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A note on lattices of idempotents in algebras

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Abstract. Let R be a unital algebra over a field K. For idempotents $e, f \in R$, we define $e \leq f$ if and only if $ef = e = fe$. Let $e \wedge f$ and $e \vee f$ denote the infimum and supremum of e and f, respectively, if they exist. Let $e' := 1-e$ for an idempotent $e \in R$. We prove the following theorem: Let $e, f \in R$ be nontrivial idempotents. Suppose that there exists $p(\lambda) \in K[\lambda]$ with zero constant term such that $p(ef) = p(fe)$ and $p(1) = 1$. Then $e \wedge f = p(ef)$ and $e \vee f = 1 - p(e'f')$.

1. Results

Throughout, R is an associative unital algebra over a field K. Let $\mathrm{Id}(R)$ denote the set of all idempotents in R. For $e, f \in \mathrm{Id}(R)$, we define $e \leq f$ if and only if $ef = e = fe$. Clearly, $e \leq f$ if and only if $e \in fRf$. Define $e < f$ if $e \leq f$ and $e \neq f$. Let $e \wedge f$ and $e \vee f$ denote the infimum and supremum of e and f, respectively, if they exist. Then $(\mathrm{Id}(R), \leq)$ forms a partially ordered set.

Following [1], two idempotents $e, f \in R$ are called generalized commuting if there exists a positive integer n such that $(ef)^n = (fe)^n$ or $(ef)^n e = (fe)^n f$. We denote by $\langle e, f\rangle_s$ the subsemigroup of the multiplicative monoid of R generated by e and f. For an idempotent $e \in R$, we set $e' := 1 - e$, the complementary idempotent of e. In [1], Călugăreanu proved the following (see [1, Theorem 7, Proposition 4 and Lemma 2]).

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Theorem 1. *Let $e, f \in \mathrm{Id}(R)$. Then both $e \wedge f$ and $e \vee f$ exist, and $e \wedge f \in \langle e, f\rangle_s$ if and only if e and f are generalized commuting idempotents. In this case, there exists a positive integer n such that $e \wedge f = (ef)^n$ and $e \vee f = 1 - (e'f')^n$.*

In a recent paper [2], Han and Park proved that for idempotents $e, f \in R$ and a positive integer n, $(ef)^n = (fe)^n$ if and only if $(e'f')^n = (f'e')^n$. Thus, Theorem 1 is easily proved by the result (see [2, Theorem 2.5]). However, there exist idempotents $e, f \in R$ such that $e \wedge f$ and $e \vee f$ exist but e, f are not generalized commuting idempotents (see the example below). In this situation, of course, $e \wedge f \notin \langle e, f\rangle_s$.

We let $\mathrm{M}_n(K)$ stand for the $n \times n$ matrix algebra over the field K and let $\mathbb{R}$ denote the field of real numbers.

Example 2. In $R := \mathrm{M}_2(\mathbb{R})$, let $e = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $e \wedge f = 0$ and $e \vee f = 1$.

Indeed, let $g \in R$ be an idempotent such that $g \le e$ and $g \le f$. Then $g(e - f) = 0$. But $e - f = \left(\begin{smallmatrix} 0 & 2 \\ -1 & 0 \end{smallmatrix}\right)$. Then $g = 0$ follows since $e - f$ is a unit in R. This proves that $e \wedge f = 0$. On the other hand, let $h \in R$ be an idempotent such that $e \le h$ and $f \le h$. Then $(1-h)e = 0 = (1-h)f$, implying $(1-h)(e-f) = 0$. Thus, $h = 1$. This proves that $e \vee f = 1$. However,

$$(ef)^n = 3^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (fe)^n = 3^{n-1} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$(ef)^n e = 3^n \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (fe)^n f = 3^n \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

for all $n \ge 1$. Thus, e and f are not generalized commuting idempotents.

In this note, we will prove some theorems to compute $e \wedge f$ and $e \vee f$ for idempotents $e, f \in R$ even when e and f are not generalized commuting idempotents. Let $K[\lambda]$ be the polynomial ring over K in the indeterminate λ. The main goal of this note is to prove the following theorem (see Section 2 for its proof).

Theorem 3. *Let $e, f \in \mathrm{Id}(R)$. Suppose that there exists $p(\lambda) \in K[\lambda]$ with zero constant term such that $p(ef) = p(fe)$ and $p(1) = 1$. Then the following hold:*

(i) *$p(ef)$ and $p(e'f')$ are idempotents.*

(ii) *$e \wedge f = p(ef)$ and $e \vee f = 1 - p(e'f')$.*

Theorem 4. *Let $e, f \in \mathrm{Id}(R)$. Suppose that there exists $q(\lambda) \in K[\lambda]$ with zero constant term such that $q(ef) = 0$ and $q(1) \neq 0$. Then $e \wedge f = 0$ and $e \vee f = 1 - q(1)^{-1} e'f'q(e'f')$.*

Proof. Let $p(\lambda) := q(1)^{-1}\lambda q(\lambda) \in K[\lambda]$. Then $p(ef) = 0$ and

$$p(fe) = q(1)^{-1} feq(fe) = q(1)^{-1} fq(ef)e = 0.$$

Also, $p(1) = 1$. In view of Theorem 3, $e \wedge f = p(ef) = 0$ and $e \vee f = 1 - p(e'f')$. □

Corollary 5. *Let $e, f \in \mathrm{Id}(R)$ be such that $(ef)^n = 0$ for some positive integer n. Then $e \wedge f = 0$ and $e \vee f = 1 - (e'f')^{n+1}$.*

Proof. Let $q(\lambda) := \lambda^n$. Thus, $q(ef) = 0$. In view of Theorem 4, $e \wedge f = 0$ and $e \vee f = 1 - e'f'q(e'f') = 1 - (e'f')^{n+1}$. □

2. Proof of Theorem 3

Recall that R is always a unital algebra over a field K. For an idempotent $e \in R$, we let $e' := 1 - e$. We begin with the following key observation.

Lemma 6. *Let $e, f \in R$ be idempotents and $p(\lambda) \in K[\lambda]$ with $p(0) = 0$. Then $p(ef) = p(fe)$ if and only if $p(e'f') = p(f'e')$.*

Proof. Clearly, it suffices to prove the "only if" part. Suppose that $p(ef) = p(fe)$. Write

$$p(\lambda) = \beta_m \lambda^m + \beta_{m-1}\lambda^{m-1} + \cdots + \beta_1 \lambda,$$

where $\beta_1, \ldots, \beta_m \in K$. Note that

$$(1 - e')(1 - f')e' = (1 - e')(-f')e' \quad \text{and} \quad (1 - f')(1 - e')f' = (1 - f')(-e')f'.$$

Thus, for a positive integer $k \geq 1$,

$$[1 - e, (ef)^k] = [e', ((1 - e')(1 - f'))^k] = -((1 - e')(1 - f'))^k e' = (1 - e')(f'e')^k$$

and so

$$\begin{aligned} [1 - e, p(ef)] &= \left[1 - e, \sum_{k=1}^{m} \beta_k (ef)^k\right] = \sum_{k=1}^{m} \beta_k [1 - e, (ef)^k] \\ &= (1 - e') \sum_{k=1}^{m} \beta_k (f'e')^k = (1 - e')p(f'e'). \end{aligned} \tag{1}$$

On the other hand,

$$[1-e, p(fe)] = e'p(fe) = e'p((1-f')(1-e')) = -p(e'f')(1-e'). \qquad (2)$$

Since $p(ef) = p(fe)$, it follows from equations (1) and (2) that

$$(1-e')p(f'e') = -p(e'f')(1-e').$$

Thus, $(1-e')p(f'e') = 0 = -p(e'f')(1-e')$ and hence

$$p(f'e') = e'p(f'e') = p(e'f')e' = p(e'f'),$$

as asserted. □

Proof of Theorem 3. Set $h := p(ef)$. Since $p(ef) = p(fe)$, we see that $he = h = eh$ and $hf = h = fh$. Thus, $(ef)^k h = h = (fe)^k h$ for any positive integer k. Write

$$p(\lambda) = \beta_m \lambda^m + \beta_{m-1}\lambda^{m-1} + \cdots + \beta_1 \lambda,$$

where $\beta_1, \ldots, \beta_m \in K$. Then, by $p(1) = 1$,

$$h^2 = hp(ef) = hp(1) = h.$$

This proves that h is an idempotent. In particular, $h \leq e$ and $h \leq f$. We claim that $h = e \wedge f$. Suppose that $g \leq e$ and $g \leq f$. Then

$$gh = gp(ef) = p(gef) = p(g) = gp(1) = g.$$

Similarly, $g = hg$. This implies that $g \leq h$. This proves that $h = e \wedge f$.

By Lemma 6, we see that $p(e'f') = p(f'e')$. Since $p(1) = 1$, it follows from (i) that $e' \wedge f' = p(e'f')$. This implies that $e \vee f = 1 - e' \wedge f' = 1 - p(e'f')$. □

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