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A note on Clifford parallelisms in characteristic two

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Abstract. It is well known that a purely inseparable field extension L/F with some extra property and degree $[L : F] = 4$ determines a Clifford parallelism on the set of lines of the three-dimensional projective space over F. By extending the ground field of this space from F to L, we establish the following geometric description of such a parallelism in terms of a distinguished 'absolute pencil of lines' of the extended space: Two lines are Clifford parallel if, and only if, there exists a line of the absolute pencil that meets both of them.

1. Introduction

A detailed survey of various old results about Clifford parallel lines in the three-dimensional elliptic space (over the real numbers) can be found in the recent article [3]. One such result is a description of Clifford parallel lines in terms of the complexified elliptic space, and it may be summarised as follows: The elliptic metric yields a hyperbolic quadric of the complex projective space; it is known as the 'absolute quadric'. Two lines of the elliptic space are Clifford parallel if, and only if, there exists a line of the absolute quadric that meets both of them (after complexification). Since there are two reguli on the absolute quadric, one actually gets two parallelisms. It is conventional to label them as the 'left' and 'right' Clifford parallelism of the elliptic space. An alternative approach uses the skew field $\mathbb{H}$ of real quaternions as underlying vector space of the elliptic space. The norm function of $\mathbb{H}$ is a quadratic form, and it yields the elliptic metric. The

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left and right Clifford parallelism arise from the left and right multiplication in $\mathbb{H}$, respectively; see [5, p. 8].

Any quaternion skew field L with arbitrary characteristic and centre F, say, can be used (as in the classical case) to define a left and a right Clifford parallelism in the three-dimensional projective space on the F-vector space L. This finding from [25] was the starting point for the research in [6] and [7], where the following was established: If the ground field F is extended in an appropriate way, then the description from above of Clifford parallel lines in terms of the two reguli on a hyperbolic quadric basically remains valid. However, the details are much more involved for an arbitrary quaternion skew field L than for the real quaternions $\mathbb{H}$.

According to [25], there is one more kind of Clifford parallelism in three-dimensional projective spaces over certain fields F of characteristic two. The algebraic definition of such a parallelism is similar to what we had in the preceding paragraphs, but now one has to use a field extension L/F with degree $[L : F] = 4$ and such that $a^2 \in F$ for all $a \in L$. So L/F is purely inseparable. Due to the commutativity of L, left parallel lines now are the same as right parallel lines.

There arises the question if also in this remaining case there is a geometric description of Clifford parallel lines similar to the one from [6] and [7]. We shall have to take several steps before we can provide an affirmative answer. First, we extend the ground field of the projective space (with underlying vector space L) from F to L. Next, we find in this extended space a distinguished plane Π which will be called the *absolute plane*. Its points are determined by the singular vectors of a quasilinear quadratic form. So the points of the absolute plane correspond, loosely speaking, to the points of the hyperbolic quadric from above. However, we must not use all lines of this plane to accomplish our task, but only those passing through a particular *absolute point* $\mathcal{A}$ of Π. This gives a single *absolute pencil of lines* which now takes the part of the two reguli on a hyperbolic quadric. Finally, our main result is Theorem 3.8: Two lines of the three-dimensional projective space over F are Clifford parallel if, and only if, there exists a line of the absolute pencil that meets both of them (after extension of the ground field).

For more information about parallelisms in general, we refer to [3], [21], the book [22], and the references therein. At one point we shall come across the double space axiom, which is part of the well established axiomatic description of Clifford parallelisms; see, among others, [18], the survey in [24], and [30]. Notions from geometry and algebra that are used without further reference can be found, for example, in [9], [10], [12], [19], [20], [29], and [31].

2. Algebraic preliminaries

Let V be a vector space over a field F. We shall also write V_F instead of V in order to clarify the ground field. If F is a subfield of a field L then V can be extended to a vector space over L by the following well known construction [9, p. 277]: The tensor product $(L \otimes_F V)_F$ can be made into a vector space over L by letting

$$a \sum_s a_s \otimes v_s := \sum_s (a a_s) \otimes v_s \quad \text{for all } a, a_s \in L,\ v_s \in V. \tag{1}$$

We use the shorthand $V_{(L)} := (L \otimes_F V)_L$ for this L-vector space. The canonical embedding of V_F in $V_{(L)}$ is given by $v \mapsto 1 \otimes v$. Suppose now that V is also an associative F-algebra with unit e. A multiplication in $L \otimes_F V$ can be defined by the formula

$$\Big(\sum_s a_s \otimes v_s\Big) \cdot \Big(\sum_t a'_t \otimes v'_t\Big) := \sum_{s,t} (a_s a'_t) \otimes (v_s v'_t)$$
$$\text{for all } a_s, a'_t \in L,\ v_s, v'_t \in V. \tag{2}$$

In this way $V_{(L)}$ turns into an associative algebra over L with unit $1 \otimes e$, and the embedding from above is a monomorphism [9, pp. 433–434].

Global assumption. From now on let F be a field of characteristic 2 and let L be an extension field of F with degree $[L : F] = 4$ and such that $a^2 \in F$ for all $a \in L$.

The field extension L/F is purely inseparable. Clearly, each $a \in L$ is a zero of the quadratic polynomial $X^2 + a^2 \in F[X]$, whence L is a *quadratic* or, in a different terminology, a *kinematic* algebra over F [23, p. 423]. The quadratic form

$$(\,\cdot\,)^2 : L_F \to F : y \mapsto y^2 \tag{3}$$

has no singular vectors. It is the *norm form* of the algebra L_F. Following [1, p. 150], (3) is a *quasilinear* quadratic form. This means that (3) is a semilinear mapping of the vector space L_F in the vector space F over its subfield $F^{(2)}$ formed by all squares, with $F \to F^{(2)} : f \mapsto f^2$ as accompanying field isomorphism; see also [12, p. 33].

We now specialise the algebra V_F from above to be L_F and obtain the four-dimensional commutative and associative L-algebra $(L \otimes_F L)_L = L_{(L)}$ with unit $1 \otimes 1$. According to (1), scalars from L act on the *first* factors of pure tensors. So, the scalar multiples of $1 \otimes 1$ comprise the one-dimensional subspace $L(1 \otimes 1) = \{x \otimes 1 \mid x \in L\}$ of $L_{(L)}$. This subspace is a first isomorphic copy of the field L within the algebra $L_{(L)}$. A second copy is given by the subset $\{1 \otimes y \mid y \in L\}$. Here the elements of L appear in their role as vectors of L_F. None of these isomorphic copies of L will be identified with L.

The multiplication in the field L is an L-bilinear mapping $L \times L \to L : (x, y) \mapsto xy$. Clearly, this mapping is also F-bilinear. By the universal property of the tensor product (for vector spaces over F) there is a unique F-linear mapping

$$\pi : L \otimes_F L \to L \quad \text{such that } (x \otimes y)^\pi = xy \quad \text{for all } x, y \in L. \tag{4}$$

Using (1) and (2), a straightforward calculation shows that $\pi : L_{(L)} \to L_L$ actually is a surjective homomorphism of unital L-algebras. Since L is a field, this implies already that

$$\Pi := \ker \pi \tag{5}$$

is a three-dimensional maximal ideal of the four-dimensional algebra $L_{(L)}$, but we easily can say more.

Lemma 2.1. *The L-algebra $L_{(L)}$ is local and quadratic. The ideal of non-invertible elements of $L_{(L)}$ is the kernel Π of the homomorphism π from (4).*

Proof. Any $g \in L_{(L)}$ can be written as $g = \sum_s a_s \otimes b_s$ with $a_s, b_s \in L$. We read off from

$$g^2 = \Big(\sum_s a_s \otimes b_s\Big)^2 = \sum_s a_s^2 \otimes b_s^2 = \Big(\sum_s a_s^2 b_s^2\Big)(1 \otimes 1) = \big(g^\pi\big)^2(1 \otimes 1)$$

that g is a zero of the polynomial $X^2 + \big(g^\pi\big)^2 \in F[X] \subset L[X]$, whence $L_{(L)}$ is quadratic. For $g \notin \Pi$ we obtain $g^{-1} = (g^\pi)^{-2} g$, whereas any $g \in \Pi$ clearly has no multiplicative inverse due to $g^2 = 0$. Thus the ideal Π comprises precisely the non-invertible elements of $L_{(L)}$. So, by definition, $L_{(L)}$ is a local algebra, and Π has the required property. $\square$

The canonically defined L-linear form π maps $1 \otimes y \mapsto y$ for all $y \in L$. Due to $F \subsetneqq L$, the form π does not arise as an extension of an F-linear form on L_F. However, the *square of* π, i.e., the *norm form*

$$L_{(L)} \to L : z \mapsto (z^\pi)^2 \tag{6}$$

is a quasilinear quadratic form extending the quasilinear quadratic form (3) from L_F to $L_{(L)}$. Indeed, the norm of $1 \otimes y$ equals y^2 for all $y \in L$. The non-zero vectors of Π are precisely the singular vectors of the norm form (6).

Let us return to multiplication. Any $b \in L$ determines the mapping

$$\mu_b : L \to L : x \mapsto xb, \tag{7}$$

i.e., the multiplication of elements of L by the fixed element b. Any such μ_b clearly is an L-linear mapping $L_L \to L_L$, but below we shall only make use of its F-linearity. Likewise, for any $h \in L_{(L)}$ there is an L-linear mapping $\mu_h : L_{(L)} \to L_{(L)} : z \mapsto zh$. The canonical extension of $\mu_b : L_F \to L_F$ from L_F to $L_{(L)}$ is the *Kronecker product* (or: *tensor product* [9, p. 245]) $\mu_1 \otimes \mu_b$ which acts on pure tensors by sending

$$x \otimes y \mapsto (x1) \otimes (yb) = (x \otimes y)(1 \otimes b). \tag{8}$$

So $\mu_1 \otimes \mu_b = \mu_{1\otimes b}$. Note that, contrary to what we had in (1), the element $b \in L$ acts on the *second* factors of pure tensors in (8).

At times it will be convenient to use coordinates (which are written as rows). To this end we first choose $i, j \in L$ such that 1, i, j are linearly independent over F. Then

$$(1, i, j, k) \quad \text{with } k := ij \tag{9}$$

is a basis of L_F and

$$(1 \otimes 1, 1 \otimes i, 1 \otimes j, 1 \otimes k) \tag{10}$$

is a basis of $L_{(L)}$. These bases allow us to replace L_F and $L_{(L)}$ with F^4 and L^4, respectively. For example, the coordinate representation of the homomorphism π from (4) is the mapping

$$L^4 \to L : (z_0, z_1, z_2, z_3) \mapsto z_0 + iz_1 + jz_2 + kz_3,$$

whence the quadratic norm form (6) has the representation

$$L^4 \to L : (z_0, z_1, z_2, z_3) \mapsto z_0^2 + i^2 z_1^2 + j^2 z_2^2 + k^2 z_3^2.$$

If we restrict the domain of the last mapping to F^4 and replace its codomain by F then the description of the quadratic norm form (3) in terms of coordinates is obtained. When working in $L_{(L)}$ it will often be more appropriate to change from the basis (10) to another basis of $L_{(L)}$, namely

$$\begin{aligned}(1 \otimes 1, p, q, r) \quad \text{with } p &:= 1 \otimes i + i \otimes 1,\\ q &:= 1 \otimes j + j \otimes 1,\\ r &:= pq = 1 \otimes k + i \otimes j + j \otimes i + k \otimes 1.\end{aligned} \tag{11}$$

For example, if $g \in L_{(L)}$ has coordinates $(g_0, g_1, g_2, g_3) \in L^4$ with respect to the basis (11) then $g^\pi = g_0$ and g_0^2 is the norm of g.

While the elements p, q, r depend on the choice of i, j in the basis (9), the span of r has a basis-free meaning:

Lemma 2.2. *Let $\mathcal{A}$ be the annihilator in $L_{(L)}$ of the maximal ideal Π. Upon choosing an arbitrary basis $(1, i, j, k)$ of L_F as in (9) and by changing to the associated basis (11) of $L_{(L)}$, there holds*

$$\mathcal{A} = Lr = L(1 \otimes k + i \otimes j + j \otimes i + k \otimes 1). \tag{12}$$

Proof. We recall from the proof of Lemma 2.1 that $z^2 = 0$ for all $z \in \Pi$. Thus the elements from (11) satisfy $pr = p^2 q = 0$, $qr = q^2 p = 0$, and $r^2 = 0$. From $\Pi = Lp \oplus Lq \oplus Lr$, all elements of Π are annihilated by r, and so $Lr \subset \mathcal{A}$. The annihilator of p clearly is a subspace of Π containing p and r. Due to $pq = r \neq 0$ and $\dim \Pi = 3$, the two-dimensional subspace $Lp \oplus Lr$ is the annihilator of p. Likewise the annihilator of q equals $Lq \oplus Lr$, whence $\mathcal{A} \subset (Lp \oplus Lr) \cap (Lq \oplus Lr) = Lr$, as required. □

Since $\mathcal{A}$ is an ideal of the commutative ring $L_{(L)}$, it may also be written as the principal ideal $L_{(L)} r$, which is generated by the element r. Another description of this ideal is $\mathcal{A} = \Pi \cdot \Pi = \{zw \mid z, w \in \Pi\}$. We noted already subsequent to (6) that Π is the set of vectors of $L_{(L)}$ with norm zero, so that $\mathcal{A}$ is also related to the norm form of $L_{(L)}$.

3. The absolute pencil

We shall view L_F as the underlying vector space of a projective space $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$. We adopt the usual geometric terms: Points, lines and planes are the subspaces of L_F with dimension one, two, and three, respectively. Incidence is symmetrised inclusion. Likewise, $L_{(L)} = (L \otimes_F L)_L$ gives rise to a projective space $\mathbb{P}(L_{(L)}) \cong \mathbb{P}_3(L)$. The canonical embedding of $\mathbb{P}(L_F)$ in $\mathbb{P}(L_{(L)})$ is given by $Fx \mapsto L(1 \otimes x)$. Those points of $\mathbb{P}(L_{(L)})$ that are images under this embedding are called *F-rational*. A subspace of $L_{(L)}$ is called *F-rational* if it is spanned by its F-rational points. If T is a subspace of L_F then its extension $T_{(L)}$ is an F-rational subspace of the same dimension, and all F-rational subspaces of $L_{(L)}$ arise in this way. The projective space $\mathbb{P}(L_{(L)})$ has two distinguished subspaces that stem from the algebra $L_{(L)}$:

Definition 3.1. We call the ideal $\mathcal{A}$ from (12) the *absolute point* and the ideal Π from (5) the *absolute plane* of the projective space $\mathbb{P}(L_{(L)})$. The set of lines through $\mathcal{A}$ that lie in the plane Π is denoted by $[\mathcal{A}, \Pi]$, and it is called the *absolute pencil*.

Take notice that here we adopt the phrase ‘absolute’ in analogy to the conventional terminology for Cayley–Klein geometries (see, for example, [14]) and not in its meaning for polarities, where a point is called ‘absolute’ if it is incident with its polar hyperplane. However, we shall encounter polarities at the very end of this section, and encourage the reader to compare our results with recent findings in [26, 3.5], [27, Thorem 6.4], and [28, 2.7, 2.8] about polarities with a surprisingly small set of ‘absolute’ points.

Proposition 3.2. *The absolute plane* Π *of the extended projective space* $\mathbb{P}(L_{(L)}) \cong \mathbb{P}_3(L)$ *has the following properties:*

(i) *The absolute plane contains no* F*-rational points.*

(ii) *Each point of the absolute plane is incident with at most one* F*-rational line.*

(iii) *Let* Fa *and* Fb *be distinct points of* $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$ *and let* $M = Fa \oplus Fb$ *be the line joining them. Then the* F*-rational line* $M_{(L)}$ *meets the absolute plane at the point* $L(a \otimes b + b \otimes a)$*.*

Proof. *Ad* (i). Assume to the contrary that there exists an F-rational point in Π. Such a point has the form $L(1 \otimes c)$ with $c \in L \setminus \{0\}$, whence $(1 \otimes c)^\pi = c \neq 0$ yields a contradiction.

Ad (ii). Suppose that a point of Π were on two distinct F-rational lines, say $M_{(L)}$ and $N_{(L)}$. Thus $M_{(L)} \cap N_{(L)}$ would be an F-rational point of Π, which is impossible by (i).

Ad (iii). Since a and b are linearly independent in L_F, the tensors $a \otimes b$ and $b \otimes a$ can be extended to a basis of $(L \otimes_F L)_F$. Hence their sum is non-zero, and $L(a \otimes b + b \otimes a)$ is a point of $\mathbb{P}(L_{(L)})$. From $(a \otimes b + b \otimes a)^\pi = 2ab = 0$, this point belongs to Π, and $a \otimes b + b \otimes a = a(1 \otimes b) + b(1 \otimes a)$ implies that it is on the F-rational line $M_{(L)}$. $\square$

Remark 3.3. From Proposition 3.2, an injective mapping of the set of lines of $\mathbb{P}(L_F)$ into the set of points of the absolute plane Π is given by $M \mapsto \Pi \cap M_{(L)}$. Its algebraic description is based on the alternating F-bilinear mapping of $L_F \times L_F$ to $(L \otimes_F L)_F$ sending (x, y) to $x \otimes y + y \otimes x$. By the universal property of the tensor product, this bilinear mapping gives rise to the *alternation operator*

$$(L \otimes_F L)_F \to (L \otimes_F L)_F : x \otimes y \mapsto x \otimes y + y \otimes x.$$

The image of this F-linear operator can be identified with the exterior square $\bigwedge^2 L_F$, whence the points of the form $L(x \otimes y + y \otimes x) = L(x \wedge y)$ provide a model of the Klein quadric (over F) within the projective plane Π (over L). A detailed description of this model is not within the scope of this article.

The following is taken from [25, Satz 1]:

Definition 3.4. Let (M, N) be a pair of lines of the projective space $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$. We say that M is *Clifford parallel* (or shortly: *parallel*) to N if there is an element $b \in L \setminus \{0\}$ such that $N = Mb$. In this case we write $M \parallel N$.

Due to the commutativity of L, the 'left' and 'right' parallel relations from [25] coincide here. The relation $\parallel$ defines indeed a parallelism on $\mathbb{P}(L_F)$, i.e., for each line M and each point Fa there is a unique line N with $Fa \subset N \parallel M$. The parallel class of M is written as $\mathcal{S}(M)$.

The Clifford parallelism on the line set of $\mathbb{P}(L_F)$ satisfies the *double space axiom* [25, p. 154]. In our setting this result reads as follows: Given lines M and N with a common point, say Fa, and arbitrary points on M and N, say Fb and Fc, the unique line $M' \parallel M$ through Fc has a point in common with the unique line $N' \parallel N$ through Fb. Let us repeat the easy proof. From $M' = Ma^{-1}c$ and $N' = Na^{-1}b$ follows that Fd with $d := a^{-1}bc$ is a common point of M' and N'. If the lines M, M', N, N' are mutually distinct then Fa, Fb, Fc, Fd constitute a tetrahedron, which one might call a *skew parallelogram*. It seems worth noting that—in analogy to a parallelogram in an affine plane over a field of characteristic two-also here the remaining two lines $Fa \oplus Fd$ and $Fb \oplus Fc$ are parallel to each other. The validity of the double space axiom implies the following result:

Proposition 3.5. *All parallel classes of the Clifford parallelism on* $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$ *are regular spreads.*

Proof. Let M, M_1, M_2 be mutually distinct parallel lines. So there is a unique regulus, say $\mathcal{R}$, containing them. Furthermore, there exists a line N in the opposite regulus of $\mathcal{R}$. Through each point of M there is a unique line $N' \parallel N$ and a unique line N'' of the opposite regulus of $\mathcal{R}$. By the double space axiom, N' meets M_1 and M_2 so that $N' = N''$. Consequently, the opposite regulus of $\mathcal{R}$ consists of mutually parallel lines. Applying the double space axiom once more yields that all lines of the regulus $\mathcal{R}$ are in the parallel class $\mathcal{S}(M)$. $\square$

Each line of the projective space $\mathbb{P}(L_F)$ has a unique parallel line, say K, through the point $F \cdot 1 = F$. Upon choosing any $i \in K \setminus F$, the line K takes the form $K = F \oplus Fi$. By our global assumption on the fields L and F from Section 2, we have $i^2 \in F$. So K is the *intermediate field* of F and L that arises from F by adjoining the element i. We denote the field K by $F[i]$ rather than $F(i)$ in order to avoid confusion with the subspace Fi of L_F. Conversely, any intermediate field K satisfying $F \subsetneqq K \subsetneqq L$ is a line through the point F, since $[L : K] = 4$ forces $[K : F] = 2$.

If K is an intermediate field as above then we may view $L_{(K)} := (K \otimes_F L)_K$ as a vector space over K which extends L_F. This vector space will usually not be treated as a structure in its own right, but as a substructure of $L_{(L)}$. Thereby we utilise that $L_{(L)}$ arises from $L_{(K)}$ (up to a canonical identification) by extending the ground field from K to L [9, pp. 278–279]. Those points of the projective space $\mathbb{P}(L_{(L)})$ that have at least one generating vector in $K \otimes_F L$ are named *K-rational*. A subspace of $L_{(L)}$ is called *K-rational* if it is spanned by its K-rational points. We are now in a position to describe the parallel class of the line $K = F[i]$ in terms of the absolute plane.

Theorem 3.6. *Let $i \in L \setminus F$. Then the following assertions hold:*

(i) *The absolute plane Π of $\mathbb{P}(L_{(L)}) \cong \mathbb{P}_3(L)$ contains a unique $F[i]$-rational line, namely the line joining the $F[i]$-rational point $L(1 \otimes i + i \otimes 1)$ with the absolute point $\mathcal{A}$.*

(ii) *The absolute point $\mathcal{A}$ is not $F[i]$-rational.*

(iii) *A line M of $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$ is Clifford parallel to the line $F[i] = F \oplus Fi$ if, and only if, the extended line $M_{(L)}$ meets the absolute plane Π at an $F[i]$-rational point.*

Proof. We extend $1, i$ to a basis $(1, i, j, k)$ of L_F as in (9) and introduce the associated basis $(1 \otimes 1, p, q, r)$ of $L_{(L)}$ from (11).

Ad (i). A point of Π is $F[i]$-rational precisely when it can be generated by a vector that belongs to the set $\big(F[i] \otimes_F L\big) \cap \Pi$. We claim that

$$\big(F[i] \otimes_F L\big) \cap \Pi = \{(1 \otimes y)p \mid y \in L\}. \tag{13}$$

Since p is in the ideal Π, so are all elements from the set on the right hand side of (13). According to (2), we have

$$(1 \otimes y)p = (1 \otimes y)(1 \otimes i + i \otimes 1) = 1 \otimes iy + i \otimes y \quad \text{for all } y \in L, \tag{14}$$

whence the right hand side of (13) is a subset of $F[i] \otimes_F L$. Conversely, for any g from the set on the left hand side of (13) there are a, b in L with $g = 1 \otimes a + i \otimes b$. Now $g \in \Pi = \ker \pi$ yields $a = ib$, and $g = (1 \otimes b)p$ follows as in (14). This verifies equation (13).

We infer from (13) that the four vectors

$$\begin{aligned}(1 \otimes 1)p &= 1 \otimes i + i \otimes 1 = p,\\ (1 \otimes i)p &= i^2 \otimes 1 + i \otimes i = ip\end{aligned}$$

$$(1 \otimes j)p = 1 \otimes k + i \otimes j = jp + r,$$
$$(1 \otimes k)p = i^2 \otimes j + i \otimes k = i(jp + r) \tag{15}$$

are all in $\big(F[i] \otimes_F L\big) \cap \Pi$. The first and the third vector from (15) are linearly independent over L, since p and r belong to the basis (11) of $L_{(L)}$. Writing $(y_0, y_1, y_2, y_3) \in F^4$ for the coordinates with respect to the basis (9) of an arbitrary $y \in L$ yields therefore

$$(1 \otimes y)p = (y_0 + iy_1)p + (y_2 + iy_3)(jp + r). \tag{16}$$

This shows that the $F[i]$-rational points of Π comprise an $F[i]$-subline of the line $Lp \oplus L(jp + r) = \mathcal{A} \oplus Lp$. So $\mathcal{A} \oplus Lp$ is the only $F[i]$-rational line in Π. Cf. also Figure 1 below.

Ad (ii). Clearly $r = jp + 1(jp + r)$. This is the only possibility to write r as a linear combination with coefficients in L of the (linearly independent) vectors p and $jp+r$. Thus $j \notin F[i]$ and (16) imply that there is no $y \in L$ such that $(1\otimes y)p$ is a non-zero vector of Lr. So the absolute point $\mathcal{A} = Lr$ is not $F[i]$-rational.

Ad (iii). Let M be a line of $\mathbb{P}(L_F)$. If $M \parallel F[i]$ then there is a $b \in L\setminus\{0\}$ with $M = F[i]\cdot b$. From $F[i] = F\oplus Fi$ follows $M = Fb\oplus F(ib)$. By Proposition 3.2 (iii), the extended line $M_{(L)}$ meets Π at the point $L(b\otimes ib + ib\otimes b)$. This point is $F[i]$-rational, because it can be rewritten as $L(1 \otimes ib + i \otimes b)$.

Conversely, suppose that $M_{(L)} \cap \Pi$ is an $F[i]$-rational point, say Lm. By (14), we may assume $m = 1 \otimes ib + i \otimes b$ for some $b \in L \setminus \{0\}$. Proposition 3.2 (iii) shows that the F-rational line $\big(F[i] \cdot b\big)_{(L)}$ passes through the point Lm. From Proposition 3.2 (ii), there is precisely one F-rational line through Lm. So we obtain $M = F[i] \cdot b$ or, said differently, $M \parallel F[i]$. □

Remark 3.7. The description of the parallel class $\mathcal{S}(F[i])$ from Theorem 3.6 can be found in the literature in various guises. It is a special case of the description of the spread that arises from the field extension $F[i]/F$ according to [16, Theorem 2]. (This spread in turn yields a pappian projective plane whose underlying field is isomorphic to the intermediate field $F[i]$.) Taking into account that $\mathcal{S}(F[i])$ is a regular spread, our result is covered by [4, Theorem 1.2]. See also [8] and [11] for related work. In order to fully establish the link with either of the cited articles, it is sufficient to consider the *intermediate projective space* $\mathbb{P}(L_{(F[i])}) \cong \mathbb{P}_3(F[i])$. This space contains the initial space $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$ as a *Baer subspace*. The $F[i]$-subline of $\mathcal{A} \oplus Lp$ mentioned in the proof above is an *indicator set* of the spread $\mathcal{S}(F[i])$. It constitutes the ‘visible’ part of the absolute plane within the intermediate projective space, whereas the absolute point remains entirely ‘invisible’.

The images of parallel classes under the Klein mapping are described in [15, Lemma 1]: These are elliptic quadrics (intersections of the Klein quadric by solids) with the following particular property: The tangent planes of any such quadric have a common line.

The essential role of the absolute pencil will come into effect in the next result, where we describe our Clifford parallelism in terms of the extended space $\mathbb{P}(L_{(L)}) \cong \mathbb{P}_3(L)$.

Theorem 3.8. *Let M and N be lines of the projective space $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$. Then M and N are Clifford parallel if, and only if, there exists a line of the absolute pencil $[\mathcal{A}, \Pi]$ that meets the extended lines $M_{(L)}$ and $N_{(L)}$.*

Proof. First, let us assume $M \parallel N$. So there exists an $i \in L \setminus F$ such that $M \parallel F[i] \parallel N$. By Theorem 3.6 (iii), the extended lines $M_{(L)}$ and $N_{(L)}$ meet the absolute plane Π at $F[i]$-rational points. Recall the notation $p = 1 \otimes i + i \otimes 1$ from (11). Since $\mathcal{A} \oplus Lp$ is the only $F[i]$-rational line in Π according to Theorem 3.6 (i), each of the $F[i]$-rational points $M_{(L)} \cap \Pi$ and $N_{(L)} \cap \Pi$ must be incident with the line $\mathcal{A} \oplus Lp \in [\mathcal{A}, \Pi]$.

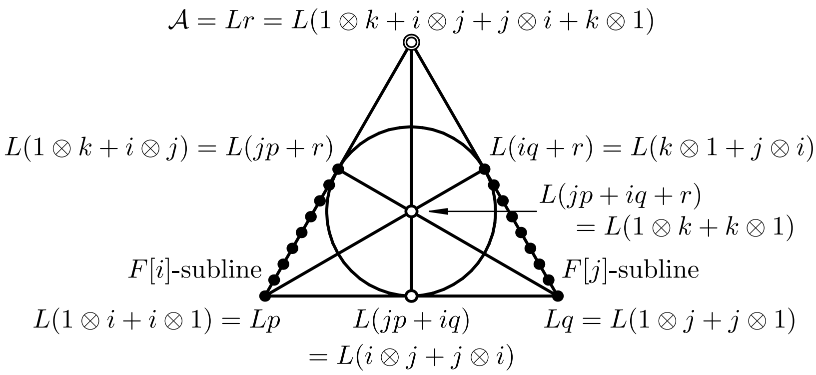

Figure 1. Sublines in the absolute plane Π.

Next, we assume $M \nparallel N$. So there are $i, j \in L \setminus F$ such that $M \parallel F[i]$ and $N \parallel F[j]$. From $F[i] \nparallel F[j]$ follows that $1, i, j$ are linearly independent over F. We extend these elements to a basis $(1, i, j, k)$ of L_F as in (9) and introduce then the basis $(1 \otimes 1, p, q, r)$ of $L_{(L)}$ from (11). By Theorem 3.6 (i), the point $M_{(L)} \cap \Pi$ belongs to the subline of $F[i]$-rational points of the line $\mathcal{A} \oplus Lp$. Moreover, from Theorem 3.6 (ii), the absolute point $\mathcal{A}$ is not $F[i]$-rational, whence $M_{(L)} \cap \Pi \neq \mathcal{A}$. Hence $\mathcal{A} \oplus Lp$ is the *only* line of the absolute pencil $[\mathcal{A}, \Pi]$ that meets $M_{(L)}$.

Exchanging M with N we obtain *mutatis mutandis*: $N_{(L)} \cap \Pi$ is an $F[j]$-rational point on the line $\mathcal{A} \oplus Lq$, the absolute point $\mathcal{A}$ is not $F[j]$-rational, and therefore $\mathcal{A} \oplus Lq$ is the *only* line of the absolute pencil $[\mathcal{A}, \Pi]$ that meets $N_{(L)}$. Since $Lp, Lq, \mathcal{A} = Lr$ are the vertices of a triangle, there is no line of the absolute pencil that meets simultaneously the extended lines $M_{(L)}$ and $N_{(L)}$. See Figure 1, where also two additional points are depicted in order to obtain a Fano subplane of the absolute plane Π. $\square$

The approach to the Clifford parallelism in Definition 3.4 makes use of the group of *Clifford translations*. These are projective collineations that arise from the multiplication maps μ_b as in (7), subject to the condition $b \neq 0$. Due to our global assumption from Section 2, the square of any Clifford translation is the identical collineation. If b has coordinates $(b_0, b_1, b_2, b_3) \in F^4$ with respect to an arbitrary basis $(1, i, j, k)$ as in (9) then the corresponding matrix of μ_b equals

$$\begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ i^2 b_1 & b_0 & i^2 b_3 & b_2 \\ j^2 b_2 & j^2 b_3 & b_0 & b_1 \\ k^2 b_3 & j^2 b_2 & i^2 b_1 & b_0 \end{pmatrix} \in \mathrm{GL}_4(F).$$

The structure of μ_b becomes more apparent from its extension $\mu_{1\otimes b}$ and by changing to the basis (11) which is associated to (9). The coordinates of $1 \otimes b$ with respect to (11) are

$$(\underbrace{b_0 + b_1 i + b_2 j + b_3 k}_{=b}, b_1 + b_3 j, b_2 + b_3 i, 1) =: (b_0', b_1', b_2', b_3') \in L^4,$$

and the matrix of $\mu_{1\otimes b}$ reads

$$\begin{pmatrix} b_0' & b_1' & b_2' & b_3' \\ 0 & b_0' & 0 & b_2' \\ 0 & 0 & b_0' & b_1' \\ 0 & 0 & 0 & b_0' \end{pmatrix} \in \mathrm{GL}_4(L). \tag{17}$$

The case when $b \neq 0$ is in F does not deserve our interest, since it gives the identical collineation. Otherwise, we may simplify matters by choosing w.l.o.g. the basis element i equal to the given $b \in L \setminus F$. As a consequence $b_1 = 1$ and $b_0 = b_2 = b_3 = 0$, which implies that the matrix from (17) turns into block diagonal form

$$\mathrm{diag}\left(\begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}, \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix} \right) \in \mathrm{GL}_4(L). \tag{18}$$

From (18) the following observations about the collineation arising from $\mu_{1\otimes i}$ are immediate: The fixed points comprise the line $\mathcal{A} \oplus Lp$. A plane is invariant precisely when it contains the line $\mathcal{A} \oplus Lp$. The restriction of the collineation to every invariant plane is an involutory (planar) elation. In particular, the restriction to the absolute plane Π has the absolute point $\mathcal{A} = Lr$ as its centre.

Remark 3.9. It is straightforward to show (e.g. in terms of Plücker coordinates or in terms of the geometric characterisation from [10, vol. II, p. 182]) that the invariant lines of the collineation given by (18) constitute a *parabolic linear congruence*. Furthermore, the F-rational lines of this congruence are exactly the extended lines of the parallel class $\mathcal{S}(F[i])$.

Our final aim is to link certain polarities with our Clifford parallelism. Let $\varphi : L \to F$ be an F-linear form. Then

$$\langle \cdot , \cdot \rangle_\varphi : L \times L \to F : (x, y) \mapsto (xy)^\varphi \tag{19}$$

is a symmetric F-bilinear form satisfying

$$\langle xb, yb\rangle_\varphi = (b^2xy)^\varphi = b^2(xy)^\varphi = b^2\langle x, y\rangle_\varphi \quad \text{for all } x, y, b \in L, \tag{20}$$

since $b^2 \in F$ holds due to our global assumption from Section 2. Letting $x = y = 1$ in (20) shows that the bilinear form $\langle \cdot , \cdot \rangle_\varphi$ is *alternating* for $1^\varphi = 0$. Likewise, the form turns out to be *anisotropic* for $1^\varphi \neq 0$.

From now on let us rule out the zero form $\varphi = 0$. Then there is a $c \in L\setminus\ker\varphi$, whence for any $a \in L \setminus \{0\}$ we obtain $\langle a, ca^{-1}\rangle_\varphi = c^\varphi \neq 0$. So $\langle \cdot , \cdot \rangle_\varphi$ is non-degenerate and determines a *projective polarity* $\perp_\varphi$ of $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$. By (20), all Clifford translations commute with this polarity, which is *null* for $1^\varphi = 0$ and *elliptic* (i.e., without self-conjugate points) otherwise. Our null polarities appear (in terms of a slightly different approach) in [13, p. 97]. It should also be noted that our elliptic polarities are *pseudo-polarities* according to the terminology used in [20] and [31].

Proposition 3.10. *For any non-zero F-linear form $\varphi : L \to F$ the associated polarity $\perp_\varphi$ of $\mathbb{P}(L_F) \cong \mathbb{P}_3(F)$ maps every line to a parallel one. If a line is fixed under $\perp_\varphi$ then so are all its parallel lines.*

Proof. Given a line M there is an $i \in L \setminus F$ and an $a \in L \setminus \{0\}$ with $M = F[i]\cdot a$. For all $y \in M^{\perp_\varphi} \setminus \{0\}$ we have $\langle M, y\rangle_\varphi = 0$ so that $M \parallel My \subset \ker\varphi$. Since $\mathcal{S}(M)$ is a spread, there cannot be two distinct lines parallel to M in the plane $\ker\varphi$. Hence, as y varies in the non-empty set $M^{\perp_\varphi} \setminus \{0\}$, the line My remains unchanged. So there is a constant $c \in L \setminus \{0\}$ such that $My = F[i] \cdot ay = F[i] \cdot c$ holds for all $y \in M^{\perp_\varphi}$. This implies $M^{\perp_\varphi} = F[i] \cdot a^{-1}c \parallel M$.

If $M = M^{\perp_\varphi}$ is satisfied for one line M then we obtain $Mb = M^{\perp_\varphi} \cdot b = (Mb)^{\perp_\varphi}$ for all $b \in L \setminus \{0\}$ by (20). So all lines from $\mathcal{S}(M)$ are fixed under $\perp_\varphi$. $\square$

Note that a self-polar line M, i.e., a line with the property $M = M^{\perp_\varphi}$, exists precisely when $\perp_\varphi$ is a null polarity. If this is the case then all such lines constitute a *general linear complex of lines*. On the other hand, any of the elliptic polarities $\perp_\varphi$ can be used to given an alternative definition of the Clifford parallelism [17, Remark 3].

Remark 3.11. Our Clifford parallelism is readily seen to be *cosymplectic* [2, Definition 3], i.e., any two distinct parallel classes $\mathcal{S}(M)$ and $\mathcal{S}(N)$ belong to a common general linear complex of lines. In order to establish this result, we may assume that both M and N are lines through the point $F \cdot 1 = F$, whence there is a non-zero F-linear form $\varphi : L \to F$ that vanishes on the plane $M + N$. The associated polarity $\perp_\varphi$ is null, and its self-polar lines comprise a linear complex which contains $\mathcal{S}(M) \cup \mathcal{S}(N)$. From this observation and from Proposition 3.5, our parallelism is also Clifford in the sense of [3, Definition 1.9].

All bilinear forms from (19) can be extended to symmetric L-bilinear forms $L_{(L)} \to L$. More generally, any L-linear form $\psi : L_{(L)} \to L$ defines a symmetric L-bilinear form in analogy to (19). Since $g^2 \in F(1 \otimes 1) \subset L(1 \otimes 1)$ holds for all elements $g \in L_{(L)}$, the analogue of (20) is satisfied too. However, such a bilinear form $\langle \cdot , \cdot \rangle_\psi$ can be degenerate for $\psi \neq 0$, and we leave a detailed exposition to the reader. From Lemma 2.2, the orthogonal subspace of the absolute point $\mathcal{A}$ contains the absolute plane Π for any choice of ψ. Therefore, when $\langle \cdot , \cdot \rangle_\psi$ is non-degenerate, the projective polarity $\perp_\psi$ will send the absolute point $\mathcal{A}$ to the absolute plane Π, and the polar planes of the points from Π will all contain the absolute point $\mathcal{A}$.

4. Future research

We are of the opinion that further investigation should prove worthwhile of those Clifford parallelisms that arise according to [25, Satz 1] from purely inseparable field extensions of degree greater than four. This task should not be confined to the finite-dimensional case. This was one motivation for avoiding, wherever possible, the use of coordinates in the present paper. It is striking that, according to the classification from [25], none of those Clifford parallelisms has an analogue when the characteristic of the ground field is different from two.

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