2 \longrightarrow G_1$ and continuous functions $h_1, h_2 : G_2 \longrightarrow \mathbb{C} \setminus \{0\}$ such that $h_1(y)h_2(y) = 1$ and*

$$T_1(\gamma)(y) = h_1(y)S_1(\gamma)(\varphi(y)), \quad T_2(\omega) = h_2(y)S_2(\omega)(\varphi(y)),$$

for $\gamma \in \Gamma$, $\omega \in \Omega$ and $y \in G_2$.

Proof. By Theorem 3.10, there is a homeomporphism $\varphi : G_2 \longrightarrow G_1$ such that $|(T_1(\gamma)T_2(\omega))(y)| = |(S_1(\gamma)S_2(\omega))(\varphi(y))|$ for all $\gamma \in \Gamma$, $\omega \in \Omega$ and $y \in G_2$. In fact, $\varphi = \psi^{-1}$.

Since the conditions are symmetric with respect to G_1 and G_2, so without loss of generality we can assume that G_1 is first countable. Define the map $h_1 : G_2 \longrightarrow \mathbb{C}$ as follows. Given $y \in G_2$ there is a peaking function $u \in A_p(G_1)$ with $M_u = \{\varphi(y)\}$, by Lemma 5.1. Let $\gamma \in \Gamma$ and $\omega \in \Omega$ such that $S_1(\gamma) = u$ and $S_2(\omega) = u$. By Lemma 5.2, $T_1(\gamma)(y)T_2(\omega)(y) = 1$. Set $h_1(y) := T_2(\omega)(y)^{-1}$. Assume that u' is another peaking function in $A_p(G_1)$ with $M_{u'} = \{\varphi(y)\}$. Let $\omega' \in \Omega$ such that $S_2(\omega') = u'$. Again by Lemma 5.2,

$$T_1(\gamma)(y)T_2(\omega)(y) = 1 = T_1(\gamma)(y)T_2(\omega')(y).$$

Therefore, $T_2(\omega)(y) = T_2(\omega')(y)$. Thus h_1 is well defined.

Similarly, define $h_2 : G_2 \longrightarrow \mathbb{C}$ by $h_2(y) = T_1(\gamma)(y)^{-1}$ for $y \in G_2$, where γ is an element in Γ with $S_1(\gamma) \in \mathcal{P}_{\varphi(y)}$ and $M_{S_1(\gamma)} = \{\varphi(y)\}$. Moreover, notice that $h_1(y)h_2(y) = 1$ for $y \in G_2$.

Next we give the representation of T_1. For $\gamma \in \Gamma$ and $y \in G_2$, if $S_1(\gamma)(\varphi(y)) = 0$, then choosing $\omega \in \Omega$ with $S_2(\omega) \in \mathcal{P}_{\varphi(y)}$ and $M_{S_2(\omega)} = \{\varphi(y)\}$, since

$$|(T_1(\gamma)T_2(\omega))(y)| = |(S_1(\gamma)S_2(\omega))(\varphi(y))| = 0,$$

we have $T_1(\gamma)(y) = h_1(y)S_1(\gamma)(\varphi(y)) = 0$. Now if we suppose that $S_1(\gamma)(\varphi(y)) \neq 0$, then by Lemma 5.1, there is a peaking function $u \in A_p(G_1)$ such that $M_u = M_{S_1(\gamma)u} = \{\varphi(y)\}$. Take $\omega_1 \in \Omega$ with $S_2(\omega_1) = u$. Since $R_\pi(T_1(\gamma)T_2(\omega_1)) \cap R_\pi(S_1(\gamma)S_2(\omega_1)) \neq \varnothing$, there is $y' \in G_2$ such that

$$T_1(\gamma)(y')T_2(\omega_1)(y') = S_1(\gamma)(\varphi(y)).$$

Hence

$$\begin{aligned} |S_1(\gamma)(\varphi(y))S_2(\omega_1)(\varphi(y))| &= |S_1(\gamma)(\varphi(y))| = |T_1(\gamma)(y')T_2(\omega_1)(y')| \\ &= |S_1(\gamma)(\varphi(y'))S_2(\omega_1)(\varphi(y'))|, \end{aligned}$$

and so $\varphi(y') \in M_{S_1(\gamma)u}$. Thus $\varphi(y') = \varphi(y)$, because $M_{S_1(\gamma)u} = \{\varphi(y)\}$. Therefore $y = y'$ and

$$T_1(\gamma)(y)T_2(\omega_1)(y) = S_1(\gamma)(\varphi(y)),$$

hence $T_1(\gamma)(y) = h_1(y)S_1(\gamma)(\varphi(y))$.

A similar argument shows that $T_2(\omega)(y) = h_2(y)S_2(\omega)(\varphi(y))$ for $\omega \in \Omega$ and $y \in G_2$. The same argument as in the proof of Theorem 4.1 gives the continuity of h_1 and h_2. $\square$

Corollary 5.4. *If G_1 or G_2 is first countable, and $T : A_p(G_1) \longrightarrow A_p(G_2)$ is a surjective map such that $R_\pi(T(f)T(g)) \cap R_\pi(fg) \neq \varnothing$ for all $f, g \in A_p(G_1)$, there exist a homeomorphism $\varphi : G_2 \longrightarrow G_1$ and a signum continuous function $h : G_2 \longrightarrow \{1, -1\}$ such that*

$$T(f)(y) = h(y)f(\varphi(y)) \quad (f \in A_p(G_1),\ y \in G_2).$$

We do not know if the first countability assumption can be dropped in the above results. However, in some particular cases, we could replace this assumption by certain condition making the underlying maps algebra homomorphisms (see Corollary 5.6).

Theorem 5.5. *Let $S_1 : \Gamma \longrightarrow A_p(G_1)$, $S_2 : \Omega \longrightarrow A_p(G_1)$, $T_1 : \Gamma \longrightarrow A_p(G_2)$ and $T_2 : \Omega \longrightarrow A_p(G_2)$ be jointly weakly peripherally multiplicative surjections. If $R_\pi(T_i(\cdot)) \subseteq R_\pi(S_i(\cdot))$ on the domain of T_i and S_i $(i = 1, 2)$, then $T_i(\cdot) = S_i(\cdot) \circ \varphi$, where φ is the inverse of ψ.*

Proof. If $\varphi = \psi^{-1}$, then φ is a homeomorphism from G_2 onto G_1 such that $|(T_1(\gamma)T_2(\omega))(y)| = |(S_1(\gamma)S_2(\omega))(\varphi(y))|$ for $\gamma \in \Gamma$, $\omega \in \Omega$ and $y \in G_2$, by Theorem 3.10.

We show that given $\gamma \in \Gamma$ and $y \in G_2$, then $T_1(\gamma)(y) = S_1(\gamma)(\varphi(y))$. If $S_1(\gamma)(\varphi(y)) = 0$ then from the above equation it follows that $T_1(\gamma)(y) = 0$.

Otherwise, if $S_1(\gamma)(\varphi(y)) \neq 0$, let V be an arbitrary neighborhood of y, and put $U = \varphi(V)$. Choose a peaking function $u \in A_p(G_1)$ such that $R_\pi(S_1(\gamma)u) = \{S_1(\gamma)(\varphi(y))\}$ and $M_{S_1(\gamma)u} \subseteq M_u \cap U$, by Lemma 3.1. Let $\omega \in \Omega$ be such that $S_2(\omega) = u$. Since $R_\pi(T_1(\gamma)T_2(\omega)) \cap R_\pi(S_1(\gamma)S_2(\omega)) \neq \varnothing$, then $S_1(\gamma)(\varphi(y)) \in R_\pi(T_1(\gamma)T_2(\omega))$. Hence there exists $y' \in G_2$ with $(T_1(\gamma)T_2(\omega))(y') = S_1(\gamma)(\varphi(y))$, that is,

$$T_1(\gamma)(y')T_2(\omega)(y') = S_1(\gamma)(\varphi(y)).$$

Therefore,

$$|S_1(\gamma)(\varphi(y'))S_2(\omega)(\varphi(y'))| = |T_1(\gamma)(y')T_2(\omega)(y')| = |S_1(\gamma)(\varphi(y))|,$$

which implies that $\varphi(y') \in U$ and $y' \in V$, by injectivity of φ. Now since $S_1(\gamma)$, $T_1(\gamma)$, $T_2(\omega)$ and φ are continuous and V is an arbitrary neighborhood of y,

$$T_1(\gamma)(y)T_2(\omega)(y) = S_1(\gamma)(\varphi(y)).$$

We claim that $|T_2(\omega)(y)| = |S_2(\omega)(\varphi(y))| = 1$. Let $\gamma' \in \Gamma$ be such that $S_1(\gamma') = S_2(\omega)$. Then from the equation $\|T_2(\omega)\|_\infty = 1 = \|T_1(\gamma')\|_\infty$ we have

$$1 \geq |T_2(\omega)(y)| \geq |T_2(\omega)(y)|\,|T_1(\gamma')(y)| = |S_2(\omega)(\varphi(y))S_1(\gamma')(\varphi(y))| = 1,$$

thus $|T_2(\omega)(y)| = 1$, that is $T_2(\omega)(y) \in R_\pi(T_2(\omega))$. Since

$$R_\pi(T_2(\omega)) \subseteq R_\pi(S_2(\omega)) = \{1\}$$

we have $T_2(\omega)(y) = 1$. Hence $T_1(\gamma)(y) = S_1(\gamma)(\varphi(y))$. □

The next result follows from the proof of Theorem 5.5.

Corollary 5.6. *Let $T_1, T_2 : A_p(G_1) \longrightarrow A_p(G_2)$ be surjections such that $R_\pi(T_1(f)T_2(g)) \cap R_\pi(fg) \neq \varnothing$ for $f, g \in A_p(G_1)$ satisfying any of the following conditions:*

(a) *$R_\pi(T_i(f)) \subseteq R_\pi(f)$ for all $f \in \mathcal{P}(A_p(G_1))$ $(i = 1, 2)$,*

(b) *$R_\pi(f) \subseteq R_\pi(T_i(f))$ for all $f \in \mathcal{P}(A_p(G_1))$ $(i = 1, 2)$.*

Then $T_1(f) = T_2(f) = f \circ \varphi$ for all $f \in A_p(G_1)$, where φ is the map given by Theorem 5.5. In particular, $T_1 = T_2$ is an algebra isomorphism.

Remark 5.7. It is worth mentioning our results are valid if we replace $A_p(G_1)$ and $A_p(G_2)$ by arbitrary uniformly closed function algebras A and B on locally compact Hausdorff spaces X and Y, respectively (except the last part in

Corollary 4.2 which uses the fact that Figà–Talamanca–Herz algebras are self-adjoint, while non-trivial uniformly closed function algebras are not necessarily self-adjoint). In the context of uniformly closed function algebras, the required homeomorphism is induced between Ch(A) and Ch(B), instead of the groups G_1 and G_2. Indeed, one can apply the multiplicative Bishop's lemma in the context of uniformly closed function algebras (which also holds in the non-unital case [30, Proposition 3.1]), and mimic the proofs given in Sections 3 to 5 for uniformly closed function algebras, to obtain extensions of some results on jointly norm-multiplicative maps and peripherally multiplicative maps proved in [7], [14] and [18], [21].

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