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Reconsidering a functional equation arising from number theory

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Abstract. In a recent paper Chávez and Sahoo considered the functional equation

$$f(ux - vy, uy + v(x + y)) = f(x, y)f(u, v),$$

which arose in a number theoretical context. Unfortunately one of their results is incorrect. Here we reconsider the equation on various domains. We observe that it is in fact a multiplicative Cauchy equation in disguise. We also point out some remaining open problems.

1. Introduction

In [2], Blecksmith and Brudno point out that the quadratic form $f(x, y) = x^2 + xy + y^2$ satisfies the functional equation

$$f(u, v)f(x, y) = f(ux - vy, uy + vx + vy), \quad u, v, x, y \in R, \tag{1}$$

where R is the ring of integers (or any commutative ring). They make use of this fact to explain certain interesting phenomena in number theory. Their main result is that there are integers with an arbitrarily large number of representations as sums of three fourth powers.

That paper inspired Chávez and Sahoo [3] to study the functional equation (1) for an unknown function f on $\mathbb{K} \times \mathbb{K}$, where $\mathbb{K}$ is the either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. In the complex case there is a small problem with their

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Theorem 2.3, in which f is a function from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The only solution of (1) in this case is $f = 1$, as can be seen by setting $u = v = 0$:

$$f(0,0)f(x,y) = f(0,0). \tag{2}$$

This problem can perhaps be overcome if the point $(0,0)$ is excluded from the domain of f (and from the domain of equation (1)). We provide another remedy in Section 2, in which we start with a fundamental lemma (Lemma 2.1) a bit different from the fundamental Theorem 2.1 in [3], while retaining their main idea. We apply Lemma 2.1 below in the complex case to get our first main result, Proposition 2.2. In fact the result holds not only for the field $\mathbb{C}$ but also for any field containing $\mathbb{Q}(\sqrt{-3}\,)$. We also derive the symmetric solutions of (1) in this case.

Section 3 deals with the real case. There is a serious flaw in the proof of Theorem 2.4 of [3]: The set $S = \{(z, \overline{z}) \mid z \in \mathbb{C}\}$ cannot be written as $H \times H$ for any commutative semigroup H, as is required for the application of Theorem 2.1 in [3]. In this case we take a whole new approach.

Let R be a commutative ring and let $x, y \in R$. Identifying the pair (x, y) with the matrix

$$\begin{pmatrix} x & -y \\ y & x+y \end{pmatrix},$$

we can rewrite equation (1) as

$$f(A)f(B) = f(AB)$$

for all matrices A, B from the set

$$M(R) = \left\{ \begin{pmatrix} x & -y \\ y & x+y \end{pmatrix} : x, y \in R \right\}.$$

That is, f satisfies a multiplicative Cauchy functional equation on $M(R)$.

Note that $M(R)$ is itself a commutative ring, and in particular we shall use the fact that it is a commutative monoid with respect to multiplication. Also, every matrix in $M(R)$ is a linear combination over R of the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, $M(R)$ is generated over the ground ring R by the matrix D. That is, $M(R) = R[D]$. Suppose R is a field of characteristic 0 such as $\mathbb{R}$ or $\mathbb{C}$. The characteristic polynomial of D is $X^2 - X + 1$, with roots $(1 \pm \sqrt{-3}\,)/2$. The characteristic polynomial splits into two linear factors if and only if $\sqrt{-3} \in R$, which sheds light on the importance of $\sqrt{-3}$ in the present context.

If $\sqrt{-3} \in R$, as when $R = \mathbb{C}$, then $R[D]$ is isomorphic as a ring to the product ring $R \times R$. This case is treated in Section 2.

If $\sqrt{-3} \notin R$, as when $R = \mathbb{R}$, then $R[D]$ is the quadratic field extension of R generated by D. In Section 3 we solve equation (1) on $M(\mathbb{R})$ and find more solutions in the real case than the ones given in [3].

Recall also that the determinant is a multiplicative function on any set of square matrices over a commutative ring. The determinant of a matrix in $M(R)$ is given by

$$\det \begin{pmatrix} x & -y \\ y & x+y \end{pmatrix} = x^2 + xy + y^2.$$

This provides another explanation for why the quadratic form $x^2 + xy + y^2$ is a solution of the functional equation (1).

2. The complex case

In this section our fundamental lemma is a modification of Theorem 2.1 in [3]. We remind the reader that a *monoid* T is a semigroup with identity, here designated by e. That is, there exists an element $e \in T$ such that $ex = xe = x$ for all $x \in T$.

In the sequel T_1 and T_2 are monoids with respective identity elements e_1 and e_2.

For any set A, let $A^2 = A \times A$. If $\alpha : A^2 \to T_1 \times T_2$, then we may write $\alpha = (\alpha_1, \alpha_2)$ where each component function α_i maps A^2 into T_i $(i = 1, 2)$. When we write $\alpha(x, y)\alpha(u, v)$ we are employing a componentwise multiplication, so that

$$\alpha(x, y)\alpha(u, v) = (\alpha_1(x, y)\alpha_1(u, v), \alpha_2(x, y)\alpha_2(u, v)).$$

If T, S are semigroups, then any solution $M : T \to S$ of Cauchy's multiplicative equation $M(ab) = M(a)M(b)$ is called a *semigroup morphism* or a *multiplicative function.*

Taking inspiration from Theorem 2.1 in [3], we prove the following.

Lemma 2.1. *Let A be a set, S be a commutative semigroup, T_1 and T_2 be monoids, and $\phi : A^4 \to A^2$. Suppose there is a bijection $\alpha = (\alpha_1, \alpha_2) : A^2 \to$*

$T_1 \times T_2$ such that

$$\alpha(\phi(x,y,u,v)) = \alpha(x,y)\alpha(u,v), \quad x,y,u,v \in A. \tag{3}$$

Then the general solution $f : A^2 \to S$ of

$$f(\phi(x,y,u,v)) = f(x,y)f(u,v), \quad x,y,u,v \in A, \tag{4}$$

is given by

$$f(x,y) = M_1(\alpha_1(x,y))M_2(\alpha_2(x,y)), \quad x,y \in A,$$

where $M_i : T_i \to S$ is a semigroup morphism ($i = 1, 2$).

Proof. Beginning as in [3], define $g : T_1 \times T_2 \to S$ by $g = f \circ \alpha^{-1}$. Then $f = g \circ \alpha$ and (4) becomes

$$g(\alpha(\phi(x,y,u,v))) = g(\alpha(x,y))g(\alpha(u,v)), \quad x,y,u,v \in A.$$

Now (3) yields

$$g(\alpha(x,y)\alpha(u,v)) = g(\alpha(x,y))g(\alpha(u,v)), \quad x,y,u,v \in A,$$

and since α is surjective, with $\alpha(x,y) = (p,q), \alpha(u,v) = (r,s)$ we have

$$g(pr,qs) = g(p,q)g(r,s), \quad (p,q),(r,s) \in T_1 \times T_2. \tag{5}$$

Putting $r = e_1$,$q = e_2$ here, we get $g(p,s) = g(p,e_2)g(e_1,s)$, that is

$$g(p,s) = M_1(p)M_2(s), \quad (p,s) \in T_1 \times T_2,$$

where $M_1(p) = g(p,e_2)$ and $M_2(s) = g(e_1,s)$. Furthermore, with $p = r = e_1$, resp. $q = s = e_2$ in (5) we find that M_2 and M_1 are multiplicative functions (i.e. semigroup morphisms). Finally, $f = g \circ \alpha$ yields

$$f(x,y) = g(\alpha_1(x,y),\alpha_2(x,y)) = M_1(\alpha_1(x,y))M_2(\alpha_2(x,y)).$$

We leave the verification of the converse to the reader. □

Now we use this lemma to prove a somewhat stronger result than Theorem 2.3 in [3]. Let $\mathbb{Q}(i\sqrt{3}\,)$ be the extension field of the rationals generated by adjoining $i\sqrt{3}$.

Proposition 2.2. *Let K be any field containing $\mathbb{Q}(i\sqrt{3}\,)$, and let S be a commutative semigroup. Then the general solution $f: K^2 \to S$ of (1) is given by*

$$f(x,y) = M_1\left(x + \frac{1+i\sqrt{3}}{2}y\right) M_2\left(x + \frac{1-i\sqrt{3}}{2}y\right), \tag{6}$$

where M_1, M_2 are semigroup morphisms from $(K,\cdot)$ to S.

Proof. The proof is essentially the same as in [3]. Let $A = K$ and $T_1 = T_2 = (K,\cdot)$. Define $\alpha : K^2 \to K^2$ by

$$\alpha(x,y) = \left(x + \frac{1+i\sqrt{3}}{2}y, x + \frac{1-i\sqrt{3}}{2}y\right) \tag{7}$$

and $\phi : K^4 \to K^2$ by

$$\phi(x,y,u,v) = (ux - vy, uy + vx + vy)$$

for all $x, y, u, v \in K$. It is easy to check that α is a bijection and satisfies equation (3). The rest follows from Proposition 2.1. □

Note that the product of the two arguments of the multiplicative functions M_1 and M_2 in the preceding Proposition is exactly the quadratic form $x^2 + xy + y^2$. Yet since the two multiplicative functions may differ, in general we cannot write $f(x,y)$ as a function of this quadratic form. Next we show that imposing a symmetry condition on f forces $M_1 = M_2$ and therefore f becomes a multiplicative function of the quadratic form. In order to prove this, we make some preliminary calculations.

We observe that the range of the map $z \mapsto \frac{(1+i\sqrt{3}\,)z^2+4z+(1-i\sqrt{3}\,)}{2(z^2+z+1)}$ for $z \in \mathbb{Q}(i\sqrt{3}\,)$ is all of $\mathbb{Q}(i\sqrt{3}\,)$. Indeed, $w \in \mathbb{Q}(i\sqrt{3}\,)$ is in the range if and only if $(1 + i\sqrt{3}\,)z^2 + 4z + (1 - i\sqrt{3}\,) = 2w(z^2 + z + 1)$, which can be written as

$$0 = (2w - 1 - i\sqrt{3}\,)z^2 + (2w - 4)z + (2w - 1 + i\sqrt{3}\,).$$

If $2w = 1 + i\sqrt{3}$, then the resulting linear equation has solution $z = \frac{-1+i\sqrt{3}}{2} \in \mathbb{Q}(i\sqrt{3}\,)$. If not, then the quadratic equation has discriminant $-12w^2 = (i2\sqrt{3}w)^2$ and therefore has a solution $z \in \mathbb{Q}(i\sqrt{3}\,)$. This fact will be used in the proof of the next result.

We also recall a basic fact about multiplicative functions. Suppose S is a monoid, D is an integral domain, and $M : S \to D$ is multiplicative. Putting $t = 1$ in $M(st) = M(s)M(t)$ we have $M(s)[1 - M(1)] = 0$ for all $s \in S$. Thus $M(1) = 1$ unless $M = 0$.

Corollary 2.3. *Let K be any field containing $\mathbb{Q}(i\sqrt{3}\,)$, and let D be an integral domain. The function $f : K^2 \to D$ is a solution of (1) and symmetric, that is $f(x,y) = f(y,x)$, if and only if*

$$f(x,y) = M(x^2 + xy + y^2), \tag{8}$$

where $M : K \to D$ is a multiplicative function.

Proof. Suppose f is a symmetric solution of (1). By the previous proposition with $S = (D, \cdot)$ we have the representation (6) for f, where $M_1, M_2 : K \to D$ are multiplicative maps. If $M_1 = 0$ or $M_2 = 0$, then equation (8) holds with $M = 0$. Henceforth we assume that neither M_1 nor M_2 is the zero function, so we have $M_1(1) = M_2(1) = 1$.

Now the symmetry of f gives

$$M_1\left(x + \frac{1+i\sqrt{3}}{2}y\right) M_2\left(x + \frac{1-i\sqrt{3}}{2}y\right) = M_1\left(y + \frac{1+i\sqrt{3}}{2}x\right) M_2\left(y + \frac{1-i\sqrt{3}}{2}x\right). \tag{9}$$

Putting $x = 1$ here and multiplying both sides by $M_1((y + (1 + i\sqrt{3}\,)/2)^{-1})\times M_2((1 + (1 - i\sqrt{3}\,)y/2)^{-1})$, we arrive at

$$M_1\left(\frac{(1+i\sqrt{3}\,)y^2 + 4y + (1-i\sqrt{3}\,)}{2(y^2+y+1)}\right) = M_2\left(\frac{(1+i\sqrt{3}\,)y^2 + 4y + (1-i\sqrt{3}\,)}{2(y^2+y+1)}\right)$$

for all $y \in K$. Thus by our preliminary computations this means that $M_1(w) = M_2(w)$ for all $w \in K$, hence $M_1 = M_2$. Putting $M = M_1 = M_2$ in (6), we get (8).

The converse is easily verified. □

Remark 2.4. Note that the integral domain D could be replaced in the previous corollary by $T \cup \{0\}$ where T is a cancellative monoid.

3. The real case: Reframing in terms of matrices

As observed in the introduction, representing (x,y) by a matrix allows us to rewrite our functional equation (1) as the multiplicative Cauchy equation

$$f(A)f(B) = f(AB), \tag{10}$$

In this section we assume that $R = \mathbb{R}$ (the field of real numbers), so A, B in the functional equation above belong to

$$M(\mathbb{R}) = \left\{ \begin{pmatrix} x & -y \\ y & x+y \end{pmatrix} : x, y \in \mathbb{R} \right\},$$

which is a commutative monoid under multiplication.

It will be convenient to introduce some notation here. We define

$$\Delta = \Delta(x, y) := x^2 + xy + y^2, \quad x, y \in \mathbb{R},$$

or equivalently

$$\Delta = \Delta(A) := \det A, \quad A \in M(\mathbb{R}).$$

Observe that

$$\Delta(x, y) = \left(x + \frac{y}{2}\right)^2 + \left(\frac{\sqrt{3}y}{2}\right)^2 \geq 0, \quad x, y \in \mathbb{R}.$$

That is $\Delta \geq 0$, with equality if and only if $x = y = 0$ (that is, if A is the zero matrix O). Let $M(\mathbb{R})^* = M(\mathbb{R}) \setminus \{O\}$.

Also, if $\Delta(x, y) = 1$, then the point $\left(x + \frac{y}{2}, \frac{\sqrt{3}y}{2}\right)$ is on the unit circle C in $\mathbb{R}^2$. In such a case we define $\theta = \theta(x, y)$ to be a real number for which

$$(\cos\theta, \sin\theta) = \left(x + \frac{y}{2}, \frac{\sqrt{3}y}{2}\right).$$

The value of θ is determined only up to an integer multiple of 2π. We can write equivalently $\theta = \theta(A)$ where $A \in M(\mathbb{R})$ has determinant equal to 1.

Proposition 3.1. *Let T be a commutative monoid (with operation denoted multiplicatively), and suppose the map $f : M(\mathbb{R}) \to T$ is a solution of equation* (10), *i.e. a multiplicative function. Then there exist a multiplicative function $g : (0, \infty) \to T$ and a 2π-periodic function $h : \mathbb{R} \to T$ satisfying the exponential functional equation*

$$h(s + t) = h(s)h(t), \quad s, t \in \mathbb{R},$$

such that

$$f(A) = g(\sqrt{\Delta})h\left(\theta\left(\frac{1}{\sqrt{\Delta}}A\right)\right) \tag{11}$$

for all $A \in M(\mathbb{R})$ with $\Delta > 0$.

PROOF. First, let U be the subgroup of $(M(\mathbb{R})^*, \cdot)$ consisting of the matrices of determinant 1:

$$U := \{A \in M(\mathbb{R}) : \det(A) = 1\}.$$

For each matrix $A = \left(\begin{smallmatrix} x & -y \\ y & x+y \end{smallmatrix}\right)$ in U, there exists a real number $\theta = \theta(A)$ such that $(\cos\theta, \sin\theta) = (x + y/2, \sqrt{3}y/2)$, since $(x + y/2)^2 + (\sqrt{3}y/2)^2 = 1$. The value of θ is determined up to an integer multiple of 2π. Hence we have

$$A = \begin{pmatrix} x & -y \\ y & x+y \end{pmatrix} = \begin{pmatrix} \cos\theta - \frac{\sin\theta}{\sqrt{3}} & -\frac{2\sin\theta}{\sqrt{3}} \\ \frac{2\sin\theta}{\sqrt{3}} & \cos\theta + \frac{\sin\theta}{\sqrt{3}} \end{pmatrix}.$$

Now we define a function $h : \mathbb{R} \to T$ by

$$h(\theta) := f\begin{pmatrix} \cos\theta - \frac{\sin\theta}{\sqrt{3}} & -\frac{2\sin\theta}{\sqrt{3}} \\ \frac{2\sin\theta}{\sqrt{3}} & \cos\theta + \frac{\sin\theta}{\sqrt{3}} \end{pmatrix}, \quad \theta \in \mathbb{R},$$

where h is periodic with period 2π. Then a straightforward calculation verifies that h satisfies the exponential functional equation as claimed.

Second, let D be the positive diagonal subsemigroup of $(M(\mathbb{R})^*, \cdot)$ defined by

$$D := \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x > 0 \right\},$$

and define $g : (0, \infty) \to T$ by

$$g(x) := f\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad x > 0.$$

Then another easy computation confirms that g is a multiplicative function, as claimed.

Finally, note that we can factor any matrix $A \in M(\mathbb{R})^*$ uniquely as

$$A = (\sqrt{\Delta} I)\left(\frac{1}{\sqrt{\Delta}} A\right).$$

Thus, by equation (10) we have

$$f(A) = f(\sqrt{\Delta} I) f\left(\frac{1}{\sqrt{\Delta}} A\right) = g(\sqrt{\Delta}) h\left(\theta\left(\frac{1}{\sqrt{\Delta}} A\right)\right), \quad A \in M(\mathbb{R})^*,$$

which is (11). The specification of a choice of θ is immaterial here, since h is 2π-periodic.

This completes the proof. $\square$

Returning to the original setting of equation (1), we can write the solution in the form

$$f(x,y) = g(\sqrt{x^2+xy+y^2})h(\theta),$$

for all $(x,y) \neq (0,0)$, where θ is determined (up to an integer multiple of 2π) by

$$(\cos\theta, \sin\theta) = \left(\frac{2x+y}{2\sqrt{x^2+xy+y^2}}, \frac{\sqrt{3}y}{2\sqrt{x^2+xy+y^2}} \right). \tag{12}$$

The g factor is the form of solution found in [3], since $g \circ \sqrt{\cdot}$ is multiplicative. But we will show that the h factor cannot be neglected.

In case T is the commutative monoid $(\mathbb{R}, \cdot)$ or $(\mathbb{C}, \cdot)$, the general forms of g and h in the previous proposition are well-known (see for example [1] or [4]). We will concentrate here on the real solutions, that is when $T = \mathbb{R}$.

Let us assume that our solution $f : \mathbb{R}^2 \to \mathbb{R}$ is not identically 0 for $(x,y) \neq (0,0)$. Then neither is $h : \mathbb{R} \to \mathbb{R}$ identically 0. Under such conditions, any solution h of the exponential functional equation is never 0 and we have

$$h(\theta) = e^{a(\theta)}, \quad \theta \in \mathbb{R},$$

where $a : \mathbb{R} \to \mathbb{R}$ is an *additive function*, that is a solution of Cauchy's equation

$$a(x+y) = a(x) + a(y), \quad x, y \in \mathbb{R}.$$

The fact that h is 2π-periodic just means that a is also. It is well known that there exist nontrivial (i.e. not identically 0) additive functions on $\mathbb{R}$ such that $a(2\pi) = 0$. If a is such a function, then h is not identically 1 and therefore cannot be neglected in the solution of (1).

In summary, we have the following corollary to the preceding proposition.

Corollary 3.2. *The general not identically* 1 *solution* $f : \mathbb{R}^2 \to \mathbb{R}$ *of equation* (1) *is given by* $f(0,0) = 0$ *and for* $(x,y) \neq (0,0)$ *by*

$$f(x,y) = m(x^2+xy+y^2)e^{a(\theta)} \tag{13}$$

for an arbitrary multiplicative function $m : (0,\infty) \to \mathbb{R}$ *and an arbitrary additive function* $a : \mathbb{R} \to \mathbb{R}$ *such that* $a(2\pi) = 0$. *Here* θ *is determined by equation* (12).

Proof. We have already seen how the solution form (13) arises, and why $f(0,0) = 0$ if $f \neq 1$. It only remains to check that (13) is in fact a solution of (1).

It is also easy to check that $(x,y) \mapsto m(x^2+xy+y^2)$ is a solution of (1) for any multiplicative function m.

Now consider the factor $h(x,y)=e^{a(\theta)}$. Suppose $(x,y),(u,v)\in\mathbb{R}^2\setminus\{(0,0)\}$. Let θ be a real number such that (12) holds, and let ϕ be a real number for which

$$(\cos\phi,\sin\phi)=\left(\frac{2u+v}{2\sqrt{u^2+uv+v^2}},\frac{\sqrt{3}v}{2\sqrt{u^2+uv+v^2}}\right).$$

Then $h(x,y)h(u,v)=e^{a(\theta)}e^{a(\phi)}=e^{a(\theta+\phi)}$. On the other hand,

$$\begin{aligned}
(\cos(\theta+\phi),\sin(\theta+\phi))&=(\cos\theta\cos\phi-\sin\theta\sin\phi,\sin\theta\cos\phi+\cos\theta\sin\phi)\\
&=\left(\frac{(2x+y)(2u+v)-3yv}{4\sqrt{(x^2+xy+y^2)(u^2+uv+v^2)}},\frac{\sqrt{3}[y(2u+v)+(2x+y)v]}{4\sqrt{(x^2+xy+y^2)(u^2+uv+v^2)}}\right)\\
&=\left(\frac{2xu+xv+yu-yv}{2\sqrt{(x^2+xy+y^2)(u^2+uv+v^2)}},\frac{\sqrt{3}(yu+yv+xv)}{2\sqrt{(x^2+xy+y^2)(u^2+uv+v^2)}}\right)\\
&=\left(\frac{2z+w}{\sqrt{z^2+zw+w^2}},\frac{\sqrt{3}w}{2\sqrt{z^2+zw+w^2}}\right)
\end{aligned}$$

where $(z,w)=(xu-yv,yu+xv+yv)$. Thus $e^{a(\theta+\phi)}=h(xu-yv,yu+xv+yv)$, confirming that h is indeed a solution of equation (1).

This concludes the proof. □

Now that we know there are solutions of (1) other than multiplicative functions of the quadratic form x^2+xy+y^2, we may ask what additional conditions will "kill" the h factor. It turns out that either continuity or symmetry will do the trick.

In the next result we omit the constant solutions $f=0$ and $f=1$ of (1).

Corollary 3.3. *The general continuous and nonconstant solution $f:\mathbb{R}^2\to\mathbb{R}$ of equation* (1) *is given by $f(0,0)=0$ and for $(x,y)\neq(0,0)$ by*

$$f(x,y)=(x^2+xy+y^2)^p \tag{14}$$

for an arbitrary $p>0$.

Proof. By the previous corollary, we have $f(0,0)=0$ and otherwise

$$f(x,y)=m(x^2+xy+y^2)e^{a(\theta)}$$

for a multiplicative $m:(0,\infty)\to\mathbb{R}$ and an additive $a:\mathbb{R}\to\mathbb{R}$ such that $a(2\pi)=0$. First we show that $a=0$. Indeed, choosing (x,y) such that $\Delta=1$ we have

$$f(x,y)=m(1)e^{a(\theta)}.$$

Since f is nonconstant, it follows that m is not the zero function and thus $m(1)\neq 0$. Therefore the continuity of f transfers to a. Every continuous additive function has the form $a(t) = ct$ for some constant c. Now $a(2\pi) = 0$ gives $c = 0$, hence a is the zero function.

Thus, the representation (13) reduces to

$$f(x, y) = m(x^2 + xy + y^2), \quad x, y \in \mathbb{R},$$

for a continuous multiplicative function $m : (0, \infty) \to \mathbb{R}$. The form of such a function m is either $m(x) = x^p$ for some constant $p \in \mathbb{R}$, or $m = 0$. Again $m = 0$ is ruled out by the non-constancy of f. Furthermore, the continuity of f at $(0, 0)$ forces $p > 0$, and this completes the proof. □

Remark 3.4. Observe that if we do not require continuity of f at $(0, 0)$ in the previous result, then we get (14) for any real constant p.

In the following corollary, we again exclude the constant functions $f = 0$ and $f = 1$, which are clearly also symmetric solutions of (1).

Corollary 3.5. *The general symmetric and nonconstant solution $f : \mathbb{R}^2 \to \mathbb{R}$ of equation* (1) *is given by $f(0, 0) = 0$ and for $(x, y) \neq (0, 0)$ by*

$$f(x, y) = m(x^2 + xy + y^2)$$

for an arbitrary (not identically 0*) multiplicative function $m : (0, \infty) \to \mathbb{R}$.*

Proof. By Corollary 3.2 we have that $f(0, 0) = 0$ and for $(x, y) \neq (0, 0)$ that $f(x, y)$ has the form

$$f(x, y) = m(x^2 + xy + y^2)e^{a(\theta)}$$

for arbitrary multiplicative m and additive a such that $a(2\pi) = 0$. Moreover, $m \neq 0$ since f is nonconstant. Clearly, the m factor of this representation is symmetric in x and y. Furthermore, as above we note that $m \neq 0$ implies that m never takes the value zero. It follows that the e^a factor must also be symmetric with respect to a switch of x and y. We will complete the proof by showing that this forces $a = 0$.

Recall that $\theta = \theta(x, y)$ is determined by equation (12):

$$(\cos\theta, \sin\theta) = \left(\frac{2x + y}{2}, \frac{\sqrt{3}y}{2}\right),$$

where we have taken $\Delta = x^2 + xy + y^2 = 1$ for convenience. Now symmetry of f yields $a(\theta(x, y)) = a(\theta(y, x))$. Let us restrict the point (x, y) to the first quadrant: $x, y \geq 0$. Then $\Delta = 1$ implies that

$$y = -\frac{x}{2} + \sqrt{1 - \frac{3x^2}{4}},$$

and y remains nonnegative as long as $x \leq 1$. Now for $x, y \in [0, 1]$ we can write

$$\theta(x, y) = \arcsin\left(\frac{\sqrt{3}y}{2}\right), \quad \theta(y, x) = \arcsin\left(\frac{\sqrt{3}x}{2}\right)$$

Then we have

$$\begin{aligned}
0 &= a(\theta(x, y)) - a(\theta(y, x)) \\
&= a\left(\arcsin\left(\frac{\sqrt{3}y}{2}\right)\right) - a\left(\arcsin\left(\frac{\sqrt{3}x}{2}\right)\right) \\
&= a\left(\arcsin\left(\frac{\sqrt{3}}{2}\left(-\frac{x}{2} + \sqrt{1 - \frac{3x^2}{4}}\right)\right)\right) - a\left(\arcsin\left(\frac{\sqrt{3}x}{2}\right)\right) \\
&= a\left(\arcsin\left(-\frac{t}{2} + \frac{\sqrt{3}}{2}\sqrt{1 - t^2}\right) - \arcsin(t)\right),
\end{aligned}$$

where $t = \frac{\sqrt{3}x}{2}$ runs through the interval $0 \leq t \leq \sqrt{3}/2$. Letting $\phi(t)$ denote the argument of a in the last displayed line above, we see that ϕ is continuous on $[0, \sqrt{3}/2]$, that $\phi(0) = \pi/3$, and that $\phi(\sqrt{3}/2) = -\pi/3$. Therefore the range of ϕ contains the interval $[-\pi/3, \pi/3]$. Since $a(u) = 0$ for all u in this interval, it follows that a is bounded on an interval of positive length, thus it is a linear function and indeed the zero function. $\square$

4. Some remarks and open problems

We have found the solutions of (1) for functions f mapping $\mathbb{Q}(i\sqrt{3}\,)^2$ into an integral domain, and for functions from $\mathbb{R}^2$ to $\mathbb{R}$. In view of the origins of our functional equation in number theory, it would be interesting to know the solutions of this functional equation when both the domain and range are integral domains like the integers or the Gaussian integers rather than fields.

In fact, the method of Proposition 2.2 "almost" works on the ring $R = \mathbb{Z}\left[\frac{1}{2}, i\sqrt{3}\right]$ generated by the integers, $\frac{1}{2}$, and $i\sqrt{3}$. The snag is that the function

α defined by (7) has range $\{(p,q) \in R^2 \mid \Re(p) - \Re(q) = 3r \text{ for some } r \in \mathbb{Z}[\frac{1}{2}]\}$, where the operator $\Re$ is defined on R by $\Re\big(\frac{a}{2^k} + i\sqrt{3}\frac{b}{2^n}\big) = \frac{a}{2^k}$ for $a, b, n, k \in \mathbb{Z}$. So the range of α is not of the form $T_1 \times T_2$.

Problem 4.1. *Find the solutions (or even the symmetric solutions)* $f : R^2 \to R$ *of equation* (1) *when* $R = \mathbb{Z}$ *or* $R = \mathbb{Z}[i]$.

It seems that a different approach is needed to solve these problems.

Problem 4.2. *Another open problem is to find the solutions (or even the symmetric solutions)* $f : \mathbb{Q}^2 \to \mathbb{Q}$ *of equation* (1).

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