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Double analogue of Hamburger's theorem

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Abstract. We give an analogue of Hamburger's theorem for the Euler double zeta function.

1. Introduction

Let $s = \sigma + it$, $s_1 = \sigma_1 + it_1$, $s_2 = \sigma_2 + it_2$ with $\sigma, \sigma_1, \sigma_2, t, t_1, t_2 \in \mathbb{R}$. The Riemann zeta function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{1}$$

where

$$\chi(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$$

and $\Gamma(s)$ is the gamma function. The following theorem is well-known as a characterization of $\zeta(s)$.

Hamburger's theorem (see, for example, p. 31 in [4])**.** *Let $G(s)$ be an integral function of finite order, $P(s)$ a polynomial, and $f(s) = G(s)/P(s)$, and let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be absolutely convergent for $\sigma > 1$. Let $\alpha > 0$ and

$$f(s) = \chi(s)g(1-s),$$

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where

$$g(1-s)=\sum_{n=1}^{\infty}\frac{b_n}{n^{1-s}},$$

the series being absolutely convergent for $\sigma<-\alpha$*. Then* $f(s)=C\zeta(s)$*, where* C *is a constant.*

The purpose of this paper is to give an analogue of Hamburger's theorem for the Euler double zeta function.

The Euler double zeta function $\zeta_2(s_1,s_2)$ is defined by

$$\zeta_2(s_1,s_2)=\sum_{1\le m<n}\frac{1}{m^{s_1}n^{s_2}}\quad(\sigma_1+\sigma_2>2,\ \sigma_2>1)$$

and continued meromorphically on $\mathbb{C}^2$ (see [1]). The functions $\zeta(s)$ and $\zeta_2(s_1,s_2)$ satisfy the functional relation

$$\zeta_2(s_1,s_2)+\zeta_2(s_2,s_1)=\zeta(s_1)\zeta(s_2)-\zeta(s_1+s_2) \tag{2}$$

for $s_1,s_2\in\mathbb{C}$. On the other hand MATSUMOTO obtained the following result in [3].

Let

$$g(s_1,s_2)=\zeta_2(s_1,s_2)-\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1).$$

Let

$$\Psi(a,c;x)=\frac{1}{\Gamma(a)}\int_0^{\infty e^{i\phi}}e^{-xy}y^{a-1}(1+y)^{c-a-1}dy$$

be the confluent hypergeometric function, where $\Re a>0$, $-\pi<\phi<\pi$, $|\phi+\arg x|<\pi/2$. We use the notation $\sigma_l(k)=\sum_{d|k}d^l$.

Matsumoto's theorem. *We have*

$$\frac{g(s_1,s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)}=\frac{g(1-s_2,1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)}+2i\sin\left(\frac{\pi}{2}(s_1+s_2-1)\right)F_+(s_1,s_2), \tag{3}$$

where $i=\sqrt{-1}=\exp(\pi i/2)$ *and* $F_+(u,v)$ *is the series defined by*

$$F_+(u,v)=\sum_{k=1}^{\infty}\sigma_{u+v-1}(k)\Psi(v,u+v;2\pi ik). \tag{4}$$

The series (4) *is convergent only in the region* $\Re u<0$, $\Re v>1$*, but it can be continued meromorphically to the whole* $\mathbb{C}^2$ *space.*

The equation (3) is a functional equation for $\zeta_2(s_1,s_2)$.

Moreover, Komori, Matsumoto and Tsumura obtained the following result in [2].

Let $\omega_1,\omega_2\in\mathbb{C}$ and

$$\zeta_2(s_1,s_2;\omega_1,\omega_2)=\sum_{m=1}^{\infty}\frac{1}{(m\omega_1)^{s_1}}\sum_{n=1}^{\infty}\frac{1}{(m\omega_1+n\omega_2)^{s_2}},$$

where $z^s=\exp(s\log z)$, $\log z=\log|z|+i\arg z$ and $-\pi<\arg z\le\pi$ for $z\in\mathbb{C}$. Note that $\zeta_2(s_1,s_2;1,1)=\zeta_2(s_1,s_2)$. Let

$$g_0(s_1,s_2;\omega_1,\omega_2)=\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1)\omega_1^{-1}\omega_2^{1-s_1-s_2}.$$

Theorem (Komori, Matsumoto and Tsumura). *For $\omega_1,\omega_2\in\mathbb{C}$ with $\Re\omega_1>0$, $\Re\omega_2>0$, the hyperplane*

$$\Omega_{2k+1}=\{(s_1,s_2)\in\mathbb{C}^2\mid s_1+s_2=2k+1\}\quad(k\in\mathbb{Z}\setminus\{0\})$$

is not a singular locus of $\zeta_2(s_1,s_2;\omega_1,\omega_2)$. On this hyperplane the following functional equation holds:

$$\begin{aligned}&\left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{1-s_1-s_2}{2}}\Gamma(s_2)\{\zeta_2(s_1,s_2;\omega_1,\omega_2)-g_0(s_1,s_2;\omega_1,\omega_2)\}\\&=\left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{s_1+s_2-1}{2}}\Gamma(1-s_1)\{\zeta_2(1-s_2,1-s_1;\omega_1,\omega_2)-g_0(1-s_2,1-s_1;\omega_1,\omega_2)\}\end{aligned}\tag{5}$$

for $(s_1,s_2)\in\Omega_{2k+1}$ $(k\in\mathbb{Z}\setminus\{0\})$.

The equation (5) is a functional equation for $\zeta_2(s_1,s_2;\omega_1,\omega_2)$ on the hyperplane Ω_{2k+1} $(k\in\mathbb{Z}\setminus\{0\})$. In the case $\omega_1=\omega_2=1$ we have

$$\frac{g(s_1,s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)}=\frac{g(1-s_2,1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)}$$

on the hyperplane Ω_{2k+1} $(k\in\mathbb{Z}\setminus\{0\})$. Therefore we see that

$$2i\sin\left(\frac{\pi}{2}(s_1+s_2-1)\right)F_+(s_1,s_2)=0\tag{6}$$

on the hyperplane Ω_{2k+1} $(k\in\mathbb{Z}\setminus\{0\})$.

The following is our main result. The cardinal number of the set A is denoted by $|A|$.

Theorem 1. *Let $G(s)$ be an integral function of finite order, $P(s)$ a polynomial, and $f(s) = G(s)/P(s)$, and let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be absolutely convergent for $\sigma > 1$. Let $f_2(s_1, s_2)$ be a meromorphic function on $\mathbb{C}^2$. Let

$$f_2(s_1, s_2) + f_2(s_2, s_1) = f(s_1)f(s_2) - f(s_1 + s_2) \tag{7}$$

and

$$\begin{aligned} &\frac{1}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)}\left(f_2(s_1, s_2) - \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)f(s_1+s_2-1)\right) \\ &= \frac{1}{i^{s_1+s_2-1}\Gamma(s_2)}\left(f_2(1-s_2, 1-s_1) - \frac{\Gamma(s_2)}{\Gamma(1-s_1)}\Gamma(1-s_1-s_2)f(1-s_1-s_2)\right) \\ &\quad + 2i\sin\left(\frac{\pi}{2}(s_1+s_2-1)\right)F_+(s_1, s_2) \end{aligned} \tag{8}$$

in the $\mathbb{C}^2$ space. Let $f(2) = -2\pi^2 f(-1)$ and

$$\lim_{s\to-2} \Gamma(s)f(s) = -\frac{f(3)}{8\pi^2} = -\frac{\zeta(3)}{8\pi^2}. \tag{9}$$

Assume that at least one of the following conditions (a) *or* (b) *holds.*

(a) *In the closed vertical strip $D = \{s \in \mathbb{C} \mid 2 \le \sigma \le 4\}$, $\zeta(1-s) \ll |f(1-s)|$ and $|\{s \in D \mid f(1-s) = 0\}| \le 1$.*

(b) *There exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that*

$$c = \lim_{s\to+\infty} \chi(s)f(1-s),$$

where $s \in \mathbb{R}$.

Then $f(s) = \zeta(s)$ and $f_2(s_1, s_2) = \zeta_2(s_1, s_2)$.

Note that both f and f_2 are unknown functions in Theorem 1. This implies that by using (2) and (3) we can obtain a characterization of both ζ and ζ_2.

We do not assume $f(s) = \chi(s)f(1-s)$ in Theorem 1. However, we can obtain $f(s) = \chi(s)f(1-s)$ from functional equations (7) and (8). This is a key step of the proof of Theorem 1.

It seems that the choice of special values of $f(s)$ in the assumptions of Theorem 1 can be replaced by other special values. In some sense, it is indeed possible, but there is a problem. We will explain this point after the proof of Theorem 1 (see Remark 2).

2. Lemmas for the proof of Theorem 1

In this section, we collect some auxiliary results.

Lemma 1. *Let f and g be meromorphic functions on $\mathbb{C}$. Assume that the functions f and g satisfy the functional equations*

$$f(s)f(1-s) = g(s)g(1-s) = 1 \tag{10}$$

and

$$f(s)f(k-s) = g(s)g(k-s) \tag{11}$$

for some $k \in \mathbb{R} \setminus \{1\}$. If there exists a $\sigma_0 \in \mathbb{R}$ such that $f(s)/g(s)$ is bounded in the closed vertical strip $D = \{s \in \mathbb{C} \mid \sigma_0 \leq \Re s \leq \sigma_0 + |k-1|\}$, then $f(s) = \pm g(s)$.

Proof. We define $r(s) = f(s)/g(s)$. By using (10) and (11) we have

$$r(s) = \frac{g(k-s)}{f(k-s)} = \frac{f(1-(k-s))}{g(1-(k-s))} = r(s-(k-1)),$$

namely, $r(s)$ is a periodic function with period $|k-1|$. Since $r(s)$ is bounded in D, $r(s)$ is a constant by Liouville's theorem. On the other hand, in the case $s = 1/2$, we have $f(1/2)^2 = g(1/2)^2 = 1$. This implies the lemma. □

Lemma 2. *Let $T > 0$. Let $h(s)$ be a meromorphic function on $\mathbb{C}$ and $r(s) := h(s)/h(1-s)$. Assume that $r(s+T) = r(s)$ holds for all $s \in \mathbb{C}$. If there exist*

$$\lim_{s \to +\infty, s \in \mathbb{R}} h(s)$$

and

$$\lim_{s \to +\infty, s \in \mathbb{R}} h(1-s) \neq 0,$$

then $r(s) = 1$ for all $s \in \mathbb{C}$.

Proof. We assume $s \in \mathbb{R}$ and $k \in \mathbb{N}$. We define

$$c := \lim_{s \to +\infty, s \in \mathbb{R}} h(s).$$

Since we have $r(1/2) = 1$, we obtain

$$c = \lim_{k \to +\infty} h(1/2 + kT) = \lim_{k \to +\infty} r(1/2 + kT)h(1/2 - kT) = \lim_{k \to +\infty} h(1/2 - kT).$$

Therefore we obtain $\lim_{s \to +\infty} h(1-s) = c \neq 0$. If $r(s)$ is not a constant, then there exists an x such that $r(x) \neq 1$. Hence, we have

$$c = \lim_{k \to +\infty} h(x + kT) = \lim_{k \to +\infty} r(x + kT)h(1 - x - kT) = r(x)c,$$

but this is impossible. □

Note that Lemma 1 and Lemma 2 correspond to assumptions (a) and (b) in Theorem 1, respectively.

Lemma 3. *Let $g(s_1, s_2)$ be a meromorphic function on $\mathbb{C}^2$. The solution of the functional equation*

$$g(s_1, s_2) + g(s_2, s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2) \tag{12}$$

is

$$g(s_1, s_2) = \zeta_2(s_1, s_2) + \varphi(s_1, s_2),$$

where $\varphi(s_1, s_2)$ is a meromorphic function which satisfies $\varphi(s_2, s_1) = -\varphi(s_1, s_2)$.

Proof. Let g be an arbitrary solution of (12). We define

$$F(s_1, s_2) = g(s_1, s_2) - \zeta_2(s_1, s_2).$$

By (2) and (12) we have

$$F(s_1, s_2) = g(s_1, s_2) - \zeta_2(s_1, s_2) = \zeta_2(s_2, s_1) - g(s_2, s_1).$$

This implies $F(s_2, s_1) = -F(s_1, s_2)$. Therefore we can write

$$g(s_1, s_2) = \zeta_2(s_1, s_2) + \varphi(s_1, s_2), \tag{13}$$

where $\varphi(s_1, s_2)$ is a meromorphic function which satisfies $\varphi(s_2, s_1) = -\varphi(s_1, s_2)$. On the other hand, (13) actually satisfies (12). $\square$

Remark 1. Let $f(s)$ be a meromorphic function on $\mathbb{C}$. Assume that $f(s)$ does not have a pole at $s = 0$. If $f(s)$ satisfies the functional equation

$$\zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) = f(s_1)f(s_2) - f(s_1 + s_2), \tag{14}$$

then $f(s) = \zeta(s)$.

This claim implies that $\zeta(s)$ can be characterized by the functional equation (14). We can prove this claim as follows.

By (2) we have

$$f(s_1)f(s_2) - f(s_1 + s_2) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2), \tag{15}$$

and by setting $s_1 = 0$ and $s_2 = s$, we obtain

$$f(s)(f(0) - 1) = \zeta(s)(\zeta(0) - 1).$$

Since $\zeta(0) = -1/2$, we obtain

$$f(s) = C\zeta(s), \tag{16}$$

where C is a constant. By substituting (16) into (15), we have

$$(C^2 - 1)\zeta(s_1)\zeta(s_2) = (C - 1)\zeta(s_1 + s_2),$$

and with $s_1 = s_2 = s$,

$$(C^2 - 1)\zeta(s)^2 = (C - 1)\zeta(2s),$$

which is possible if and only if $C = 1$. Hence, we obtain $f(s) = \zeta(s)$.

3. Proof of Theorem 1

Now, we prove our main result.

Proof. We define $C(s_1, s_2) = \Gamma(s_2)/\Gamma(1 - s_1)$. In the case $s_1 + s_2 = 3$ we have

$$C(s_1, s_2) = \frac{\Gamma(3 - s_1)}{\Gamma(1 - s_1)} = (s_1 - 1)(s_1 - 2).$$

By this relation, we see $C(s_2, s_1) = C(s_1, s_2)$ in the case $s_1 + s_2 = 3$. On the other hand, we can easily see $\chi(s)\chi(3 - s) = -4\pi^2((s - 1)(s - 2))^{-1}$ by the definition of $\chi(s)$. Therefore, in the case $s_1 + s_2 = 3$, we obtain $\chi(s_1)\chi(s_2) = -4\pi^2(C(s_1, s_2))^{-1}$. Now, we assume $s_1 + s_2 = 3$. By (6) we obtain

$$-\frac{1}{4\pi^2}C(s_1, s_2)(f_2(s_1, s_2) - C(s_1, s_2)^{-1}f(2)) = f_2(1 - s_2, 1 - s_1) + C(s_1, s_2)\frac{f(3)}{8\pi^2}.$$

By interchanging s_1 and s_2, we obtain

$$-\frac{1}{4\pi^2}C(s_1, s_2)(f_2(s_2, s_1) - C(s_1, s_2)^{-1}f(2)) = f_2(1 - s_1, 1 - s_2) + C(s_1, s_2)\frac{f(3)}{8\pi^2}.$$

By adding the last two equations and using (7), we obtain

$$\begin{aligned} &-\frac{1}{4\pi^2}C(s_1, s_2)(f(s_1)f(s_2) - f(3) - 2f(2)C(s_1, s_2)^{-1}) \\ &\qquad = f(1 - s_1)f(1 - s_2) - f(-1) + 2C(s_1, s_2)\frac{f(3)}{8\pi^2}, \end{aligned}$$

namely,

$$\begin{aligned} f(s_1)f(3-s_1) &= -4\pi^2(C(s_1,s_2))^{-1}f(1-s_1)f(s_1-2) \\ &= \chi(s_1)\chi(3-s_1)f(1-s_1)f(s_1-2) \end{aligned} \tag{17}$$

by $f(2) = -2\pi^2 f(-1)$ and (9). If we define $K(s) = f(s)/f(1-s)$, then we have $K(s)K(1-s) = 1$ and, by (17),

$$\chi(s)\chi(3-s) = \frac{f(s)f(3-s)}{f(1-s)f(s-2)} = K(s)K(3-s). \tag{18}$$

On the other hand, if we define $r(s) = K(s)/\chi(s)$ and $h(s) = f(s)/\zeta(s)$, then we have

$$r(s) = \frac{f(s)}{\zeta(s)} \cdot \frac{\zeta(1-s)}{f(1-s)} \tag{19}$$

by the definition of $r(s)$,

$$r(s) = \frac{\chi(3-s)}{K(3-s)} = r(s-2) \tag{20}$$

by (18) and the definition of $r(s)$ and

$$h(s) = r(s)h(1-s) \tag{21}$$

by (1) and the definition of $K(s)$.

First we assume that (a) holds. Since $\zeta(s) \gg 1$ and $f(s) \ll 1$ in the case $\sigma \geq 2$, $f(s)/\zeta(s)$ is bounded in D. By (a) and (9), $f'(-2) \neq 0$ and $f(1-s) = 0$ in D if and only if $s = 3$. Therefore $\zeta(1-s)/f(1-s)$ is bounded in D, namely, by (19), $r(s)$ is also bounded in D. Hence, we obtain $K(s) = \pm\chi(s)$ by setting $f = K$ and $g = \chi$ in Lemma 1, and we obtain $K(s) = \chi(s)$ by $K(1/2) = \chi(1/2) = 1$. This implies $f = \zeta$ by Hamburger's theorem and (9).

Next we assume that (b) holds. Note that

$$h(s) = \sum_{n=1}^{\infty} \frac{\sum_{d|n} a_d \mu(n/d)}{n^s}$$

holds, where μ is the Möbius function. By (b) we have

$$\lim_{s\to+\infty} h(1-s) = \lim_{s\to+\infty} \frac{\chi(s)f(1-s)}{\zeta(s)} = c \neq 0$$

for $s \in \mathbb{R}$. Since (20) and (21) hold, we obtain $K(s) = \chi(s)$ by Lemma 2. This implies $f = \zeta$ by Hamburger's theorem and (9).

Hereafter, we assume $s_1, s_2 \in \mathbb{C}$, namely, we do not assume $s_1 + s_2 = 3$. If $f = \zeta$, then, by Lemma 3, we can write

$$f_2(s_1, s_2) = \zeta_2(s_1, s_2) + \varphi(s_1, s_2),$$

where φ is a meromorphic function which satisfies $\varphi(s_2, s_1) = -\varphi(s_1, s_2)$. The remaining task is to prove $\varphi = 0$. Note that the pair $f_2 = \zeta_2$ and $f = \zeta$ is a solution of (8) by Matsumoto's theorem. By subtracting (3) from (8) we obtain

$$\frac{\varphi(s_1, s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} = \frac{\varphi(1-s_2, 1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)}.$$

If we assume $\varphi \neq 0$, then we can define

$$G(s_1, s_2) = \frac{\varphi(s_1, s_2)}{\varphi(1-s_2, 1-s_1)} = \frac{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)},$$

and we have

$$G(s_2, s_1) = \frac{-\varphi(s_1, s_2)}{-\varphi(1-s_2, 1-s_1)} = G(s_1, s_2).$$

However, this implies that

$$\frac{\Gamma(1-s_1)}{\Gamma(s_2)} = \frac{\Gamma(1-s_2)}{\Gamma(s_1)}$$

holds, namely, $\sin \pi s_1 = \sin \pi s_2$ holds for all $s_1, s_2 \in \mathbb{C}$. This is impossible. This completes the proof. $\square$

Remark 2. We guess that if assumption (b) holds, then the choice of special values of $f(s)$ in Theorem 1 can be replaced by other special values, namely, we choose hyperplane $s_1 + s_2 = 2k + 1$ $(0 \neq k \in \mathbb{Z})$instead of the hyperplane $s_1 + s_2 = 3$ in the proof of Theorem 1. However, if assumption (b) does not hold, then assumption (a) must be replaced by a more complicate assumption, because we use (9) when we determine the zeros of $f(1-s)$ in D.

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