Publ. Math. Debrecen
86/1-2 (2015), 169–182
DOI: 10.5486/PMD.2015.7043

Dual multiplier Banach algebras and Connes-amenability

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Abstract. Following M. Daws, we consider a Banach algebra B for which the multiplier algebra $M(B)$ is a dual Banach algebra in the sense of V. Runde, and show that under certain continuity condition, B is amenable if and only if $M(B)$ is Connes-amenable. As a result, we conclude that for a discrete amenable group G, the Fourier–Stieltjes algebra $B(G)$ is Connes-amenable if and only if G is abelian by finite.

1. Introduction

In 1924, based on earlier work of Giuseppe Vitali, Stefan Banach and Alfred Tarski [1] proposed a "paradoxical decomposition": there exists a decomposition of a ball in $\mathbb{R}^3$ into a finite number of disjoint subsets, which can be put back together to yield two identical copies of the original ball. Later John von Neumann showed that such a paradoxical decomposition is impossible for the disk in $\mathbb{R}^2$ (using only Euclidean congruences) and studied the possibility of paradoxical decomposition for more general groups of equivalences. In the course of investigation on Banach–Tarski phenomenon, von Neumann discovered the notion of amenable groups (groups with an invariant mean) and showed that paradoxical decomposition arises only for non-amenable groups of equivalences [24]. The original treatment of von Neumann was only for the discrete case, and it was Mahlon Day who extended the work of von Neumann to locally compact groups [3].

Mathematics Subject Classification: 46H20.
Key words and phrases: amenability, Connes-amenability, dual multiplier algebra, normal virtual diagonal.
The second author was partly supported by a grant of IPM (No. 91430215).

Later it was shown that the notion of amenability could be rephrased in terms of homological algebra, whose origins traces back to investigations in combinatorial topology (later called algebraic topology) and abstract algebra. This is mainly done by forming appropriate chain complexes and studying their homology and cohomology. In 1972, Barry E. Johnson put everything in a solid framework. Johnson defined amenability of Banach algebras and showed that a locally compact group G is amenable if and only if the group algebra $L^1(G)$ is amenable as a Banach algebra [12]. Investigating the notion of amenability for operator algebras, it was observed by Alain Connes and Uffe Haagerup that the Johnson notion of amenability for Banach algebras fits well to the subcategory of C^*-algebras. For von Neumann algebras, it was already observed by Johnson, in a joint paper with Richard V. Kadison and John V. Ringrose, that it is more suitable to take into account the w^*-topology of the algebra induced by its (unique) predual [15]. This new notion of amenability was called Connes-amenability by A. Ya. Helemskii [9].

It is quite natural to ask if the concept of Connes-amenability could be defined for Banach algebras with a predual. Before one can do this, there are few technical problems which should be taken care of: the predual is not unique (hence one predual should be fixed) and the corresponding module maps are not necessarily w^*-continuous. This led Volker Runde to define Connes-amenability for a suitable subclass, called dual Banach algebras [19]. Examples of dual Banach algebras include, von Neumann algebras, measure algebras and Fourier–Stieltjes algebras.

In the original work of Johnson, it was shown that amenability of a Banach algebra is equivalent to the existence of an appropriate element in the second dual of the projective tensor product of the algebra by itself. Johnson called this a virtual diagonal and also suggested an approximate analog for the diagonal [11]. Runde added a normality condition and used the idea due to Edward G. Effros [5] (for von Neumann algebras) to show that a dual Banach algebra with a normal virtual diagonal is Connes-amenable, and noticed that the converse is not true in general [20], [21], [23].

The idea of the present paper originated from the simple observation that the measure algebra $M(G)$ is the multiplier algebra of the group algebra $L^1(G)$. It was natural to ask if, in general, amenability of a Banach algebra B (with bounded approximate identity) is related to Connes-amenability of its multiplier algebra $M(B)$. For this to make sense, one has to find suitable conditions under which $M(B)$ is a dual Banach space. These conditions where stated in a (not so well known) paper by Emmanuel O. Oshobi and John S. Pym [18]. Although this is not explicitly stated in [18], it is easy to see that under natural conditions on B set by the authors, the multiplier algebra $M(B)$ is a dual Banach algebra [18, Theorems 3.4 and 3.8].

In [10], the authors proved that under the natural conditions set by [18], amenability of a Banach algebra B is equivalent to Connes-amenability of the multiplier algebra $M(B)$, as well as to the existence of a normal virtual diagonal. The proof was straightforward and relatively short. Part of the conditions set by [18] is a natural compatibility condition (see Assumption 2.1) which holds for $B = L^1(G)$ when G is compact and for $B = A(G)$ when G is discrete.

The present paper started following an observation by MATTHEW DAWS in [4] who set conditions, under which $M(B)$ is a dual Banach algebra. We noticed that adding a natural continuity condition (Assumption 3.4) our proof in [10] carries over this more general setting and for a Banach algebra B with bounded approximate identity, amenability of B, Connes-amenability of $M(B)$, and existence of a normal virtual diagonal for $M(B)$ are equivalent (Theorem 3.5).

2. Preliminaries

For a Banach algebra B and a Banach B-bimodule E, a continuous linear map $D : B \to E$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in B)$$

is called a *derivation* from B into E. The space of all derivations of B into E is denoted by $\mathfrak{Z}^1(B, E)$. For each $x \in E$, the map $a \mapsto a \cdot x - x \cdot a$ is a derivation, and these maps form the space $\mathfrak{N}^1(B, E)$ of *inner* derivations. The quotient space $\mathfrak{H}^1(B, E) = \mathfrak{Z}^1(B, E)/\mathfrak{N}^1(B, E)$ is the first cohomology group of B with coefficients in E, and B is called *amenable* if $\mathfrak{H}^1(B, E^*) = \{0\}$, for every Banach B-bimodule E [11], [12], where E^* is the dual Banach B-bimodule whose actions are defined by

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \quad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle \quad (a \in B,\ x \in E,\ x^* \in E^*).$$

A Banach algebra B is called a *dual Banach algebra* if it is dual as a Banach B-bimodule. One can see that a Banach algebra which is also a dual space is a dual Banach algebra if and only if the multiplication map is separately w^*-continuous [19]. Examples of dual Banach algebras include all von Neumann algebras, the algebra $B(E) = (E \hat{\otimes} E^*)^*$ of all bounded operators on a reflexive

Banach space E, the measure algebra $M(G) = C_0(G)^*$, the Fourier–Stieljes algebra $B(G) = C^*(G)^*$, and the second dual B^{**} of an Arens regular Banach algebra B.

Let B be a Banach algebra. We use the notations $B\hat{\otimes}B$ and $\mathfrak{L}^2(B;\mathbb{C})$ to denote the projective tensor product of B with itself and the space of bounded bilinear forms on $B \times B$, respectively. A dual Banach B-bimodule E is called *normal* if for each $x \in E$, the maps $a \mapsto a \cdot x$ and $b \mapsto x \cdot b$ from B into E are w^*-continuous, and *Connes-amenable* if for every normal dual Banach B-bimodule E, every w^*-continuous derivation $D : B \to E$ is inner [19]. If $\Delta_B : B\hat{\otimes}B \to B$ is the diagonal operator induced by $a \otimes b \mapsto ab$, $a, b \in B$, then since the multiplication in B is separately w^*-continuous, $\Delta_B^* B_* \subseteq \mathfrak{L}^2_{w^*}(B;\mathbb{C})$, where B_* is a closed submodule of B^* such that $B = B_*^*$, and $\mathfrak{L}^2_{w^*}(B;\mathbb{C})$ is the set of all separately w^*-continuous elements of $\mathfrak{L}^2(B;\mathbb{C}) \cong (B\hat{\otimes}B)^*$. Also, $\mathfrak{L}^2_{w^*}(B;\mathbb{C})$ is a closed submodule of $\mathfrak{L}^2(B;\mathbb{C})$; in particular, $\mathfrak{L}^2_{w^*}(B;\mathbb{C})^*$ carries a canonical Banach B-bimodule structure which makes it a quotient module of $(B\hat{\otimes}B)^{**}$. Taking the adjoint of $\Delta_B^*|_{B_*}$, we may lift Δ_B to a B-bimodule homomorphism Δ_{w^*} on $\mathfrak{L}^2_{w^*}(B;\mathbb{C})^*$. An element $M \in \mathfrak{L}^2_{w^*}(B;\mathbb{C})^*$ is called a *normal, virtual diagonal* for B if

$$a \cdot M = M \cdot a, \; a\Delta_{w^*}M = a \quad (a \in B).$$

A *double multiplier* τ on an algebra B consists of a pair of mappings of B into itself denoted by $b \mapsto b\tau$ and $b \mapsto \tau b$ $(b \in B)$ such that $(a\tau)b = a(\tau b)$, for all $a, b \in B$. Every element of B by the natural map, $b \mapsto (L_b, R_b)$, gives rise to a double multiplier where L_b and R_b are left and right multiplications respectively.

The algebra B is called *faithful* if the only element $b \in B$ such that $abc = 0$, for all $a, c \in B$, is $b = 0$. The set $M(B)$ of all double multipliers on B is an algebra (under composition of maps) with identity $1_{M(B)} = (\mathrm{id}_B, \mathrm{id}_B)$. When B is faithful, the natural map from B into $M(B)$ is injective and B is an ideal of $M(B)$. Also in this case, each double multiplier τ is a *left multiplier*, that is, $\tau(ab) = (\tau a)b$ for all $a, b \in B$. A double multiplier is also a *right multiplier* which is defined similarly. The set of all left multipliers and right multipliers on B are denoted by $L(B)$ and $R(B)$ respectively.

One can see easily that a normed algebra B with a bounded approximate identity is faithful. If B is complete and faithful, then left and right multipliers are continuous linear operators [14]. For a double multiplier τ, the norms

$$\|\tau\|_{R(B)} = \sup\{\|a\tau\|_B : \|a\|_B \leq 1\}$$

and

$$\|\tau\|_{L(B)} = \sup\{\|\tau b\|_B : \|b\|_B \leq 1\}.$$

are finite. If we consider $M(B)$ with the following norm

$$\|\tau\|_{M(B)} = \max\{\|\tau\|_{R(B)}, \|\tau\|_{L(B)}\}$$

the inclusion map $B \to M(B)$ is norm decreasing and if moreover B has a bounded approximate identity $(e_\alpha)_\alpha$ with bound M, then

$$\|b\|_B = \lim_\alpha \|e_\alpha b\|_B \leq M\|b\|_{L(B)},$$

The theory of double multipliers is due to B. E. Johnson and for more information one can see [14] and [16].

Following [18], the authors observed in [10] that $M(B)$ is a dual Banach algebra whenever the following assumption holds:

Assumption 2.1. *There exists a continuous linear injection $B^* \to M(B)$ such that for each $u, v \in B^*$ and $b \in B$,*

$$\langle u, vb\rangle_B = \langle v, bu\rangle_B.$$

In this case, the predual of $M(B)$ is a subspace of $M(B)^*$. Indeed, $M(B){=}A^*$, where

$$A = cls_{M(B)^*}\,\mathrm{span}\{a \circ u : a \in B,\ u \in B^*\}$$

and $a \circ u$ is an element of $M(B)^*$ defined by $\langle a \circ u, \tau\rangle_{M(B)} = \langle u, \tau a\rangle_B$.

In general it is not easy to verify Assumption 2.1 for typical Banach algebras such as $L^1(G)$ and $A(G)$. In [10] the authors verified this for these algebras in the compact and discrete case, respectively.

M. Daws in [4, Theorem 7.1] gave the following set of assumptions for the multiplier algebra $M(B)$ to be a dual Banach algebra.

Theorem 2.2. *Let B be a Banach algebra such that $\{ab : a, b \in B\}$ is linearly dense in B, (A, A_*) be a dual Banach algebra, and $\imath : B \to A$ be an isometric homomorphism such that $\imath(B)$ is an essential ideal in A. If the induced map $\theta : A \to M(B)$ is injective, then there is a w^*-topology on $M(B)$ making $M(B)$ a dual Banach algebra.*

Let us briefly review the structure of the predual of $M(B)$ in the above theorem. Consider $(B\hat{\otimes}A_*) \oplus_1 (B\hat{\otimes}A_*)$ with dual space $\mathfrak{L}(B, A) \oplus_\infty \mathfrak{L}(B, A)$. The closure $\bar{X}$ of

$$X = \mathrm{span}\{(b \otimes \mu \cdot \imath(a)) \oplus (-a \otimes \imath(b) \cdot \mu) : a, b \in B,\ \mu \in A_*\}$$

is a closed linear subspace of $(B\hat{\otimes}A_*) \oplus_1 (B\hat{\otimes}A_*)$ and $\bar{X}^\perp = X^\perp$ is a w^*-closed subspace of $\mathfrak{L}(B,A) \oplus_\infty \mathfrak{L}(B,A)$. For each element $(S,T) \in X^\perp$, there exists $L, R \in \mathfrak{L}(B)$ such that $T = \imath L$ and $S = \imath R$ and $(L,R) \in M(B)$. Now $M(B)$ is isomorphic to $X^\perp$ and therefore a dual Banach space with predual

$$Y = \frac{(B\hat{\otimes}A_*) \oplus_1 (B\hat{\otimes}A_*)}{\bar{X}},$$

where the w^*- topology is given by the embedding $M(B) \to (B\hat{\otimes}A_* \oplus_1 B\hat{\otimes}A_*)^*$ given by

$$\langle (L,R), (a \otimes \mu) \oplus (b \otimes \lambda) \rangle = \langle \imath L(a), \mu \rangle + \langle \imath R(b), \lambda \rangle$$

for $L, R \in \mathfrak{L}(B)$, $a, b \in B$ and $\lambda, \mu \in A_*$.

Note that under a natural assumption the w^*-topology in the above theorem is unique [4, Theorem 7.2].

Theorem 2.3. *Let A and B be as above, and $\theta : A \to M(B)$ be the induced map. There is one and only one w^*-topology on $M(B)$ such that*

i) *$M(B)$ is a dual Banach algebra,*

ii) *For a bounded net (a_α) in A, $a_\alpha \to a$ weak* in A if and only if $\theta(a_\alpha) \to \theta(a)$ weak* in $M(B)$.*

3. Connes-amenability of M(B)

Throughout this section B is a Banach algebra satisfying conditions of Theorem 2.2. In particular $M(B)$ is a dual Banach algebra and we may ask when it is Connes-amenable.

Each element $\lambda \in A_*$ may be considered as an element of B^* via

$$\langle \lambda, b \rangle_B = \langle \imath(b), \lambda \rangle_{A_*} \quad (b \in B).$$

One can easily see that A_* is a B- submodule of B^* with the following module actions

$$\langle \lambda \cdot a, b \rangle_B = \langle \imath(ab), \lambda \rangle_{A_*}, \ \langle a \cdot \lambda, b \rangle_B = \langle \imath(ba), \lambda \rangle_{A_*} \quad (a, b \in B, \ \lambda \in A_*).$$

Lemma 3.1. *Let $\sim : \mathfrak{L}^2_{w^*}(M(B), \mathbb{C}) \to \mathfrak{L}^2(B; \mathbb{C})$ be defined by $\tilde{\psi} := \psi|_{B \times B}$, for $\psi \in \mathfrak{L}^2_{w^*}(M(B), \mathbb{C}))$. Then*

i) *$\sim$ is a continuous linear map.*

ii) $\Delta^*_{M(B)}((a\otimes\lambda)\oplus(b\otimes\mu)+\bar{X})^\sim=\Delta^*_B(a\cdot\lambda+\mu\cdot b)$, $(a,b\in B,\ \lambda,\mu\in A_*)$.

iii) $(\psi\cdot\tau)^\sim=\tilde{\psi}\cdot\tau;(\tau\cdot\psi)^\sim=\tau\cdot\tilde{\psi}$, $(\tau\in M(B),\ \psi\in\mathfrak{L}^2_{w^*}(M(B);\mathbb{C}))$.

We note that in (ii), $(a\otimes\lambda)\oplus_1(b\otimes\mu)+\bar{X}$ is an element of $Y(\subseteq M(B)^*)$ and since the product in $M(B)$ is separately w^*- continuous, we have $\Delta^*_{M(B)}(Y)\subseteq\mathfrak{L}^2_{w^*}(M(B);\mathbb{C})$.

Proof. It is straightforward to see that $\sim$ is a continuous linear map. To show (ii), let $c,d\in B$ and ϕ be the natural map from B into $M(B)$. Then

$$\begin{aligned}
&\langle\Delta^*_{M(B)}((a\otimes\lambda)\oplus(b\otimes\mu)+\bar{X})^\sim,(c,d)\rangle_{B\times B}\\
&\quad=\langle\Delta^*_{M(B)}((a\otimes\lambda)\oplus(b\otimes\mu)+\bar{X}),(\phi(c),\phi(d))\rangle_{M(B)\times M(B)}\\
&\quad=\langle(a\otimes\lambda)\oplus(b\otimes\mu)+\bar{X},\phi(cd)\rangle_{M(B)}\\
&\quad=\langle(a\otimes\lambda)\oplus(b\otimes\mu)+\bar{X},(L_{cd},R_{cd})\rangle_{M(B)}\\
&\quad=\langle(a\otimes\lambda)\oplus(b\otimes\mu),(L_{cd},R_{cd})\rangle_{M(B)}\\
&\quad=\langle(L_{cd},R_{cd}),(a\otimes\lambda)\oplus(b\otimes\mu)\rangle_{(B\hat{\otimes}A_*)\oplus_1(B\hat{\otimes}A_*)}\\
&\quad=\langle\imath L_{cd}(a),\lambda\rangle_{A_*}+\langle\imath R_{cd}(b),\mu\rangle_{A_*}=\langle\imath(cda),\lambda\rangle_{A_*}+\langle\imath(bcd),\mu\rangle_{A_*}\\
&\quad=\langle a\cdot\lambda,cd\rangle_B+\langle\mu\cdot b,cd\rangle_B=\langle a\cdot\lambda+\mu\cdot b,\Delta_B(c,d)\rangle_B\\
&\quad=\langle\Delta^*_B(a\cdot\lambda+\mu\cdot b),(c,d)\rangle_{B\times B}.
\end{aligned}$$

To prove (iii), for each $\tau\in M(B)$ and $b\in B$, we have $\tau\phi(b)=\phi(\tau b)$. If $\tau\in M(B)$ and $\psi\in\mathfrak{L}^2_{w^*}(M(B);\mathbb{C})$, and $b,c\in B$, then

$$\begin{aligned}
\langle(\psi.\tau)^\sim,(b,c)\rangle_{B\times B}&=\langle\psi.\tau,(\phi(b),\phi(c))\rangle_{M(B)\times M(B)}\\
&=\langle\psi,(\phi(\tau b),\phi(c))\rangle_{M(B)\times M(B)}=\langle\tilde{\psi}.\tau,(b,c)\rangle_{B\times B}.
\end{aligned}$$

Thus $(\psi.\tau)^\sim=\tilde{\psi}.\tau$. The second equality is proved similarly. □

Definition 3.2. A net (τ_α) in $M(B)$ converges in *weak strict topology* (wst) to τ in $M(B)$ if $\langle\phi,(\tau_\alpha-\tau)b\rangle_B\to0$ for each $\phi\in B^*$, $b\in B$.

We recall that the induced map $\theta:A\to M(B)$ is defined by $\theta(a)=(L_a,R_a)$ where L_a and R_a as operators on B are defined by

$$L_a(b)=\imath^{-1}(a\imath(b)),\quad R_a(b)=\imath^{-1}(\imath(b)a)\ (a\in A,\ b\in B).$$

Theorem 3.3. *Let B and (A,A_*) be as in Theorem 2.2. Then the followings are equivalent:*

i) $\theta : A \to M(B)$ *is* w^*-wst*-continuous;*

ii) $\imath(B) \cdot A^* \subseteq A_*$;

iii) *For each* $b \in B$ *and* $\lambda \in B^*$, *there is* $\omega \in A_*$ *with*

$$\langle \lambda, \imath^{-1}(a\imath(b)) \rangle_B = \langle a, \omega \rangle_{A_*} \quad (a \in A).$$

PROOF. To prove (i) $\Longrightarrow$ (ii), let $b \in B$ and $\mu \in A^*$ be such that $\imath(b) \cdot \mu$ does not belong to A_*. Then, by Hahn–Banach and Goldestein's theorem, there is a bounded net (a_α) in A such that $a_\alpha \to 0$ in w^*-topology, but $\langle \imath(b) \cdot \mu, a_\alpha \rangle_A \to 1$. That is

$$1 = \lim_\alpha \langle \mu, a_\alpha \imath(b) \rangle_A = \langle \lambda, \theta(a_\alpha) b \rangle_B$$

by assumption $\theta(a_\alpha) \to 0$ in wst-topology, a contradiction.

(ii) $\Longrightarrow$ (iii). Let $b \in B$ and $\lambda \in B^*$ be arbitrary. $\sigma_\lambda : \imath(B) \to \mathbb{C}$ defined by $\langle \sigma_\lambda, \imath(c) \rangle_A = \langle \lambda, c \rangle_B$, for each $c \in B$, is a continuous linear functional. By Hahn–Banah theorem there is $\mu \in A^*$ such that $\mu|_{\imath(B)}$ is σ_λ. Then

$$\langle \lambda, \imath^{-1}(a\imath(b)) \rangle_B = \langle \mu, a\imath(b) \rangle_A = \langle \imath(b) \cdot \mu, a \rangle_A \quad (a \in A)$$

set $\omega = \imath(b) \cdot \mu$, by assumption ω is in A_*.

(iii) $\Longrightarrow$ (ii). Let (a_α) be a net in A with $a_\alpha \to 0$ in w^*-topology. Let $b \in B$ and $\lambda \in B^*$, by assumption, there exists $\omega \in A_*$ such that

$$\langle \lambda, \imath^{-1}(a\imath(b)) \rangle_B = \langle a, \omega \rangle_{A_*} \quad (a \in A)$$

then

$$\lim_\alpha \langle \lambda, \theta(a_\alpha) b \rangle_B = \lim_\alpha \langle \lambda, \imath^{-1}(a_\alpha \imath(b)) \rangle_B = \lim_\alpha \langle a_\alpha, \omega \rangle_{A_*} = 0.$$

So θ is w^*-wst-continuous. □

Assumption 3.4. *Let* B *and* (A, A_*) *be as in Theorem 2.2. Assume further that one of the equivalent conditions of Theorem 3.3 holds.*

Theorem 3.5. *If* B *has a bounded approximate identity and satisfies Assumption 3.4, then the followings are equivalent:*

i) B *is amenable;*

ii) $M(B)$ *is Connes-amenable;*

iii) $M(B)$ *has a normal, virtual diagonal.*

Proof. (ii) $\Longrightarrow$ (i). Let $M(B)$ be Connes-amenable, E be a pseudo unital Banach B-bimodule and $D : B \to E^*$ be a derivation. Since B is a closed ideal of $M(B)$ with a bounded approximate identity, there exists a unique extension of D to a derivation $\tilde{D} : M(B) \to E^*$ such that $\tilde{D}$ is continuous with respect to the strict topology on $M(B)$ and the w^*-topology on E^*. We show that E^* is a normal, dual Banach $M(B)$-bimodule, and $\tilde{D} : M(B) \to E^*$ is w^*-w^*-continuous.

Claim 1. For $\phi \in E^*$ and $y \in E$, the mapping $\phi_y : A \to \mathbb{C}$ defined by $a \mapsto \langle \phi, \theta(a) \cdot y \rangle_E$ belongs to A_*.

It suffices to show that ϕ_y is w^*-continuous. Let $a_\alpha \stackrel{w^*}{\to} 0$ in A, by Assumption 3.3, $\theta(a_\alpha) \stackrel{wst}{\to} 0$ in $M(B)$. On the other hand, there exist $c \in B$ and $z \in E$ such that $y = c \cdot z$. Since $\theta(a_\alpha)c \stackrel{w}{\to} 0$ in B, we have

$$\begin{aligned}\lim_\alpha \langle \phi_y, a_\alpha \rangle_A &= \lim_\alpha \langle \phi, \theta(a_\alpha) \cdot y \rangle_E = \lim_\alpha \langle \phi, \theta(a_\alpha) \cdot c \cdot z \rangle_E \\ &= \lim_\alpha \langle \phi, \theta(a_\alpha)c \cdot z \rangle_E = \lim_\alpha \langle \phi^z, \theta(a_\alpha)c \rangle_{M(B)} \\ &= \lim_\alpha \langle \phi^z|_B, \theta(a_\alpha)c \rangle_B = 0,\end{aligned}$$

where ϕ^z is an element of $M(B)^*$ defined by $\tau \mapsto \langle \phi, \tau \cdot z \rangle_E$ and $\phi^z|_B$ is in B^*.

Claim 2. For $\phi \in E^*$ and $y \in E$, as above, if $\phi^y : M(B) \to \mathbb{C}$ is defined by $\tau \mapsto \langle \phi, \tau \cdot y \rangle_E$ and $\jmath : B \to M(B)$ is the canonical mapping $\jmath(b) = \langle L_b, R_b \rangle$, then $\theta \circ \imath = \jmath$ and $\phi_y \circ \imath = \phi^y \circ \jmath$.

Let $b \in B$, then for each $c \in B$ we have

$$L_{\imath(b)}(c) = \imath^{-1}(\imath(b)\imath(c)) = bc = L_b(c)$$

and

$$R_{\imath(b)}(c) = \imath^{-1}(\imath(c)\imath(b)) = cb = R_b(c).$$

Therefore $(L_b, R_b) = (L_{\imath(b)}, R_{\imath(b)})$, for each $b \in B$, thus

$$\theta \circ \imath(b) = \theta(\imath(b)) = (L_{\imath(b)}, R_{\imath(b)}) = (L_b, R_b) = \jmath(b).$$

Also for each $b \in B$

$$\begin{aligned}\langle \phi_y \circ \imath, b \rangle_B &= \langle \phi_y, \imath(b) \rangle_A = \langle \phi, \theta \circ \imath(b) \cdot y \rangle_E = \langle \phi, \jmath(b) \cdot y \rangle_E \\ &= \langle \phi^y, \jmath(b) \rangle_{M(B)} = \langle \phi^y \circ \jmath, b \rangle_B.\end{aligned}$$

Claim 3. If $\tau_\alpha \stackrel{w^*}{\to} \tau$ in $M(B)$, then for each $\lambda \in A_*$ and $b \in B$ we have

$$\lim_\alpha \langle \lambda, \tau_\alpha b \rangle_B = \langle \lambda, \tau b \rangle_B.$$

For each $(L,R)\in M(B)$ and each $b\in B$, we have

$$(L,R)\theta(\imath(b)) = (LL_{\imath(b)}, R_{\imath(b)}R) = (L_{L(b)}, R_{L(b)}) = \jmath(L(b)) = \theta(\imath(L(b))).$$

Let $\tau_\alpha = \langle L_\alpha, R_\alpha\rangle$, $\tau = \langle L, R\rangle$ and $\tau_\alpha \overset{w^*}{\to} \tau$. For each $b\in B$ we have

$$\lim_\alpha \theta(\imath(L_\alpha(b))) = \lim_\alpha (L_\alpha, R_\alpha)\theta(\imath(b)) = (L,R)\theta(\imath(b)) = \theta(\imath(L(b))).$$

Therefore, by Theorem 2.3, $\imath L_\alpha(b) \overset{w^*}{\to} \imath L(b)$ in A for each $b\in B$. Hence for each $\lambda\in A_*$ and $b\in B$,

$$\begin{aligned}\lim_\alpha\langle\lambda, \tau_\alpha b\rangle_B &= \lim_\alpha\langle\lambda, L_\alpha(b)\rangle_B = \lim_\alpha\langle\imath L_\alpha(b), \lambda\rangle_{A_*}\\ &= \langle \imath L(b), \lambda\rangle_{A_*} = \langle\lambda, L(b)\rangle_B = \langle\lambda, \tau b\rangle_B.\end{aligned}$$

Now we are ready to show that E^* is a normal dual Banach $M(B)$-bimodule. If $\tau_\alpha \overset{w^*}{\to} \tau$ in $M(B)$, then for each $x\in E$ and $\phi\in E^*$, there exist $b\in B$ and $y\in E$ such that $x = b\cdot y$ and

$$\begin{aligned}\lim_\alpha\langle\phi\cdot\tau_\alpha, x\rangle_E &= \lim_\alpha\langle\phi, \tau_\alpha\cdot x\rangle_E = \lim_\alpha\langle\phi, \tau_\alpha\cdot(b\cdot y)\rangle_E\\ &= \lim_\alpha\langle\phi, \tau_\alpha b\cdot y\rangle_E = \lim_\alpha\langle\phi^y, \tau_\alpha b\rangle_B\\ &= \lim_\alpha\langle\phi^y, \jmath(\tau_\alpha b)\rangle_{M(B)} = \lim_\alpha\langle\phi^y\circ\jmath, \tau_\alpha b\rangle_B\\ &= \lim_\alpha\langle\phi_y\circ\imath, \tau_\alpha b\rangle_B = \lim_\alpha\langle\phi_y, \imath(\tau_\alpha b)\rangle_A\\ &= \lim_\alpha\langle\imath(\tau_\alpha b), \phi_y\rangle_{A_*} = \lim_\alpha\langle\phi_y, \tau_\alpha b\rangle_B\\ &= \langle\phi_y, \tau b\rangle_B = \langle\phi, \tau b\cdot y\rangle_E = \langle\phi, \tau\cdot(b\cdot y)\rangle_E = \langle\phi\cdot\tau, x\rangle_E.\end{aligned}$$

To prove that $\tilde{D}$ is w^*-w^*-continuous, let $\tau_\alpha\to 0$ in w^*-topology. For $x\in E$, take $y\in E$ and $b\in B$ with $x = b\cdot y$. Then

$$\langle\tilde{D}\tau_\alpha, x\rangle_E = \langle\tilde{D}(\tau_\alpha b) - \tau_\alpha\cdot\tilde{D}b, y\rangle_E = \langle D(\tau_\alpha b), y\rangle_E - \langle\tau_\alpha\cdot Db, y\rangle_E \to 0.$$

Since $M(B)$ is Connes-amenable, $\tilde{D}$ and so D is an inner derivation, therefore B is amenable.

(i) $\Longrightarrow$ (iii). Since B is amenable, it has a virtual diagonal $M\in\mathfrak{L}^2(B;\mathbb{C})^*$ such that $a\cdot M = M\cdot a$ and $a\cdot\Delta_B^{**}M = a$ for $a\in B$. Define $\tilde{M}:\mathfrak{L}^2_{w^*}(M(B);\mathbb{C})\to\mathbb{C}$ by $\langle\tilde{M},\psi\rangle = \langle M,\tilde{\psi}\rangle$, for $\psi\in\mathfrak{L}^2_{w^*}(M(B);\mathbb{C}))$. Then $\tilde{M}$ is linear and by Lemma 3.1 (i), $\tilde{M}\in\mathfrak{L}^2_{w^*}(M(B);\mathbb{C})^*$. To prove that $\tilde{M}$ is a normal, virtual diagonal for $M(B)$, it suffices to show that for each $\tau\in M(B)$, $\tilde{M}\cdot\tau = \tau\cdot\tilde{M}$ and

$\tau\Delta_{w^*}\tilde{M} = \tau$. By density of B in $M(B)$, for each $\tau \in M(B)$ there exists a net $(a_\alpha)_\alpha \subseteq B$ such that $a_\alpha \to \tau$ in the strict topology. Since $\mathfrak{L}^2(B;\mathbb{C})^*$ is a pseudo unital Banach B-bimodule, there exist $a \in B$ and $M' \in \mathfrak{L}^2(B;\mathbb{C})^*$ such that $M = a \cdot M'$. Thus $a_\alpha a \to \tau a$ in the norm topology, and $a_\alpha a \cdot M \to \tau a \cdot M'$ in the w^*-topology. Next, let $b \in B$ and $M'' \in \mathfrak{L}^2(B;\mathbb{C})^*$ be such that $M = M'' \cdot b$, then since $M'' \cdot ba_\alpha \to M'' \cdot b\tau$ in the w^*-topology, we have $\tau.M = M \cdot \tau$, from which and Lemma 3.2 (iii) it follows that $\tau \cdot \tilde{M} = \tilde{M} \cdot \tau$. To complete the proof, we need to show that $\Delta_{w^*}\tilde{M}$ is the unit $1_{M(B)}$ of $M(B) = Y^*$. Let $(a \otimes \lambda) \oplus (b \otimes \mu) + \bar{X}$ be an arbitrary element of Y. Then by Lemma 3.2 (ii)

$$\begin{aligned}
\langle \Delta_{w^*}\tilde{M}, (a \otimes \lambda) \oplus (b \otimes \mu) + \bar{X}\rangle_Y \\
&= \langle \tilde{M}, \Delta^*_{M(B)}((a \otimes \lambda) \oplus (b \otimes \mu) + \bar{X})\rangle_{\mathfrak{L}^2_{w^*}(M(B);\mathbb{C})} \\
&= \langle M, \Delta^*_{M(B)}((a \otimes \lambda) \oplus (b \otimes \mu) + \bar{X})^{\sim}\rangle_{\mathfrak{L}^2(B;\mathbb{C})} \\
&= \langle M, \Delta^*_{B}(a \cdot \lambda + \mu \cdot b)\rangle_{\mathfrak{L}^2(B;\mathbb{C})} = \langle \Delta^{**}_B(M), a \cdot \lambda + \mu \cdot b)\rangle_{B^*} \\
&= \langle \Delta^{**}_B(M) \cdot a, \lambda\rangle_{A_*} + \langle b \cdot \Delta^{**}_B(M), \mu\rangle_{A_*} \\
&= \langle a, \lambda\rangle_{A*} + \langle b, \mu\rangle_{A_*} = \langle \lambda, a\rangle_B + \langle \mu, b\rangle_B = \langle \imath(a), \lambda\rangle_{A_*} + \langle \imath(b), \mu\rangle_{A_*} \\
&= \langle (I_B, I_B), (a \otimes \lambda) \oplus (b \otimes \mu) + \bar{X})\rangle_Y \\
&= \langle 1_{M(B)}, (a \otimes \lambda) \oplus (b \otimes \mu) + \bar{X})\rangle_Y.
\end{aligned}$$

(iii) $\Longrightarrow$ (ii). This holds for any dual Banach algebra [19]. □

Example 3.6. Let G be a compact group and $B = L^1(G)$ then $M(B) = M(G) = C(G)^*$, hence $A = M(G)$ and $A_* = C(G)$. The map $\imath : B \to A$ is the canonical embedding which takes $f \in L^1(G)$ to $f\lambda$, where λ is the left Haar measure on G and $\theta : A \to M(B)$ is the identity map. We show that θ satisfies Assumption 3.3. If $\mu_\alpha \overset{w^*}{\to} 0$ in $M(G)$, by the principle of uniform boundedness, $(\mu_\alpha)_\alpha$ is norm bounded, say $\|\mu_\alpha\| \leq M$, for each α. We claim that $\mu_\alpha \to 0$ in wst-topology in $M(G)$. For $\phi \in L^\infty(G)$ and $f \in C(G)$, by [7, 2.39 (d)], since G is compact we have $\phi * f \in C(G)$. Therefore

$$\lim_\alpha \langle \mu_\alpha * f, \phi\rangle = \langle \mu_\alpha, \phi * f\rangle = 0,$$

and $\theta = \mathrm{id}_{M(G)}$ is w^*-wst-continuous. Now since $B = L^1(G)$ is amenable, $M(G)$ is Connes amenable. Indeed by a result of V. Runde [19], [20], $M(G)$ is Connes amenable for each locally compact amenable group G

One should note that the identity map on $M(G)$ is not w^*-wst-continuous in general. Take $G = \mathbb{R}$ and (x_n) a sequence in $\mathbb{R}$ such that $x_n \to \infty$. Then for each

$f \in C_0(\mathbb{R})$, $\langle \delta_{x_n}, f\rangle = f(x_n) \to 0$, hence $\delta_{x_n} \to 0$ in the w^*-topology on $M(\mathbb{R})$, but for the constant function $1 \in L^\infty(\mathbb{R})$ and each $f \in C_0(\mathbb{R})$, $1 * f$ is a constant function with value $c_f = \int_G f(y)dy$, hence

$$\langle \delta_{x_n} * f, 1\rangle = \langle \delta_{x_n}, 1 * f\rangle = (1 * f)(x_n) = c_f.$$

Thus δ_{x_n} does not tend to zero in the wst-topology on $M(\mathbb{R})$.

Example 3.7. For a locally compact group G, let $A(G)$, $B(G)$ and $VN(G)$ be the Fourier algebra, the Fourier–Stieltjes algebra and the group von Neumann algebra of G [6]. Let G be a discrete amenable group. We claim that the followings are equivalent:

i) $A(G)$ is amenable;

ii) $B(G)$ is Connes-amenable;

iii) $B(G)$ has a normal virtual diagonal.

Since G is amenable, by [17], $M(A(G)) = B(G)$. Let $C^*(G)$ be the full group C^*-algebra of G with dual space $B(G)$. Take $B = A(G)$, $A = B(G)$ and $A_* = C^*(G)$. By [6, Proposition 3.4] $A(G)$ is an ideal in $B(G)$. The induced map from $B(G)$ into $M(A(G)) = B(G)$ is $\theta = \mathrm{id}_{B(G)}$. We show that θ Satisfies Assumption 3.3.

Since G is amenable, $A(G)$ has a bounded approximate identity which may be chosen to be contained in $A(G) \cap C_c(G)$ [17]. Let $u_\alpha \overset{w^*}{\to} 0$ in $B(G)$, then by the principle of uniform boundedness, $(u_\alpha)_\alpha$ is norm bounded, say $\|u_\alpha\| \le M$. Since G is discrete, $\{\delta_x : x \in G\} \subseteq L^1(G) \subseteq C^*(G)$, and for each $x, y \in G$

$$\lim_\alpha \langle u_\alpha \delta_x, \delta_y\rangle = \lim_\alpha \delta_x(y) u_\alpha(y) = \lim_\alpha u_\alpha(x) = \lim_\alpha \langle u_\alpha, \delta_x\rangle = 0$$

If $\phi = \sum_{y\in G} \alpha_y \delta_y$, then for each $x \in G$,

$$\lim_\alpha \langle u_\alpha \delta_x, \phi\rangle = \lim_\alpha \sum_{y\in G} \alpha_y \delta_x(y) u_\alpha(y) = \lim_\alpha u_\alpha(x) = 0.$$

Finally for each $T \in VN(G)$,

$$\lim_\alpha \Big\langle u_\alpha \Big(\sum_{x\in G} \beta_x \delta_x\Big), T\Big\rangle = 0.$$

Now let $g \in A(G)$ and $T \in VN(G)$ and $\epsilon > 0$ be given. Since $A(G) \cap C_c(G)$ is norm dense in $A(G)$, we may choose $f \in A(G) \cap C_c(G)$ such that $\|g - f\| < \epsilon$ and

$$|\langle u_\alpha g, T\rangle| \le |\langle u_\alpha (g - f), T\rangle| + |\langle u_\alpha f, T\rangle| \le M\epsilon\|T\| + \epsilon,$$

for large α, therefore $\theta = \mathrm{id}_{B(G)}$ satisfies Assumption 3.3. Now $A(G)$ is amenable if and only if G is abelian by finite [8]. Hence for a discrete amenable group G, $B(G)$ is Connes-amenable if and only if G is abelian by finite. This was proved by V. Runde using a different method in [22].

Let us have a closer look to Theorem 3.3. Assume that B is a Banach algebra such that its multiplier algebra is already a dual Banach algebra. Take $A = M(B)$ and let A_* be the existing predual of $M(B)$. Then $\imath$ is the usual inclusion, if ii) of Theorem 3.3 holds, then for each $b \in B$, $\mu \in M(B)^*$, there is $\omega \in M(B)_*$ with

$$\langle \mu, (L,R)b\rangle_{M(B)} = \langle \mu, L(b)\rangle_B = \langle (L,R), \omega\rangle_{M(B)_*} \quad ((L,R) \in M(B)).$$

Notice that all that matters is the value of μ restricted to B. So we really want $B \cdot B^* \subseteq M(B)_*$. Although, this condition seems very strong, there exist some Banach algebras satisfies in such condition.

Example 3.8. Let E be a reflexive Banach space with the approximation property, then set $B = \mathfrak{K}(E)$, the compact operators on E, so $M(B) = \mathfrak{L}(E) = \mathfrak{K}(E)^{**}$. We might then take $A = \mathfrak{L}(E)$ and $A_* = E\hat{\otimes}E^*$. Again, we then want $\mathfrak{K}(E) \cdot (E\hat{\otimes}E^*) \subseteq E\hat{\otimes}E^*$ which is always true. So we recover a result of Runde [22]: $\mathfrak{K}(E)$ is amenable if and only if $\mathfrak{L}(E)$ is Connes-amenable.

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(Received April 20, 2014; revised October 1, 2014)