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On generalised pseudo symmetric manifolds

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Dedicated to the memory of Professor Béla Barna

1. Introduction

The notion of a pseudo symmetric manifold was introduced by the author in an earlier paper [1]. A non-flat Riemannian manifold (M^n, g) $(n \geq 2)$ was called pseudo symmetric if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,W) &= 2A(X)R(Y,Z,W) + A(Y)R(X,Z,W)\\ &+ A(Z)R(Y,X,W) + A(W)R(Y,Z,X) + g(R(Y,Z,W),X)P;\\ &X,Y,Z,P \in \chi(M^n)\end{aligned} \tag{1}$$

where A is a non-zero 1-form, ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and P is a vector field given by

$$g(X,P) = A(X) \qquad \forall X. \tag{2}$$

The 1-form A was called the associated 1-form of the manifold and such an n-dimensional manifold was denoted by $(PS)_n$. The vector field P defined by (2) is called the basic vector field corresponding to the associated 1-form A.

The object of this paper is to study a type of non-flat Riemannian manifold $(M^n, g)(n > 2)$ whose curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,W) &= 2A(X)R(Y,Z,W) + B(Y)R(X,Z,W)\\ &+ C(Z)R(Y,X,W) + D(W)R(Y,Z,X) + g(R(Y,Z,W),X)P\end{aligned} \tag{3}$$

where A, B, C, D are non-zero 1-forms and ∇ and P have the meaning already mentioned.

Such a manifold shall be called a generalised pseudo symmetric manifold, A, B, C, D shall be called its associated 1-forms and an n-dimensional manifold of this kind shall be denoted by $G(PS)_n$. Let

$$(4)\qquad \begin{aligned} &g(X,\lambda) = B(X), g(X,\mu) = C(X) \quad \text{and} \\ &g(X,\nu) = D(X) \qquad \forall X \in \chi(M) \end{aligned}$$

Then $P, \lambda, \mu, \nu \in \chi(M)$ shall be called the basic vector fields of $G(PS)_n$ corresponding to the associated 1-forms A, B, C, D, respectively. If, in particular, $B = C = D = A$, then the manifold defined by (3) reduces to a pseudo symmetric manifold defined by (1). This justifies the name "Generalized pseudo symmetric manifold" and the use of the symbol $G(PS)_n$. It may be mentioned in this connection that following my paper [1], Tamássy and Binh [2] studied a type of Riemannian manifold (M, g) whose curvature tensor R satisfies the condition

$$(5)\qquad \begin{aligned} (\nabla_X R)(Y,Z,W) &= \alpha(X)R(Y,Z,W) + \beta(Y)R(X,Z,W) \\ &+ \gamma(Z)R(Y,X,W) + \delta(W)R(Y,Z,X) + g(R(Y,Z,W),X)F \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are 1-forms and F any vector field. They called such a manifold weakly symmetric. (5) is a little weaker assumption than (3). (5) gives (3) if α and F are related by $g(X,F) = \alpha\ \forall X$. Though the definition of a $G(PS)_n$ is similar to that of a weakly symmetric manifold mentioned above, our study of a $G(PS)_n$ is different from that of Tamássy and Binh.

In this paper the question whether a $G(PS)_n$ can be of constant curvature has been answered. Considering an Einstein $G(PS)_n$ it is shown that such a manifold is necessarily of zero scalar curvature under a certain condition. Further, an interesting result of paper [1] for a conformally flat $(PS)_n$ has been generalised for a conformally flat $G(PS)_n$. Finally, it is shown that if a $G(PS)_n$ admits a parallel vector field which is not orthogonal to the basic vector field P, then the manifold cannot be conformally flat.

1. Preliminaries

Let L be the symmetric endomorphism of the tangent space at each point of a $G(PS)_n$ corresponding to the Ricci tensor S of type $(0,2)$. Then

$$(1.1)\qquad g(LX,Y) = S(X,Y) \qquad \forall X, Y \in \chi(M).$$

Further, let

$$(1.2)\qquad \begin{aligned} \bar{A}(X) &= A(LX), \quad \bar{B}(X) = B(LX), \\ \bar{C}(X) &= C(LX), \quad \bar{D}(X) = D(LX). \end{aligned}$$

Then the 1-forms $\bar{A},\ \bar{B},\ \bar{C},\ \bar{D}$ shall be called the auxiliary associated 1-forms of a $G(PS)_n$ corresponding to the forms $A,\ B,\ C,\ D$, respectively. Establishing the inner product of both sides of (3) with a vector $U \in \chi(M^n)$ and then contracting over Z and W, we get

$$\begin{aligned}(\nabla_X S)(Y,U) &= 2A(X)S(Y,U) + B(Y)S(X,U)\\ &\quad + C(R(X,Y,U)) + D(R(X,U,Y)) + A(U)S(X,Y).\end{aligned} \tag{1.3}$$

Next, contracting (1.3) over Y and U, we obtain

$$dr(X) = 2A(X)r + S(X,P) + S(X,\lambda) + S(X,\mu) + S(X,\nu), \tag{1.4}$$

where r is the scalar curvature. Using the notations $P+\lambda+\mu+\nu=\rho$ and $A+B+C+D=E$, we obtain $\bar{E}(X)=E(LX)=S(X,\rho)$ and then (1.4) takes the form

$$dr(X) = 2A(X)r + S(X,\rho) = 2A(X)r + \bar{E}(X). \tag{1.4$'$}$$

From (1.4$'$) we get

$$ddr(X,Y) = 2rdA(X,Y) + 2\bar{E}(X)A(Y) - 2\bar{E}(Y)A(X) + d\bar{E}(X,Y).$$

Since $ddr(X)=0$, we obtain

$$rdA(X,Y) + [\bar{E}(X)A(Y) - \bar{E}(Y)A(X)] + \frac{1}{2}d\bar{E}(X,Y) = 0. \tag{1.5}$$

These formulas will be used in the sequel.

2. $G(PS)_n$ of non-zero constant scalar curvature

We suppose that in a $G(PS)_n$ the scalar curvature r is a constant different from zero. Then from (1.4$'$) we get

$$2A(X)r + \bar{E}(X) = 0 \quad \text{or} \quad \bar{E}(X) = -2A(X)r.$$

From this it follows that

$$S(X,\rho) = -2rA(X). \tag{2.1}$$

If, in particular, $B=C=D=A$, then $S(X,\rho)=4\bar{A}(X)$ and (2.1) takes the form $4\bar{A}(X)=-2rA(X)$, from which we get

Theorem 1. *In a $G(PS)_n$ of non-zero constant scalar curvature, in which $B=C=D=A$ we obtain.*

$$\bar{A}(X) = -\left(\frac{r}{2}\right)A(X).$$

This result has already been obtained in paper [1] of the author.

3. Einstein $G(PS)_n$ $(n > 2)$

In this section we suppose that a $G(PS)_n$ is an Einstein manifold. Then

$$S(X,Y) = \frac{r}{n} g(X,Y). \tag{3.1}$$

It is known [3] that in an Einstein manifold (M^n, g) $(n > 2) r$ is constant. Hence in this case $dr(X) = 0$. Therefore from (1.4) it follows that

$$2A(X)r + S(X,P) + S(X,\lambda) + S(X,\mu) + S(X,\nu) = 0.$$

Or, in virtue of (3.1) we have

$$2A(X)r + \frac{r}{n}[A(X) + B(X) + C(X) + D(X)] = 0,$$

or

$$[(2n+1)A(X) + B(X) + C(X) + D(X)]r = 0. \tag{3.2}$$

From (3.2) it follows that if

$$(2n+1)A(X) + B(X) + C(X) + D(X) \neq 0, \quad \text{then } r = 0. \tag{3.3}$$

This leads to the following theorem:

Theorem 2. *An Einstein $G(PS)_n$ satisfying the condition (3.3) is of zero scalar curvature.*

If, in particular, $B = C = D = A$, then the $G(PS)_n$ reduces to a $(PS)_n$ and the expression $(2n+1)A(X)+B(X)+C(X)+D(X)$ takes the form $2(n+2)A(X)$ which is not zero, because $A(X) \neq 0$. Thus it follows that an Einstein $(PS)_n$ with $n > 2$ is of zero scalar curvature — a result already proved by the author in his paper [1]. It is known that a manifold of constant curvature is an Einstein manifold, but the converse is not, in general, true. The question therefore arises whether a $G(PS)_n$ can be of constant curvature.

Suppose that a $G(PS)_n$ is of constant curvature. Then we can write

$$R(X,Y,Z) = \kappa[g(Y,Z)X - g(X,Z)Y] \tag{3.4}$$

where κ is constant. Being of constant curvature, the $G(PS)_n$ under consideration is an Einstein manifold. Hence if (3.3) holds, then according to Theorem 2, $r = 0$. Therefore $\kappa = 0$, because from (3.4) we easily get $r = \kappa n(n-1)$ by contraction.

Consequently, from (3.4) it follows that $R(X,Y,Z) = 0$, that is, the manifold is flat. But this is not admissible by the definition of a $G(PS)_n$. Therefore in answer to the question raised above we can state the following theorem:

Theorem 3. *A $G(PS)_n$ satisfying the condition (3.3) cannot be of constant curvature.*

Since a 3-dimensional Einstein manifold is of constant curvature ([3] p. 293), we can state the following corollary of Theorem 3.

Corollary. *An Einstein $G(PS)_n$ satisfying the condition $7A(X) + B(X) + C(X) + D(X) \neq 0$ does not exist.*

4. Conformally flat $G(PS)_n$ $(n \geq 3)$

It has been proved by the author elsewhere [1] that in a conformally flat $(PS)_n$, the associated 1-form A is proportional to the auxiliary associated 1-form $\bar{A}$. It is therefore narural to enquire about the nature of generalisation of this result for a conformally flat $G(PS)_n$. An answer to this enquiry is given in this section.

It is known ([4] p. 91) that in a conformally flat $(M^n, g)(n \geq 3)$

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Z)g(X,Y)]. \tag{4.1}$$

In virtue of (1.4) the equation (4.1) can be written as follows:

$$\begin{aligned} 2(n-1)[(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) &= 2r[A(X)g(Y,Z) - A(Z)g(X,Y)] \\ &+ [\bar{A}(X)g(Y,Z) - \bar{A}(Z)g(X,Y)] + [\bar{B}(X)g(Y,Z) - \bar{B}(Z)g(X,Y)] \\ &+ [\bar{C}(X)g(Y,Z) - \bar{C}(Z)g(X,Y)] + [\bar{D}(X)g(Y,Z) - \bar{D}(Z)g(X,Y)] \end{aligned} \tag{4.2}$$

Again in virtue of (1.3)

$$\begin{aligned} (\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) &= A(X)S(Y,Z) - A(Z)S(Y,X) \\ &+ C(R(X,Z,Y)) + 2D(R(X,Z,Y)). \end{aligned}$$

Therefore

$$\begin{aligned} &2(n-1)[(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X)] \\ &= 2(n-1)A(X)S(Y,Z) - 2(n-1)A(Z)S(Y,X) \\ &+ 2(n-1)C(R(X,Z,Y) + 4(n-1)D(R(X,Z,Y)). \end{aligned} \tag{4.3}$$

From (4.2) and (4.3) we get

$$
\begin{aligned}
&2(n-1)A(X)S(Y,Z) - 2(n-1)A(Z)S(Y,X)\\
&+ 2(n-1)C(R(X,Z,Y)) + 4(n-1)D(R(X,Z,Y))\\
&= 2r[A(X)g(Y,Z) - A(Z)g(X,Y)]\\
&+ [\bar{A}(X)g(Y,Z) - \bar{A}(Z)g(X,Y)]\\
&+ [\bar{B}(X)g(Y,Z) - \bar{B}(Z)g(X,Y)]\\
&+ [\bar{C}(X)g(Y,Z) - \bar{C}(Z)g(X,Y)]\\
&+ [\bar{D}(X)g(Y,Z) - \bar{D}(Z)g(X,Y)]
\end{aligned}
\tag{4.4}
$$

Again in a conformally flat (M^n, g) $(n > 2)$

$$
\begin{aligned}
R(X,Z,Y,W) &= \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Y)g(Z,W)\\
&+ S(X,W)g(Y,Z) - S(Z,W)g(X,Y)]\\
&+ \frac{r}{(n-1)(n-2)}[g(X,Y)g(Z,W) - g(Y,Z)g(X,W)]
\end{aligned}
\tag{4.5}
$$

where

$$
R(X,Z,Y,W) = g[R(X,Z,Y),W]
\tag{4.6}
$$

In virtue of (4.5) we get

$$
\begin{aligned}
&2(n-1)[C(R(X,Z,Y)) + 2D(R(X,Z,Y))]\\
&= \frac{2(n-1)}{n-2}[S(Y,Z)\{C(X) + 2D(X)\} - S(X,Y)\{C(Z) + 2D(Z)\}\\
&+ g(Y,Z)\{\bar{C}(X) + 2\bar{D}(X)\} - g(X,Y)\{\bar{C}(Z) + 2\bar{D}(Z)\}]\\
&+ \frac{2r}{n-2}[g(X,Y)\{C(Z) + 2D(Z)\} - g(Y,Z)\{C(X) + 2D(X)\}].
\end{aligned}
\tag{4.7}
$$

In virtue of (4.7) we can express (4.4) as follows:

$$
\begin{aligned}
& 2(n-1)S(Y,Z)[(n-2)A(X)+C(X)+2D(X)] \\
& -2(n-1)S(X,Y)[(n-2)A(Z)+C(Z)+2D(Z)] \\
& +g(Y,Z)[2(n-1)\bar{C}(X)+4(n-1)\bar{D}(X) \\
& -(n-2)\{\bar{A}(X)+\bar{B}(X)+\bar{C}(X)+\bar{D}(X)\}] \\
& -g(X,Y)[2(n-1)\bar{C}(Z)+4(n-1)\bar{D}(Z) \\
& -(n-2)\{\bar{A}(Z)+\bar{B}(Z)+\bar{C}(Z)+\bar{D}(Z)\}] \\
& =2r[g(Y,Z)\{(n-2)A(X)+2C(X)+4D(X)\} \\
& -g(X,Y)\{(n-2)A(Z)+2C(Z)+4D(Z)\}]
\end{aligned} \tag{4.8}
$$

Putting $Y=P$ in (4.8) we obtain

$$
\begin{aligned}
& (n-1)[2C(X)\bar{A}(Z)-2C(Z)\bar{A}(X)+4D(X)\bar{A}(Z)-4D(Z)\bar{A}(X)] \\
& +(n-1)[2\bar{C}(X)A(Z)-2\bar{C}(Z)A(X)+4\bar{D}(X)A(Z)-4\bar{D}(Z)A(X)] \\
& -(n-2)[A(Z)\{(-2n+1)\bar{A}(X)+\bar{B}(X)+\bar{C}(X)+\bar{D}(X)\} \\
& -A(X)\{(2n+1)\bar{A}(Z)+\bar{B}(Z)+\bar{C}(Z)+\bar{D}(Z)\}] \\
& =2r[2C(X)A(Z)-2C(Z)A(X)+4D(X)A(Z)-4D(Z)A(X)].
\end{aligned} \tag{4.9}
$$

This is the required generalization.

We can therefore state the following theorem:

Theorem 4. *In a conformally flat $G(PS)_n$ the associated and the auxiliary associated 1-forms satisfy the relation (4.9).*

If, in particular, $B=C=D=A$, then the $G(PS)_n$ reduces to a $(PS)_n$ and the relation (4.9) takes the form

$$\bar{A}(X)A(Z)-\bar{A}(Z)A(X)=0,$$

a result already proved by the author elsewhere [1].

5. $G(PS)_n$ admitting a parallel vector field

In this section we suppose that a $G(PS)_n$ admits a parallel vector field V ([3] p. 124, [5] p. 322]).

Then

$$(\nabla_X V)=0 \quad \forall X\in\chi(G(PS)_n). \tag{5.1}$$

Applying Ricci identity to (5.1) we get

$$R(X,Y,V)=0. \tag{5.2}$$

From (5.2) it follows that

$$R(X,Y,Z,V)=0. \tag{5.3}$$

In virtue of (5.3) we get

$$S(X,V)=0. \tag{5.4}$$

Now, by (5.1) and (5.4)

$$(\nabla_X S)(Y,V)=\nabla_X S(Y,V)-S(\nabla_X Y,V)-S(Y,\nabla_X V)=0 \tag{5.5}$$

Again from (1.3) we get by (5.3) and (5.4)

$$\begin{aligned}(\nabla_X S)(Y,V)&=2A(X)S(Y,V)+B(Y)S(X,V)\\&+R(Y,X,\mu,V)+R(Y,\nu,X,Y)+A(V)S(Y,X)=A(V)S(Y,X).\end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we obtain

$$A(V)S(Y,X)=0, \tag{5.7}$$

If $A(V)\neq 0$, i.e., if $g(P,V)\neq 0$, then from (5.7) we get $S(Y,X)=0$. Hence

$$\tilde{C}(X,Y,Z)=R(X,Y,Z),$$

where $\tilde{C}$ is Weyl's conformal curvature tensor. Therefore $\tilde{C}(X,Y,Z)\neq 0$, for otherwise $R(X,Y,Z)$ will be zero implying that the manifold is flat which is inadmissible. Hence we can state the following theorem.

Theorem 5. *If a $G(PS)_n$ admits a parallel vector field which is not orthogonal to the basis vector field P, then the manifold cannot be conformally flat.*

References

[1] M. C. Chaki, On pseudo symmetric manifolds Analele Stiint, *Univ. Al-I. Cuza* **33** (1987), 53–58.

[2] L. Tamássy and T. Q. Binh, On weakly symmetric and weakly projective symmetric Riemannian Manifolds, *Colloq. Math. János Bolyai* **56** (1989), 663–670.

[3] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1., *Interscience Publishers* (1983).

[4] L. P. Eisenhart, Riemannian Geometry. (1926).

[5] J. A. Schouten, Ricci Calculus (2nd Ed.), *Springer-Verlag* (1954).

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