Publ. Math. Debrecen
86/1-2 (2015), 99–117
DOI: 10.5486/PMD.2015.6084

Characterization of additive maps ξ-Lie derivable at zero on von Neumann Algebras

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Abstract. Let $\mathcal{M}$ be any von Neumann algebra with the center $\mathcal{Z}(\mathcal{M})$. For any scalar ξ, denote by $[A,B]_\xi = AB - \xi BA$ the ξ-Lie product of $A, B \in \mathcal{M}$. Assume that $L : \mathcal{M} \to \mathcal{M}$ is an additive map. It is shown that, if $\mathcal{M}$ has no central summands of type I_1 or type I_2, then L satisfies $L([A,B]) = [L(A),B] + [A, L(B)]$ whenever $[A,B] = 0$ if and only if there exists an element $Z_0 \in \mathcal{Z}(\mathcal{M})$, an additive map $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ and an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ such that $L(A) = \varphi(A)+h(A)+Z_0A$ for all $A \in \mathcal{M}$; if $\mathcal{M}$ has no central summands of type I_1, then L satisfies $L([A,B]_\xi) = [L(A),B]_\xi+[A,L(B)]_\xi$ whenever $[A,B]_\xi = 0$ with $\xi \neq 1$ if and only if $L(I) \in \mathcal{Z}(\mathcal{M})$ and there exists an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ such that $\varphi(\xi A) = \xi\varphi(A)$ and $L(A) = \varphi(A) + L(I)A$ for all $A \in \mathcal{M}$. A result in [22] is improved for prime algebra case.

1. Introduction

Let $\mathcal{R}$ be an associative ring. Recall that an additive map δ on $\mathcal{R}$ is called an additive derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all A, $B \in \mathcal{R}$; is called an additive Jordan derivation if $\delta(AB+BA) = \delta(A)B+A\delta(B)+\delta(B)A+B\delta(A)$ for all $A, B \in \mathcal{R}$ (equivalently, $\delta(A^2) = \delta(A)A+A\delta(A)$ for all $A \in \mathcal{R}$ if the characteristic of $\mathcal{R}$ is not 2); is called a Lie derivation if $\delta([A,B]) = [\delta(A),B] + [A,\delta(B)]$ for all $A, B \in \mathcal{R}$, where $[A,B] = AB - BA$ is the Lie product of A and B. The structure of derivations, Jordan derivations and Lie derivations had been studied

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Mathematics Subject Classification: 47B47, 46L10.
Key words and phrases: Von Neumann algebras, Lie derivations, ξ-Lie derivations, derivations, additive maps.
This work is partially supported by National Natural Science Foundation of China (11171249,11101250), Program for the Outstanding Innovative Teams of Higher Learning Institutions of Shanxi and Youth Foundation of Shanxi Province (2012021004).

intensively for many years (for example, see [1], [3], [4], [7], [10], [16] and the references therein).

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$. For a scalar $\xi \in \mathbb{F}$ and for $A, B \in \mathcal{A}$, we say that A commutes with B up to a factor ξ if $AB = \xi BA$. The notion of commutativity up to a factor for pairs of operators is an important concept and has been studied in the context of operator algebras and quantum groups. Motivated by this, a binary operation $[A, B]_\xi = AB - \xi BA$, called ξ-Lie product of A and B, was introduced in [19]. An additive map $L : \mathcal{A} \to \mathcal{A}$ is called an additive ξ-Lie derivation if $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for all $A, B \in \mathcal{A}$. This conception unifies the above three notions. It is clear that a ξ-Lie derivation is a derivation if $\xi = 0$; is a Lie derivation if $\xi = 1$; is a Jordan derivation if $\xi = -1$. The structure of ξ-Lie derivations on various operator algebras was also discussed by several authors (see [9], [14], [19], [23], [25]).

Recently, the question of under what conditions an additive map becomes a derivation attracted much attention of many researchers (see [5], [8], [12], [13], [18] and the references therein). For ξ-Lie derivations, an additive (a linear) map L on $\mathcal{A}$ is said to be ξ-Lie derivable at a point $Z \in \mathcal{A}$ if $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for any $A, B \in \mathcal{A}$ with $[A, B]_\xi = Z$. Clearly, this definition is only valid for ξ-Lie commutators, that is, the elements of the form $Z = [A, B]_\xi$. For instance, if $Z = I$ and $\xi = 1$, as the unit I may not be a commutator in general, there is no sense to define that L is Lie derivable at I. Also, L is a ξ-Lie derivation if and only if it is ξ-Lie derivable at every ξ-Lie commutator Z. If $Z \in \mathcal{A}$ satisfies that, for any additive map $L : \mathcal{A} \to \mathcal{A}$, L is ξ-Lie derivable at the point Z will imply that L is a ξ-Lie derivation, we say that Z is a full ξ-Lie derivable point of $\mathcal{A}$. The following problem is natural.

Problem. *How to characterize the additive (linear) maps that are ξ-Lie derivable at some ξ-Lie commutator? Are there any ξ-Lie commutators that are full ξ-Lie derivable points?*

Since zero is a ξ-Lie commutator for any ξ and any algebra, as a start, the above problem has been attacked by several researchers for maps ξ-Lie derivable at zero. Qi and Hou [20] characterized the linear maps between $\mathcal{J}$-subspace lattice algebras that Lie derivable at zero. In [22] they consider further the additive maps ξ-Lie derivable at zero in pure algebra frame. Let $\mathcal{A}$ be a unital prime algebra over a field $\mathbb{F}$ containing a non-trivial idempotent P. Denote by $\mathcal{Z}(\mathcal{A})$ and $\mathcal{C}$ the center of $\mathcal{A}$ and the extended centroid of $\mathcal{A}$, respectively. Assume that $\xi \in \mathbb{F}$ and $L : \mathcal{A} \to \mathcal{A}$ is an additive map. Qi, Cui and Hou [22] showed that, if L is ξ-Lie derivable at zero, then there exists an additive derivation $\tau : \mathcal{A} \to \mathcal{C}$ such that (1) if $\xi = 0$, then $L(I) \in \mathcal{Z}(\mathcal{A})$ and $L(A) = \tau(A) + L(I)A$ for all $A \in \mathcal{A}$; (2) if $\xi = 1$ and $\mathcal{A}$ is of characteristic not 2 with $\deg(\mathcal{A}) > 2$, then $L(A) = \tau(A) + \alpha A + \nu(A)$ for all $A \in \mathcal{A}$, where $\alpha \in \mathcal{C}$ and ν is an additive map from $\mathcal{A}$ into $\mathcal{C}$; (3) if $\xi = -1$, $L(I) \in \mathcal{Z}(\mathcal{A})$ and $\mathcal{A}$ is of characteristic not 2, then $L(A) = \tau(A) + L(I)A$ for all $A \in \mathcal{A}$; (4) if $\xi \neq 0, \pm 1$, $L(I) \in \mathcal{Z}(\mathcal{A})$ and $L(\xi A) = \xi L(A)$ for each $A \in \mathcal{A}$, then $L(A) = \tau(A) + L(I)A$ for all $A \in \mathcal{A}$. Since factor von Neumann algebras are prime, as a consequence of the result for prime algebras, all additive maps ξ-Lie derivable at zero on factor von Neumann algebras are characterized. However the proof in [22] for factor von Neumann algebras is not valid anymore for non-factor von Neumann algebras. So, it is natural to ask what happens if the concerned von Neumann algebra is not a factor.

Let X be a Banach space with $\dim X \geq 3$ and $\mathcal{B}(X)$ the algebra of all bounded linear operators acting on X. We mention here that, Lu and Jing in [15] introduced another kind of Lie derivable at zero product (idempotent product) for a linear map and showed that, if $\delta : \mathcal{B}(X) \to \mathcal{B}(X)$ is a linear map satisfying $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for any $A, B \in \mathcal{B}(X)$ with $AB = 0$ (resp. $AB = P$, where P is a fixed nontrivial idempotent), then $\delta = \tau + \nu$, where τ is a derivation of $\mathcal{B}(X)$ and $\nu : \mathcal{B}(X) \to \mathbb{C}I$ is a linear map vanishing at commutators $[A, B]$ with $AB = 0$ (resp. $AB = P$), particularly, δ is a Lie derivation if ν vanishes on all commutators. Later, this result was generalized to the additive maps on triangular algebras, prime rings and general von Neumann algebras in [11], [21] and [24] respectively. Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1 and $L : \mathcal{M} \to \mathcal{M}$ be an additive map. In [24], Qi and Hou showed that, L satisfies $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for any A, B with $AB = 0$ if and only if there exists an additive derivation φ and an additive map $f : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ that vanishes each commutator $[A, B]$ whenever $AB = 0$, such that (1) $\xi = 1$, $L = \varphi + f$; (2) $\xi = -1$, $L = \varphi$; (3) $\xi = 0$, $L(A) = \varphi(A) + L(I)A$ for all A; (4) $\xi \notin \{0, \pm 1\}$, $\varphi(\xi A) = \xi L(A)$ and $L(A) = \varphi(A) + L(I)A$ for all A.

The purpose of the present paper is to give a complete characterization of additive maps ξ-Lie derivable at zero on von Neumann algebras without central summands of type I_1 for any scalar ξ. We remark here that the question of characterizing additive maps δ that are ξ-Lie derivable at zero is relatively more difficult than the question of characterizing additive maps satisfying $\delta([A, B]_\xi) = [\delta(A), B]_\xi + [A, \delta(B)]_\xi$ for any $A, B \in \mathcal{M}$ with $AB = 0$ since it is more difficult to find A, B satisfying $[A, B]_\xi = 0$, of course, the conclusions are also different.

The paper is organized as follows. Let $\mathcal{M}$ be a von Neumann algebra with the center $\mathcal{Z}(\mathcal{M})$ and $L : \mathcal{M} \to \mathcal{M}$ an additive map. In Section 2, we show that, if $\mathcal{M}$

has no central summands of type I_1 or type I_2, then L is Lie derivable at zero if and only if there exists an element $Z_0 \in \mathcal{Z}(\mathcal{M})$, an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ and an additive map $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ such that $L(A) = \varphi(A) + h(A) + Z_0A$ for all $A \in \mathcal{M}$ (Theorem 2.1). There are counterexamples to illustrate that the condition "$\mathcal{M}$ has no central summands of type I_1 or type I_2" can not be replaced simply by "$\mathcal{M}$ has no central summands of type I_1". Section 3 is devoted to discussing additive maps ξ-Lie derivable at zero with $\xi \neq 1$. Assume that $\mathcal{M}$ is a von Neumann algebra without central summands of type I_1. It is shown that L satisfies $L(AB - \xi BA) = L(A)B - \xi BL(A) + AL(B) - \xi L(B)A$ whenever $AB - \xi BA = 0$ if and only if $L(I) \in \mathcal{Z}(\mathcal{M})$ and there exists an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ with $\varphi(\xi A) = \xi\varphi(A)$ for each A such that $L(A) = \varphi(A) + L(I)A$ for all $A \in \mathcal{M}$ (Theorem 3.1). Thus, zero is not a full ξ-Lie derivable point, though they have a structure very close to ξ-Lie derivation. From these results, one can easily get a characterization of additive ξ-Lie derivations. As a consequence, our approach also enables us to improve the result (4) in [22] mentioned in the third paragraph by omitting the assumptions "$L(I) \in \mathcal{Z}(\mathcal{A})$ and $L(\xi A) = \xi L(A)$ for each $A \in \mathcal{A}$".

2. Additive maps Lie derivable at zero

In this section, we discuss additive maps Lie derivable at zero on von Neumann algebras. The following is our main result in this section.

Theorem 2.1. *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1 or type I_2. Suppose that $L : \mathcal{M} \to \mathcal{M}$ is an additive map. Then L is Lie derivable at zero, that is, L satisfies $L([A, B]) = [L(A), B] + [A, L(B)]$ for any $A, B \in \mathcal{M}$ with $[A, B] = 0$, if and only if there exists an element $Z_0 \in \mathcal{Z}(\mathcal{M})$, the center of $\mathcal{M}$, an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ and an additive map $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ such that $L(A) = \varphi(A) + h(A) + Z_0A$ for all $A \in \mathcal{M}$.*

We remark that the conditions "without central summands of type I_1 or type I_2" in Theorem 2.1 can not be deleted simply because (i) every additive map on a commutative ring is a Lie derivation; and (ii) it is shown in [20, Proposition 2.5] that a linear map L on 2 by 2 matrix algebra is Lie derivable at zero if and only if $L(I) = \lambda I$ for some scalar λ.

To prove Theorem 2.1, we need several lemmas.

Lemma 2.2 ([2, Lemma 2]). *Let $\mathcal{M}$ be a von Neumann algebra with no* *central summands of type I_1 or type I_2. Then the ideal $\mathcal{I}$ of $\mathcal{M}$ generated algebraically by $\{[A^2, C]B[A, C] - [A, C]B[A^2, C] : A, B, C \in \mathcal{M}\}$ is equal to $\mathcal{M}$.*

A ring $\mathcal{R}$ is said to be semiprime if, for any $A \in \mathcal{R}$, $A\mathcal{R}A = \{0\}$ will imply that $A = 0$; to be torsion-free if, for any $A \in \mathcal{R}$ and any positive integer n, $nA = 0$ will imply that $A = 0$. Every von Neumann algebra is semiprime and torsion-free.

Lemma 2.3 ([2, Lemma 6]). *Let $\mathcal{R}$ be a semiprime torsion-free ring and G an additive group. Suppose that maps $\epsilon : G \times G \to \mathcal{R}$ and $\tau : G \times G \times G \to \mathcal{R}$ are additive in each argument. If $\epsilon(A, A)\mathcal{R}\tau(A, A, A) = \{0\}$ for every $A \in G$, then $\epsilon(B, B)\mathcal{R}\tau(A, A, A) = \{0\}$ for all $A, B \in G$.*

Recall that a map q from a ring $\mathcal{R}$ into itself is commuting if $[q(A), A] = 0$ for all $A \in \mathcal{R}$; is a trace of a biadditive map if there exists a biadditive map $g : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ such that $q(A) = g(A, A)$ for all $A \in \mathcal{R}$.

The following lemma is crucial for proving our main result.

Lemma 2.4 ([2, Theorem 2]). *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1 or type I_2. Let q be a trace of a biadditive map. If q is commuting, then $q(A) = \lambda A^2 + \mu(A)A + \nu(A)$ for all $A \in \mathcal{M}$, where $\lambda \in \mathcal{Z}(\mathcal{M})$ and μ, ν are maps of $\mathcal{M}$ into $\mathcal{Z}(\mathcal{M})$ with μ additive.*

Now we are at a position to give our proof of Theorem 2.1.

Proof of Theorem 2.1. The "if" part is obvious. For the "only if" part, assume that $L : \mathcal{M} \to \mathcal{M}$ is an additive map Lie derivable at zero, that is,

$$[L(A), B] + [A, L(B)] = 0 \quad \text{for all } A, B \in \mathcal{M} \text{ with } [A, B] = 0. \tag{2.1}$$

Take $B = A^2$ in equation (2.1) and we get $L(A)A^2 - A^2L(A) + AL(A^2) - L(A^2)A = 0$. This yields

$$[L(A^2) - L(A)A - AL(A), A] = 0 \quad \text{for all } A \in \mathcal{M}. \tag{2.2}$$

For any $A, B \in \mathcal{M}$, write $\delta(A, B) = L(AB) - L(A)B - AL(B)$. It is obvious that $\delta : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a biadditive map and $\delta(A, A)$ is a trace of the biadditive map δ by equation (2.2). It follows from Lemma 2.4 that there exist an element $Z \in \mathcal{Z}(\mathcal{M})$, an additive map $\mu : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ and a map $\nu : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ such that $\delta(A, A) = ZA^2 + \mu(A)A + \nu(A)$, that is,

$$L(A^2) - L(A)A - AL(A) = ZA^2 + \mu(A)A + \nu(A) \quad \text{holds for all } A \in \mathcal{M}. \tag{2.3}$$

Now define two maps $\varphi : \mathcal{M} \to \mathcal{M}$ and $\epsilon : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ as follows:

$$\varphi(A) = L(A) + \frac{1}{2}\mu(A) + ZA \quad \text{for all} \quad A \in \mathcal{M}$$

and

$$\epsilon(A,B)=\varphi(AB+BA)-\varphi(A)B-A\varphi(B)-\varphi(B)A-B\varphi(A) \text{ for all } A,B\in\mathcal{M}.$$

Clearly, φ is additive and ϵ is biadditive. By the definition of φ and equation (2.3), we have

$$\begin{aligned}\varphi(A^2)&=L(A^2)+\frac{1}{2}\mu(A^2)+ZA^2\\&=L(A)A+AL(A)+\mu(A)A+\nu(A)+\frac{1}{2}\mu(A^2)+2ZA^2\end{aligned}$$

and

$$\varphi(A)A+A\varphi(A)=L(A)A+AL(A)+\mu(A)A+2ZA^2.$$

The above two equations imply $\varphi(A^2)-\varphi(A)A-A\varphi(A)=\nu(A)+\frac{1}{2}\mu(A^2)\in\mathcal{Z}(\mathcal{M})$. Replacing A by $A+B$ in the relation, one obtains

$$\varphi(AB+BA)-\varphi(A)B-A\varphi(B)-\varphi(B)A-B\varphi(A)\in\mathcal{Z}(\mathcal{M}),$$

which implies that ϵ maps $\mathcal{M}\times\mathcal{M}$ into $\mathcal{Z}(\mathcal{M})$.

Claim. $\epsilon(A,A)=0$ for all $A\in\mathcal{M}$.

For any $A\in\mathcal{M}$, by the definition of ϵ, we have

$$\begin{aligned}&2\varphi(A^4)=2\varphi(A^2A^2)=2\varphi(A^2)A^2+2A^2\varphi(A^2)+\epsilon(A^2,A^2)\\&=2\varphi(A)A^3+2A\varphi(A)A^2+2A^2\varphi(A)A+2A^3\varphi(A)+2\epsilon(A,A)A^2+\epsilon(A^2,A^2)\end{aligned}$$

and

$$\begin{aligned}4\varphi(A^4)&=2\varphi(A^3A+AA^3)\\&=2\varphi(A^3)A+2A^3\varphi(A)+2\varphi(A)A^3+2A\varphi(A^3)+2\epsilon(A^3,A)\\&=\varphi(A^2A+AA^2)A+2A^3\varphi(A)+2\varphi(A)A^3+A\varphi(A^2A+AA^2)+2\epsilon(A^3,A)\\&=4\varphi(A)A^3+4A\varphi(A)A^2+4A^2\varphi(A)A+4A^3\varphi(A)\\&\quad+2\epsilon(A^3,A)+2\epsilon(A,A)A^2+2\epsilon(A^2,A)A.\end{aligned}$$

Comparing the above two equations gives

$$\epsilon(A,A)A^2-\epsilon(A^2,A)A=\epsilon(A^3,A)-\epsilon(A^2,A^2)\in\mathcal{Z}(\mathcal{M}) \tag{2.4}$$

for all $A\in\mathcal{M}$. Take any $A,C\in\mathcal{M}$. By equation(2.4), it is easily checked that $\epsilon(A,A)[A^2,C]=\epsilon(A^2,A)[A,C]$. It follows that

$$\epsilon(A,A)([A^2,C]X[A,C]-[A,C]X[A^2,C])=0 \quad \text{for all } A,C,X\in\mathcal{M}. \tag{2.5}$$

Now fix X and C. Define a map $\phi : \mathcal{M}\times\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ as follows:

$$\phi(A_1, A_2, A_3) = [A_1A_2, C]X[A_3, C] - [A_1, C]X[A_2A_3, C] \quad \text{for all} \quad A_1, A_2, A_3 \in \mathcal{M}.$$

It is obvious that ϕ is additive in each argument and

$$\phi(A, A, A) = [A^2, C]X[A, C] - [A, C]X[A^2, C].$$

Thus, by equation (2.5), one gets $\epsilon(A, A)\phi(A, A, A) = 0$, and so $\epsilon(A, A)\mathcal{M}\phi(A, A, A) = \{0\}$ for each $A \in \mathcal{M}$ as ϵ maps into the center. It follows from Lemma 2.3 that $\epsilon(B, B)\mathcal{M}\phi(A, A, A) = \{0\}$ for all $A, B \in \mathcal{M}$. Now by using Lemma 2.2, we obtain that $\epsilon(B, B) = 0$ for all $B \in \mathcal{M}$, as desired.

Note that $\epsilon(A, A) = 2(\varphi(A^2) - \varphi(A)A - A\varphi(A))$ for every $A \in \mathcal{M}$. So by claim, $\varphi(A^2) = \varphi(A)A + A\varphi(A)$ holds for all $A \in \mathcal{M}$, that is, φ is an additive Jordan derivation. Since every additive Jordan derivation on a 2-torsion free semiprime ring is an additive derivation ([1, Theorem 1]) and every von Neumann algebra is semiprime, φ is in fact an additive derivation. Let $h = -\frac{1}{2}\mu$ and $Z_0 = -Z$. It follows from the definition of φ that

$$L(A) = \varphi(A) + Z_0A + h(A) \tag{2.6}$$

for all A, where $Z_0 \in \mathcal{Z}(\mathcal{M})$, φ is an additive derivation and $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ is an additive map. The proof of the theorem is complete. □

By Theorem 2.1, we get a characterization of Lie derivations, which is also given in [24, Corollary 2.5] as a corollary of [24, Theorem 2.1].

Corollary 2.5. *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1 or type I_2. Suppose that $L : \mathcal{M} \to \mathcal{M}$ is an additive map. Then L is a Lie derivation if and only if there exists an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ and an additive map $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ vanishing on each commutator such that $L(A) = \varphi(A) + h(A)$ for all $A \in \mathcal{M}$.*

Proof. The "if" part is obvious. For the "only if" part, assume that L is an additive Lie derivation. Then, L is Lie derivable at zero and hence, by Theorem 2.1, has the form equation (2.6). Thus we have

$$\begin{aligned} L(AB - BA) &= \varphi(AB - BA) + h(AB - BA) + Z_0(AB - BA) \\ &= \varphi(A)B + A\varphi(B) - \varphi(B)A - B\varphi(A) + h(AB - BA) + Z_0(AB - BA) \end{aligned} \tag{2.7}$$

and on the other hand,

$$\begin{aligned} L(AB-BA) &= L(A)B - BL(A) + AL(B) - L(B)A \\ &= \varphi(A)B + A\varphi(B) - \varphi(B)A - B\varphi(A) + 2Z_0(AB-BA) \end{aligned} \quad (2.8)$$

for all $A, B \in \mathcal{M}$. Comparing equations (2.7)–(2.8), we see that $Z_0(AB-BA) = h(AB-BA) \in \mathcal{Z}(\mathcal{M})$ for any A, B. Note that, for any $A \in \mathcal{M}$ and any projection $P \in \mathcal{M}$, $PA(I-P) = [P, P + PA(I-P)]$ is a commutator. Thus $h(PA(I-P)) = Z_0PA(I-P) = PZ_0A(I-P) \in \mathcal{Z}(\mathcal{M})$ holds for any projection P and any element A in $\mathcal{M}$. It follows that $PZ_0A(I-P) = PPZ_0A(I-P) = PZ_0A(I-P)P = 0$ for all projection P and all A in $\mathcal{M}$. This forces that Z_0A commutes with each projection and hence is an element in the center of $\mathcal{M}$. So we have $Z_0\mathcal{M} \subseteq \mathcal{Z}(\mathcal{M})$. Since $\mathcal{M}$ has no central summand of type I_1, we must have $Z_0 = 0$, and consequently, $L = \varphi + h$ with h vanishing on all commutators. □

By Corollary 2.5, the additive maps Lie derivable at zero are very close to Lie derivations, but, not Lie derivations in general. Thus zero is not a full Lie derivable point of the von Neumann algebra $\mathcal{M}$.

3. Additive maps ξ-Lie derivable at zero with $\xi \neq 1$

In this section, we will give a characterization of additive maps ξ-Lie derivable at zero on von Neumann algebras without central summands of type I_1. Here $\xi \neq 1$.

The following is the main result of this section.

Theorem 3.1. *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. Suppose that $L : \mathcal{M} \to \mathcal{M}$ is an additive map and ξ is a scalar with $\xi \neq 1$. Then L is ξ-Lie derivable at zero (that is, L satisfies $L([A,B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for any $A, B \in \mathcal{M}$ with $[A,B]_\xi = 0$) if and only if $L(I) \in \mathcal{Z}(\mathcal{M})$ and there exists an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ such that $\varphi(\xi A) = \xi\varphi(A)$ and $L(A) = \varphi(A) + L(I)A$ for all $A \in \mathcal{M}$.*

Before proving Theorem 3.1, we need some notations. Let $\mathcal{M}$ be any von Neumann algebra and $A \in \mathcal{M}$. Recall that the central carrier of A, denoted by $\overline{A}$, is the intersection of all central projections P such that $PA = A$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{M}) : S = S^*,\ S \leq A\}$. Particularly, if $A = P$ is a projection, it is clear that $\underline{P}$ is the largest central projection $\leq P$. A projection P is called core-free if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I-P} = I$.

We first give two useful lemmas which are needed to prove Theorem 3.1.

Lemma 3.2 ([17, Lemma 4]). *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. Then each nonzero central projection $C \in \mathcal{M}$ is the carrier of a core-free projection in $\mathcal{M}$. Particularly, there exists a nonzero core-free projection $P \in \mathcal{M}$ with $\overline{P} = I$.*

In fact, $\mathcal{M}$ is a von Neumann algebra without central summands of type I_1 if and only if it has a projection P with $\underline{P} = 0$ and $\overline{P} = I$.

Lemma 3.3 ([17]). *Let $\mathcal{M}$ be a von Neumann algebra. For projections $P, Q \in \mathcal{M}$, if $\overline{P} = \overline{Q} \neq 0$ and $P + Q = I$, then $T \in \mathcal{M}$ commutes with PXQ and QXP for all $X \in \mathcal{M}$ implies $T \in \mathcal{Z}(\mathcal{M})$.*

For more properties of core-free projections, see [17].

Proof of Theorem 3.1. The "if" part is easily checked. We only need to give the proof of the "only if" part.

Assume that $\xi \neq 1$ and $L : \mathcal{M} \to \mathcal{M}$ is an additive map satisfying $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for any $A, B \in \mathcal{M}$ with $[A, B]_\xi = 0$. We will prove the "only if" part by several claims.

By Lemma 3.2, we can find a core-free projection $P \in \mathcal{M}$ with $\overline{P} = I$. In the sequel fix such a projection P. By the definitions of core and central carrier, we have $\underline{P} = \underline{I - P} = 0$ and $\overline{I - P} = I$. For the convenience, write $P_1 = P$, $P_2 = I - P$ and $\mathcal{M}_{ij} = P_i \mathcal{M} P_j$, $i, j \in \{1, 2\}$. Then $\mathcal{M} = \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}$.

Claim 1. $P_1 L(I) P_2 = P_2 L(I) P_1 = 0$ and $P_2 L(P_1) P_2 = P_1 L(P_2) P_1 = 0$.

Since $[P_1, P_2]_\xi = [P_2, P_1]_\xi = 0$, we have $[L(P_1), P_2]_\xi + [P_1, L(P_2)]_\xi = 0$ and $[L(P_2), P_1]_\xi + [P_2, L(P_1)]_\xi = 0$, that is,

$$L(P_1)P_2 - \xi P_2 L(P_1) + P_1 L(P_2) - \xi L(P_2) P_1 = 0 \tag{3.1}$$

and

$$L(P_2)P_1 - \xi P_1 L(P_2) + P_2 L(P_1) - \xi L(P_1) P_2 = 0. \tag{3.2}$$

Multiplying by P_1 and P_2 from the left and the right respectively in equation (3.1), and multiplying by P_2 and P_1 from the left and the right respectively in equation (3.2), one gets $P_1 L(P_1) P_2 + P_1 L(P_2) P_2 = 0$ and $P_2 L(P_2) P_1 + P_2 L(P_1) P_1 = 0$, which imply

$$P_1 L(I) P_2 = 0 \quad \text{and} \quad P_2 L(I) P_1 = 0.$$

Multiplying by P_2 and P_1 from both sides in equation (3.1) and equation (3.2), respectively, one gets $P_2 L(P_1) P_2 - \xi P_2 L(P_1) P_2 = 0$ and $P_1 L(P_2) P_1 - \xi P_1 L(P_2) P_1 = 0$. It follows from the assumption $\xi \neq 1$ that

$$P_2 L(P_1) P_2 = 0 \quad \text{and} \quad P_1 L(P_2) P_1 = 0.$$

The claim holds.

Now define a map $\delta : \mathcal{M} \to \mathcal{M}$ by $\delta(A) = L(A) + SA - AS$ for each $A \in \mathcal{M}$, where $S = P_1L(P_1)P_2 - P_2L(P_1)P_1$. It is easily verified that δ is also an additive map ξ-Lie derivable at zero, that is, δ satisfies

$$\delta([A, B]_\xi) = [\delta(A), B]_\xi + [A, \delta(B)]_\xi \quad for\ A, B \in \mathcal{M} \text{ with } [A, B]_\xi = 0.$$

Moreover, $P_1\delta(I)P_2 = P_2\delta(I)P_1 = P_2\delta(P_1)P_2 = P_1\delta(P_2)P_1 = 0$ by Claim 1. Thus we get

$$\begin{aligned}\delta(P_1) &= L(P_1) + SP_1 - P_1S = P_1L(P_1)P_1 \\ &= P_1\delta(P_1)P_1 - P_1(SP_1 - P_1S)P_1 = P_1\delta(P_1)P_1 \in \mathcal{M}_{11}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\delta(P_2) &= L(P_2) + SP_2 - P_2S = P_2L(P_2)P_2 \\ &= P_2\delta(P_2)P_2 - P_2(SP_2 - P_2S)P_2 = P_2\delta(P_2)P_2 \in \mathcal{M}_{22}.\end{aligned} \tag{3.4}$$

Claim 2. $\delta(\mathcal{M}_{ii}) \subseteq \mathcal{M}_{ii}$, $i = 1, 2$.

We only give the proof for $\mathcal{M}_{11}$. The proof for $\mathcal{M}_{22}$ is similar.

Take any $A_{11} \in \mathcal{M}_{11}$. Since $[A_{11}, P_2]_\xi = 0$, we have $[\delta(A_{11}), P_2]_\xi + [A_{11}, \delta(P_2)]_\xi = 0$. This and equation (3.4) yield

$$\delta(A_{11})P_2 - \xi P_2\delta(A_{11}) = 0. \tag{3.5}$$

Multiplying by P_1 from the left side in equation (3.5), one gets

$$P_1\delta(A_{11})P_2 = 0; \tag{3.6}$$

multiplying by P_2 from both sides in equation (3.5), one gets $(1-\xi)P_2\delta(A_{11})P_2 = 0$, which implies

$$P_2\delta(A_{11})P_2 = 0. \tag{3.7}$$

Note that $[P_2, A_{11}]_\xi = 0$. Then $[\delta(P_2), A_{11}]_\xi + [P_2, \delta(A_{11})]_\xi = 0$, which and equation (3.4) give $P_2\delta(A_{11}) - \xi\delta(A_{11})P_2 = 0$. Then, multiplying P_1 from the right side in this equation, one gets

$$P_2\delta(A_{11})P_1 = 0. \tag{3.8}$$

Combining equations (3.6)–(3.8), we achieve that $\delta(A_{11}) \in \mathcal{M}_{11}$. So the claim is true.

Claim 3. For any $A_{ij} \in \mathcal{M}_{ij}$, $1 \le i \ne j \le 2$, the following statements hold.

(1) If $\xi \neq -1$, then $\delta(A_{ij}) \in \mathcal{M}_{ij}$.

(2) If $\xi = -1$, then $P_i\delta(A_{ij})P_i = P_j\delta(A_{ij})P_j = 0$ and $\delta(A_{ij})A_{ij} + A_{ij}\delta(A_{ij}) = 0$.

For any $A_{ij} \in \mathcal{M}_{ij}$ $(i \neq j)$, since $[A_{ij}, A_{ij}]_\xi = 0$, we have $[\delta(A_{ij}), A_{ij}]_\xi + [A_{ij}, \delta(A_{ij})]_\xi = 0$, that is,

$$\begin{aligned}\delta(A_{ij})A_{ij} - \xi A_{ij}\delta(A_{ij}) + A_{ij}\delta(A_{ij}) - \xi\delta(A_{ij})A_{ij}\\ = (1-\xi)(\delta(A_{ij})A_{ij} + A_{ij}\delta(A_{ij})) = 0.\end{aligned}$$

It follows from the assumption $\xi \neq 1$ that

$$\delta(A_{ij})A_{ij} + A_{ij}\delta(A_{ij}) = 0 \quad \text{for all } A_{ij} \in \mathcal{M}_{ij}. \tag{3.9}$$

Since $[P_i + A_{ij}, A_{ij} - P_j]_\xi = 0$, by equations (3.3)–(3.4) and equation (3.9), we have

$$\begin{aligned}0 &= [\delta(P_i) + \delta(A_{ij}), A_{ij} - P_j]_\xi + [P_i + A_{ij}, \delta(A_{ij}) - \delta(P_j)]_\xi\\ &= \delta(P_i)A_{ij} - \delta(A_{ij})P_j + \xi P_j\delta(A_{ij}) + P_i\delta(A_{ij}) - A_{ij}\delta(P_j) - \xi\delta(A_{ij})P_i.\end{aligned}$$

Multiplying by P_i and P_j from both sides in the above equation, respectively, and noting that equations (3.3)–(3.4), one gets $(1-\xi)P_i\delta(A_{ij})P_i = 0$ and $(1-\xi)P_j\delta(A_{ij})P_j = 0$, and so

$$P_i\delta(A_{ij})P_i = 0 \quad \text{and} \quad P_j\delta(A_{ij})P_j = 0. \tag{3.10}$$

Now, combining equations (3.9) and (3.10), we see that (2) is true.

To prove (1), one needs to check further that $P_j\delta(A_{ij})P_i = 0$ whenever $\xi \neq -1$. In fact, since $[A_{ij}, P_i + \xi P_j]_\xi = 0$, by Claim 2, we have

$$\begin{aligned}0 &= [\delta(A_{ij}), P_i + \xi P_j]_\xi + [A_{ij}, \delta(P_i) + \delta(\xi P_j)]_\xi\\ &= \delta(A_{ij})P_i + \xi\delta(A_{ij})P_j - \xi P_i\delta(A_{ij}) - \xi^2 P_j\delta(A_{ij}) + A_{ij}\delta(\xi P_j) - \xi\delta(P_i)A_{ij}.\end{aligned}$$

Multiplying by P_j and P_i from the left and the right side respectively in the above equation, one obtains $(1-\xi^2)P_j\delta(A_{ij})P_i = 0$, which implies $P_j\delta(A_{ij})P_i = 0$ as $\xi \neq -1$. This and equation (3.10) yield $\delta(A_{ij}) = P_i\delta(A_{ij})P_j \in \mathcal{M}_{ij}$ whenever $\xi \neq -1$, and so (1) holds.

Claim 4. For any $A_{ij} \in \mathcal{M}_{ij}$ $(1 \leq i \neq j \leq 2)$, we have $\delta(P_i)A_{ij} = A_{ij}\delta(P_j)$.

Take any $A_{ij} \in \mathcal{M}_{ij}$ $(1 \leq i \neq j \leq 2)$. Since $[P_i + A_{ij}, A_{ij} - P_j]_\xi = 0$, by equations (3.3)–(3.4) and Claim 3, we have

$$0 = [\delta(P_i) + \delta(A_{ij}), A_{ij} - P_j]_\xi + [P_i + A_{ij}, \delta(A_{ij}) - \delta(P_j)]_\xi$$

$$\begin{aligned}&= \delta(P_i)A_{ij} + \delta(A_{ij})A_{ij} - \delta(A_{ij})P_j - \xi A_{ij}\delta(A_{ij}) + \xi P_j\delta(A_{ij})\\&\quad + P_i\delta(A_{ij}) + A_{ij}\delta(A_{ij}) - A_{ij}\delta(P_j) - \xi\delta(A_{ij})P_i - \xi\delta(A_{ij})A_{ij}\\&= P_i\delta(P_i)A_{ij} + \xi P_j\delta(A_{ij})P_i - A_{ij}\delta(P_j)P_j - \xi P_j\delta(A_{ij})P_i.\end{aligned}$$

This implies that $P_i\delta(P_i)A_{ij} - A_{ij}\delta(P_j)P_j = 0$, that is, $\delta(P_i)A_{ij} = A_{ij}\delta(P_j)$ holds for all $A_{ij} \in \mathcal{M}_{ij}$.

Claim 5. $\delta(\xi I) = \xi\delta(I)$.

For any $A_{ij} \in \mathcal{M}_{ij}$ $(1 \le i \ne j \le 2)$, since $[\xi P_i + P_j, A_{ij}]_\xi = 0$, by Claim 2, we have

$$\begin{aligned}0 &= [\delta(\xi P_i) + \delta(P_j), A_{ij}]_\xi + [\xi P_i + P_j, \delta(A_{ij})]_\xi\\&= \delta(\xi P_i)A_{ij} - \xi A_{ij}\delta(P_j) + \xi P_i\delta(A_{ij}) + P_j\delta(A_{ij}) - \xi^2\delta(A_{ij})P_i - \xi\delta(A_{ij})P_j.\end{aligned}$$

Multiplying by P_i and P_j from the left and the right side respectively in the above equation, by Claim 2 and Claim 4, one can get

$$\delta(\xi P_i)A_{ij} = P_i\delta(\xi P_i)A_{ij} = \xi A_{ij}\delta(P_j)P_j = \xi A_{ij}\delta(P_j) = \xi\delta(P_i)A_{ij}.$$

That is, $(\delta(\xi P_i) - \xi\delta(P_i))P_iAP_j = 0$ for all $A \in \mathcal{M}$. Note that $\overline{P_j} = I$. It follows from the definition of the central carrier that $\operatorname{span}\{TP_j(x) : T \in \mathcal{M},\ x \in H\}$ is dense in H. So $\delta(\xi P_i) = \xi\delta(P_i)$ for $i = 1, 2$. Thus we obtain

$$\delta(\xi I) = \delta(\xi P_1 + \xi P_2) = \xi\delta(P_1) + \xi\delta(P_2) = \xi\delta(I).$$

Claim 6. $\delta(I) \in \mathcal{Z}(\mathcal{M})$.

By Claim 4, we have proved that, for any $A_{12} \in \mathcal{M}_{12}$ and $A_{21} \in \mathcal{M}_{21}$,

$$\delta(P_1)A_{12} = A_{12}\delta(P_2) \quad \text{and} \quad \delta(P_2)A_{21} = A_{21}\delta(P_1).$$

So, by using equations (3.3)–(3.4), we obtain

$$\delta(I)A_{12} = \delta(P_1)A_{12} = A_{12}\delta(P_2) = A_{12}\delta(I)$$

and

$$\delta(I)A_{21} = \delta(P_2)A_{21} = A_{21}\delta(P_1) = A_{21}\delta(I).$$

It follows from Lemma 3.3 that $\delta(I) \in \mathcal{Z}(\mathcal{M})$. The claim is true.

Now, note that $\delta(A) = L(A) + SA - AS$ for each $A \in \mathcal{M}$. By Claim 6, one has proved that

$$L(I) = \delta(I) \in \mathcal{Z}(\mathcal{M}). \tag{3.11}$$

Claim 7. If $\xi \neq -1$, then there exists an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ satisfying $\varphi(\xi A) = \xi\varphi(A)$ for each $A \in \mathcal{M}$ such that $L(A) = \varphi(A) + L(I)A$ for all $A \in \mathcal{M}$.

We will complete the proof of Claim 7 by several steps.

Step 1. For any $A_{ii} \in \mathcal{M}_{ii}$ and $B_{ij} \in \mathcal{M}_{ij}$, we have $\delta(A_{ii}B_{ij}) = \delta(A_{ii})B_{ij} + A_{ii}\delta(B_{ij}) - A_{ii}B_{ij}\delta(I)$, $1 \leq i \neq j \leq 2$.

Let $1 \leq i \neq j \leq 2$. For any $A_{ii} \in \mathcal{M}_{ii}$ and $B_{ij} \in \mathcal{M}_{ij}$, since $[A_{ii} + A_{ii}B_{ij}, B_{ij} - P_j]_\xi = 0$, by Claim 2 and Claim 3(1), we have

$$\begin{aligned} 0 &= [\delta(A_{ii}) + \delta(A_{ii}B_{ij}), B_{ij} - P_j]_\xi + [A_{ii} + A_{ii}B_{ij}, \delta(B_{ij}) - \delta(P_j)]_\xi \\ &= \delta(A_{ii})B_{ij} - \delta(A_{ii}B_{ij}) + A_{ii}\delta(B_{ij}) - A_{ii}B_{ij}\delta(P_j). \end{aligned}$$

Note that $A_{ii}B_{ij}\delta(P_j) = A_{ii}B_{ij}\delta(I)$ by equations (3.3)–(3.4). So Step 1 holds.

Step 2. For any $A_{ii} \in \mathcal{M}_{ii}$ and $B_{ij} \in \mathcal{M}_{ij}$, we have $\delta(A_{ij}B_{jj}) = \delta(A_{ij})B_{jj} + A_{ij}\delta(B_{jj}) - A_{ij}B_{jj}\delta(I)$, $1 \leq i \neq j \leq 2$.

As $[A_{ij}B_{jj} + B_{jj}, A_{ij} - P_i]_\xi = 0$, the assertion of Step 2 follows from Claim 2 and Claim 3(1) immediately.

Step 3. For any $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$, we have $\delta(A_{ii}B_{ii}) = \delta(A_{ii})B_{ii} + A_{ii}\delta(B_{ii}) - A_{ii}B_{ii}\delta(I)$, $i = 1, 2$.

Let $i \neq j$. Take any $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$ and any $S_{ij} \in \mathcal{M}_{ij}$. By Step 1, we get

$$\delta(A_{ii}B_{ii}S_{ij}) = \delta(A_{ii}B_{ii})S_{ij} + A_{ii}B_{ii}\delta(S_{ij}) - A_{ii}B_{ii}S_{ij}\delta(I)$$

and

$$\begin{aligned} \delta(A_{ii}B_{ii}S_{ij}) &= \delta(A_{ii})B_{ii}S_{ij} + A_{ii}\delta(B_{ii}S_{ij}) - A_{ii}B_{ii}S_{ij}\delta(I) \\ &= \delta(A_{ii})B_{ii}S_{ij} + A_{ii}\delta(B_{ii})S_{ij} + A_{ii}B_{ii}\delta(S_{ij}) - 2A_{ii}B_{ii}S_{ij}\delta(I). \end{aligned}$$

Comparing the above two equations, and by Claims 2 and 6, one obtains that

$$(\delta(A_{ii}B_{ii}) - \delta(A_{ii})B_{ii} - A_{ii}\delta(B_{ii}) + A_{ii}B_{ii}\delta(I))SP_j = 0$$

holds for all $S \in \mathcal{M}$. It follows from the fact $\overline{P}_j = I$ that

$$\delta(A_{ii}B_{ii}) = \delta(A_{ii})B_{ii} + A_{ii}\delta(B_{ii}) - A_{ii}B_{ii}\delta(I).$$

Step 4. For any $A_{ij} \in \mathcal{M}_{ij}$ and $B_{ji} \in \mathcal{M}_{ji}$, we have $\delta(A_{ij}B_{ji}) = \delta(A_{ij})B_{ji} + A_{ij}\delta(B_{ji}) - A_{ij}B_{ji}\delta(I)$, $1 \leq i \neq j \leq 2$.

Take any $A_{ij} \in \mathcal{M}_{ij}$ and $B_{ji} \in \mathcal{M}_{ji}$. Note that

$$[-A_{12}B_{21} + A_{12} + B_{21} - P_2, P_1 + A_{12} + B_{21} + B_{21}A_{12}]_\xi = 0.$$

By the definition of δ, Claim 2, Claim 3(1), Claim 4, Claim 6 and Steps 1-2, one gets

$$\begin{aligned}
0 &= [-\delta(A_{12}B_{21}) + \delta(A_{12}) + \delta(B_{21}) - \delta(P_2), P_1 + A_{12} + B_{21} + B_{21}A_{12}]_\xi \\
&\quad + [-A_{12}B_{21} + A_{12} + B_{21} - P_2, \delta(P_1) + \delta(A_{12}) + \delta(B_{21}) + \delta(B_{21}A_{12})]_\xi \\
&= -\delta(A_{12}B_{21}) - \delta(A_{12}B_{21})A_{12} + \delta(A_{12})B_{21} + \delta(A_{12})B_{21}A_{12} + \delta(B_{21}) \\
&\quad + \delta(B_{21})A_{12} - \delta(P_2)B_{21} - \delta(P_2)B_{21}A_{12} + \xi\delta(A_{12}B_{21}) - \xi\delta(A_{12}) - \xi A_{12}\delta(B_{21}) \\
&\quad + \xi A_{12}\delta(P_2) + \xi B_{21}\delta(A_{12}B_{21}) - \xi B_{21}\delta(A_{12}) - \xi B_{21}A_{21}\delta(B_{21}) + \xi B_{21}A_{12}\delta(P_2) \\
&\quad - A_{12}B_{21}\delta(P_1) - A_{12}B_{21}\delta(A_{12}) + A_{12}\delta(B_{21}) + A_{12}\delta(B_{21}A_{12}) + B_{21}\delta(P_1) \\
&\quad + B_{21}\delta(A_{12}) - \delta(B_{21}) - \delta(B_{21}A_{12}) + \xi\delta(P_1)A_{12}B_{21} - \xi\delta(P_1)A_{12} - \xi\delta(A_{12})B_{21} \\
&\quad + \xi\delta(A_{12}) + \xi\delta(B_{21})A_{12}B_{21} - \xi\delta(B_{21})A_{12} - \xi\delta(B_{21}A_{12})B_{21} + \xi\delta(B_{21}A_{12}) \\
&= (\xi-1)\delta(A_{12}B_{21}) + (1-\xi)\delta(A_{12})B_{21} + (1-\xi)A_{12}\delta(B_{21}) + (\xi-1)A_{12}B_{21}\delta(I) \\
&\quad + (\xi-1)\delta(B_{21}A_{12}) + (1-\xi)\delta(B_{21})A_{12} + (1-\xi)B_{21}\delta(A_{12}) + (\xi-1)B_{21}A_{12}\delta(I).
\end{aligned}$$

As $\xi \neq 1$, the above equation implies that

$$\delta(A_{12}B_{21}) = \delta(A_{12})B_{21} + A_{12}\delta(B_{21}) - A_{12}B_{21}\delta(I)$$

and

$$\delta(B_{21}A_{12}) = \delta(B_{21})A_{12} - B_{21}\delta(A_{12}) + B_{21}A_{12}\delta(I).$$

Step 5. For any $A, B \in \mathcal{M}$, we have $\delta(AB) = \delta(A)B + A\delta(B) - \delta(I)AB$.

For any $A = A_{11} + A_{12} + A_{21} + A_{22}$, $B = B_{11} + B_{12} + B_{21} + B_{22} \in \mathcal{M}$, by the additivity of δ, Claim 6 and Steps 1-4, one can easily check that $\delta(AB) = \delta(A)B + A\delta(B) - \delta(I)AB$.

Step 6. There exists an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ satisfying $\varphi(\xi A) = \xi\varphi(A)$ for each A such that $L(A) = \varphi(A) + L(I)A$ for all $A \in \mathcal{M}$.

Define a map $\varphi : \mathcal{M} \to \mathcal{M}$ by $\varphi(A) = L(A) - L(I)A = \delta(A) - SA + AS - L(I)A$ for all $A \in \mathcal{M}$. Obviously, φ is additive. Moreover, for any A, $B \in \mathcal{M}$, by Step 5 and equation (3.11), one achieves

$$\begin{aligned}
\varphi(AB) &= \delta(AB) - SAB + ABS - L(I)AB \\
&= \delta(A)B + A\delta(B) - 2L(I)AB - SAB + ABS \\
&= (\delta(A) - L(I)A - SA + AS)B + A(\delta(B) - L(I)B + BS - SB) \\
&= \varphi(A)B + A\varphi(B),
\end{aligned}$$

that is, φ is an additive derivation.

Now we show that $\varphi(\xi A) = \xi\varphi(A)$ for each $A \in \mathcal{M}$. In fact, since $\delta(\xi I) = \xi\delta(I)$ (Claim 5), we get $\varphi(\xi I) = \xi\varphi(I) = 0$. Hence, for any $A \in \mathcal{M}$, we have

$$\varphi(\xi A) = \varphi(\xi IA) = \varphi(\xi I)A + \xi\varphi(A) = \xi\varphi(A).$$

Step 6, combining equation (3.11), ensures that Claim 7 holds.

Claim 8. If $\xi = -1$, then there exists an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ such that $L(A) = \varphi(A) + L(I)A$ for all $A \in \mathcal{M}$.

In this case, we will first show that $\delta(A^2) = \delta(A)A + A\delta(A) - \delta(I)A^2$ for all $A \in \mathcal{M}$. This will be done by several steps.

Step 1. For any $A_{ij} \in \mathcal{M}_{ij}$ and $A_{jj} \in \mathcal{M}_{jj}$ $(1 \le i \ne j \le 2)$, we have

$$\delta(A_{ij}A_{jj}) = \delta(A_{ij})A_{jj} + A_{ij}\delta(A_{jj}) + A_{jj}\delta(A_{ij}) - \delta(I)A_{ij}A_{jj}.$$

Let $1 \le i \ne j \le 2$. Taking any $A_{ij} \in \mathcal{M}_{ij}$, $A_{jj} \in \mathcal{M}_{jj}$, and noting that

$$[A_{ij}A_{jj}+A_{jj}, A_{ij}-P_i]_{-1} = (A_{ij}A_{jj}+A_{jj})(A_{ij}-P_i)+(A_{ij}-P_i)(A_{ij}A_{jj}+A_{jj}) = 0,$$

by Claim 2, we have

$$\begin{aligned} 0 &= [\delta(A_{ij}A_{jj}) + \delta(A_{jj}), A_{ij} - P_i]_{-1} + [A_{ij}A_{jj} + A_{jj}, \delta(A_{ij}) - \delta(P_i)]_{-1} \\ &= \delta(A_{ij}A_{jj})A_{ij} - \delta(A_{ij}A_{jj})P_i + A_{ij}\delta(A_{ij}A_{jj}) + A_{ij}\delta(A_{jj}) - P_i\delta(A_{ij}A_{jj}) \\ &\quad + A_{ij}A_{jj}\delta(A_{ij}) + A_{jj}\delta(A_{ij}) + \delta(A_{ij})A_{ij}A_{jj} + \delta(A_{ij})A_{jj} - \delta(P_i)A_{ij}A_{jj}. \end{aligned} \tag{3.12}$$

Multiplying by P_i and P_j from the left and the right side in equation (3.12) respectively, by Claim 2 and Claim 3(2), we get

$$\begin{aligned} P_i\delta(A_{ij}A_{jj})P_j &= P_i\delta(A_{ij})A_{jj} + A_{ij}\delta(A_{jj}) - \delta(P_i)A_{ij}A_{jj} \\ &= \delta(A_{ij})A_{jj} + A_{ij}\delta(A_{jj}) - \delta(I)A_{ij}A_{jj}. \end{aligned}$$

Similarly, multiplying by P_j and P_i from the left and the right side in equation (3.12) respectively, one can get

$$P_j\delta(A_{ij}A_{jj})P_i = A_{jj}\delta(A_{ij})P_i = A_{jj}\delta(A_{ij}).$$

So

$$\begin{aligned} \delta(A_{ij}A_{jj}) &= P_i\delta(A_{ij}A_{jj})P_j + P_j\delta(A_{ij}A_{jj})P_i \\ &= \delta(A_{ij})A_{jj} + A_{ij}\delta(A_{jj}) + A_{jj}\delta(A_{ij}) - \delta(I)A_{ij}A_{jj}. \end{aligned}$$

Step 2. For any $A_{ii} \in \mathcal{M}_{ii}$ and $A_{ij} \in \mathcal{M}_{ij}$ $(1 \le i \ne j \le 2)$, we have

$$\delta(A_{ii}A_{ij}) = \delta(A_{ii})A_{ij} + A_{ii}\delta(A_{ij}) + \delta(A_{ij})A_{ii} - \delta(I)A_{ii}A_{ij}.$$

As $[A_{ii}A_{ij} + A_{ii}, A_{ij} - P_j]_{-1} = 0$, by the same argument as that of Step 1, one can check that this step is true.

Step 3. For any $A_{ii} \in \mathcal{M}_{ii}$, we have $\delta(A_{ii}^2) = \delta(A_{ii})A_{ii} + A_{ii}\delta(A_{ii}) - A_{ii}^2\delta(I)$, $i = 1, 2$.

Let $j \neq i$. For any $A_{ii} \in \mathcal{M}_{ii}$ and any $S_{ij} \in \mathcal{M}_{ij}$, by using Step 2 and calculating $\delta(A_{ii}A_{ii}S_{ij})$ by two different ways, one can easily check that the step is true.

Step 4. For any $A_{ij} \in \mathcal{M}_{ij}$ and $A_{ji} \in \mathcal{M}_{ji}$ $(1 \leq i \neq j \leq 2)$, we have

$$\delta(A_{ij}A_{ji}) = \delta(A_{ij})A_{ji} + A_{ij}\delta(A_{ji}) - A_{ij}A_{ji}\delta(I).$$

In fact, for any $A_{ij} \in \mathcal{M}_{ij}$ and $A_{ji} \in \mathcal{M}_{ji}$ $(1 \leq i \neq j \leq 2)$, since

$$[A_{ij}A_{ji} + A_{ij} + A_{ji} + P_j, P_i + A_{ij} - A_{ji} + A_{ji}A_{ij}]_{-1} = 0,$$

the assertion of this step follows from Claim 2, Claim 3(2) and Steps 1–3 in Claim 8.

Step 5. For any $A \in \mathcal{M}$, we have $\delta(A^2) = \delta(A)A + A\delta(A) - \delta(I)A^2$.

For any $A = A_{11} + A_{12} + A_{21} + A_{22} \in \mathcal{M}$, by the additivity of δ, Claim 5 and Steps 1-4 in Case 2, it is easily checked that $\delta(A^2) = \delta(A)A + A\delta(A) - \delta(I)A^2$.

Step 6. There exists an additive derivation $\varphi : \mathcal{M} \to \mathcal{M}$ such that $L(A) = \varphi(A) + L(I)A$ holds for all $A \in \mathcal{M}$.

Define a map $\varphi : \mathcal{M} \to \mathcal{M}$ by $\varphi(A) = L(A) - L(I)A = \delta(A) - SA + AS - L(I)A$ for all $A \in \mathcal{M}$. It is clear that φ is additive and $L(A) = \varphi(A) + L(I)A$ for each A. Moreover, by using Step 5 of Claim 8, one can check that φ is a Jordan derivation, that is, $\varphi(A^2) = \varphi(A)A + A\varphi(A)$ for all $A \in \mathcal{M}$. By [1], φ is an additive derivation. This and equation (3.11) imply that Claim 8 holds.

Since $\varphi(-1A) = -\varphi(A)$ for each A as φ is additive, Claims 7-8 and equation (3.11) together ensure that Theorem 3.1 is true, finishing the proof. □

By Theorem 3.1 we get a characterization of additive ξ-Lie derivations immediately.

Corollary 3.4. *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. Suppose that $L : \mathcal{M} \to \mathcal{M}$ is an additive map and ξ is a scalar with $\xi \neq 1$. Then the following statements are equivalent.*

(1) *L is a ξ-Lie derivation.*

(2) *L is a derivation with $L(\xi A) = \xi L(A)$ for all $A \in \mathcal{M}$.*

(3) *L is ξ-Lie derivable at zero and $L(I) = 0$.*

Finally, we remark that, by a similar approach to the proof of Theorem 3.1, we can improve further the main result [22, Theorem 2.1] omitting the assumptions $L(I) \in \mathcal{Z}(\mathcal{A})$ and $L(\xi A) = \xi L(A)$ for each A in the prime algebra case.

Theorem 3.5. *Let $\mathcal{A}$ be a unital prime algebra over a field $\mathbb{F}$ containing a non-trivial idempotent P. Denote by $\mathcal{Z}(\mathcal{A})$ and $\mathcal{C}$ the center of $\mathcal{A}$ and the extended centroid of $\mathcal{A}$, respectively. Assume that $\xi \in \mathbb{F}$ and $L : \mathcal{A} \to \mathcal{A}$ is an additive map. If L is ξ-Lie derivable at zero, then there exists an additive derivation $\tau : \mathcal{A} \to \mathcal{C}$ such that*

(1) *if $\xi = 0$, then $L(I) \in \mathcal{Z}(\mathcal{A})$ and $L(A) = \tau(A) + L(I)A$ for all $A \in \mathcal{A}$;*

(2) *if $\xi = 1$ and $\mathcal{A}$ is of characteristic not 2 with $\deg(\mathcal{A}) > 2$, then $L(A) = \tau(A) + \alpha A + \nu(A)$ for all $A \in \mathcal{A}$, where $\alpha \in \mathcal{C}$ and ν is an additive map from $\mathcal{A}$ into $\mathcal{C}$;*

(3) *if $\xi = -1$ and $\mathcal{A}$ is of characteristic not 2, then $L(I) \in \mathcal{Z}(\mathcal{A})$ and $L(A) = \tau(A) + L(I)A$ for all $A \in \mathcal{A}$;*

(4) *if $\xi \neq 0, \pm 1$, then $L(I) \in \mathcal{Z}(\mathcal{A})$, $L(\xi A) = \xi L(A)$ and $L(A) = \tau(A) + L(I)A$ for all $A \in \mathcal{A}$.*

Proof. Assume that $L : \mathcal{A} \to \mathcal{A}$ is ξ-Lie derivable at zero. We need only to show that, for $\xi \neq 1$, $L(I) \in \mathcal{Z}(\mathcal{A})$, $L(\xi A) = \xi L(A)$.

Using the same notations as that in the proof of Theorem 3.1, but replacing $\mathcal{M}$ by the prime algebra $\mathcal{A}$, Claims 1–4 are still true. Thus, similar to the argument in Claim 5, we have

$$\delta(\xi P_i)A_{ij} = P_i\delta(\xi P_i)A_{ij} = \xi A_{ij}\delta(P_j)P_j = \xi A_{ij}\delta(P_j) = \xi\delta(P_i)A_{ij}.$$

So $(\delta(\xi P_i) - \xi\delta(P_i))P_iAP_j = 0$ for all $A \in \mathcal{A}$. Since $\mathcal{A}$ is prime, we must have $\delta(\xi P_i) - \xi\delta(P_i) = (\delta(\xi P_i) - \xi\delta(P_i))P_i = 0$, that is, $\delta(\xi P_i) = \xi\delta(P_i)$ for $i = 1, 2$. Thus we obtain

$$\delta(\xi I) = \delta(\xi P_1 + \xi P_2) = \xi\delta(P_1) + \xi\delta(P_2) = \xi\delta(I).$$

Then, a similar argument as in Claim 6 gives

$$\delta(I)A_{12} = \delta(P_1)A_{12} = A_{12}\delta(P_2) = A_{12}\delta(I).$$

This implies that

$$\delta(P_1)AP_2 = P_1A\delta(P_2) \quad \text{for all } A \in \mathcal{A}.$$

Again, as $\mathcal{A}$ is prime, it follows from [6, Theorem A.7] that there exists some central element $\lambda \in \mathcal{C}$ such that $\delta(P_i) = \lambda P_i$. Hence $\delta(I) = \delta(P_1) + \delta(P_2) =$

$\lambda(P_1 + P_2) = \lambda \in \mathcal{C} \cap \mathcal{A} = \mathcal{Z}(\mathcal{A})$. Thus $L(I) = \delta(I) \in \mathcal{Z}(\mathcal{A})$ and $L(\xi I) = \xi L(I)$.

Now, similar to the proof of Claim 8, and using the primeness of $\mathcal{A}$ where needs, one can show that, for any $A \in \mathcal{A}$, we have $\varphi(\xi A) = \varphi(\xi I A) = \varphi(\xi I)A + \xi\varphi(A) = \xi\varphi(A)$. This, together with the facts proved before, entails that $L(\xi A) = \xi L(A)$ for all $A \in \mathcal{A}$. □

Acknowledgement. The authors wish to give their thanks to the referees for helpful comments and suggestions.

References

[1] M. Brešar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.* **104** (1988), 1003–1006.

[2] M. Brešar and C. R. Miers, Commutativity preserving mappings of von Neumann algebras, *Canad. J. Math.* **45** (1993), 695–708.

[3] M. Brešar, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, *Trans. Amer. Math. Soc* **335** (1993), 525–546.

[4] M. Brešar, Jordan derivations revisited, *Math. Proc. Cambridge Philos. Soc.* **139** (2005), 411–425.

[5] M. Brešar, Characterizing homomorphisms, derivations and multipliers in rings with idempotents, *Proc. R. Soc. Edinb., Sect. A* **137** (2007), 9–21.

[6] M. Brešar, M. A. Chebotar and W. S. Martindale III, Functional Identities, *Birkhäuser Basel*, 2007.

[7] I. N. Herstein, Jordan derivations of prime rings, *Proc. Amer. Math. Soc.* **8** (1957), 1104–1110.

[8] J.-C. Hou and X.-F. Qi, Additive maps derivable at some points on $\mathcal{J}$-subspace lattice algebras, *Lin. Alg. Appl.* **429** (2008), 1851–1863.

[9] M.-Y. Huang and J.-H. Zhang, ξ-Lie derivable maps on triangular algebras by Jordan zero products, *Acta Math. Sin., Chinese Series* **55** (2012), 953–960.

[10] B. E. Johnson, Symmetric amenability and the nonexistence of Lie and Jordan derivations, *Math. Proc. Cambridge Philos. Soc.* **120** (1996), 455–473.

[11] P. Ji and W. Qi, Characterizations of Lie derivations of triangular algebras, *Lin. Alg. Appl.* **435** (2011), 1137–1146.

[12] W. Jing, S.-J. Lu and P.-T. Li, Characterisations of derivations on some operator algebras, *Bull. Austral. Math. Soc.* **66** (2002), 227–232.

[13] T.-K. Lee, Generalized skew derivations characterized by acting on zero products, *Pacific J. Math.* **216** (2004), 293–301.

[14] L. Liu and G.-X. Ji, ξ-Lie derivable maps and generalized ξ-Lie derivable maps on standard operator algebras, `http://www.paper.edu.cn/releasepaper/content/201205-96 [1].`

[15] F.-Y. Lu and W. Ji, Characterizations of Lie derivations of $\mathcal{B}(X)$, *Lin. Alg. Appl.* **432** (2009), 89–99.

[16] M. Mathieu and A. R. Villena, The structure of Lie derivations on C^*-algebras, *J. Funct. Anal.* **202** (2003), 504–525.

[17] C. R. Miers, Lie homomorphisms of operator algebras, *Pacific J. Math.* **38** (1971), 717–735.

[18] Z. Pan, Derivablemaps and derivational points, *Lin. Alg. Appl.* **436** (2012), 4251–4260.

[19] X.-F. Qi and J.-C. Hou, Additive Lie (ξ-Lie) derivations and generalized Lie (ξ-Lie) derivations on Nest algebras, *Lin. Alg. Appl.* **431** (2009), 843–854.

[20] X.-F. Qi and J.-C. Hou, Linear maps Lie derivable at zero on $\mathcal{J}$-subspace lattice algebras, *Studia Math.* **197** (2010), 157–169.

[21] X.-F. Qi and J.-C. Hou, Characterization of Lie derivations on prime rings, *Communication in Algebras* **39** (2011), 3824–3835.

[22] X.-F. Qi, J.-L. Cui and J.-C. Hou, Characterizing additive ξ-Lie derivations of prime algebras by ξ-Lie zero products, *Lin. Alg. Appl.* **434** (2011), 669–682.

[23] X.-F. Qi and J.-C. Hou, Additive Lie (ξ-Lie) derivations and generalized Lie (ξ-Lie) derivations on prime algebras, *Acta Math. Sin., English Series* **29** (2013), 383–392.

[24] X.-F. Qi and J.-C. Hou, Characterization of Lie derivations on von Neumann algebras, *Linear Alg. Appl.* **438** (2013), 533–548.

[25] W. Yang and J. Zhu, Characterizations of additive (generalized) ξ-Lie (α, β) derivations on triangular algebras, *Linear and Multilinear Algebras* **61** (2013), 811–830.

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(Received February 8, 2014; revised July 16, 2014)