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Generalized derivations which extend the concept of Jordan homomorphism

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Abstract. Let R be a prime ring, U the right Utumi quotient ring of R, C its extended centroid, I a non-zero right ideal of R, $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C, F, G two generalized derivations of R, $m \geq 1$ a fixed integer. Denote $f(I)$ the set of all evaluations of the polynomial $f(x_1,\dots,x_n)$ in I. If $F(u^m) = G(u)^m$, for any $u \in f(I)$, then we describe all possible forms of F and G.

1. Introduction

In all that follows let R be a prime ring, $Z(R)$ the center of R, R be the right Utumi quotient ring of R and $C = Z(U)$ be the center of U. C is usually called the *extended centroid* of R and is a field when R is a prime ring. It should be remarked that U is a centrally closed prime C-algebra.

We recall that an additive map d on R is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. We now also recall the definition of a generalized derivation of R. Let R be an associative ring and d be a derivation of R. An additive map $G : R \longrightarrow R$ is called a *generalized derivation* of R if

$$G(xy) = G(x)y + xd(y)$$

for all $x, y \in R$. For fixed elements a and b of R, the map $G : R \to R$ defined as $G(x) = ax + xb$ for all $x \in R$ is a generalized derivation of R. A generalized derivation of this form is called an *inner* generalized derivation. The definition of generalized derivations is a unified notion of derivations and centralizers, which

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have been investigated by many researchers from various view points (see [14], [20]). We would like to point out that one of the leading roles in the development of the theory of generalized derivations is played by the inner generalized derivations.

We say that an additive map F acts as a homomorphism on a subset $T \subseteq R$, if $F(xy) = F(x)F(y)$, for all $x, y \in T$; F acts as an anti-homomorphism on T, if $F(xy) = F(y)F(x)$, for all $x, y \in T$; finally F acts as a Jordan homomorphism on T if $F(x^2) = F(x)^2$, for all $x \in T$. Obviously any additive map, which is a homomorphism or an anti-homomorphism, is a Jordan homomorphism. On the other hand, in [15] (p. 50) Herstein proves that in case R is a prime ring of characteristic different from 2, any Jordan homomorphism on R is either a homomorphism or an anti-homomorphism of R.

In [5, Theorem 3] Bell and Kappe prove that if d is a derivation of a prime ring R which acts as a homomorphism or anti-homomorphism on a non-zero right ideal of R, then $d = 0$ on R.

In [28] Wang and You extend this result to a Lie ideal L of a prime ring R with characteristic different from 2. They prove that there is no non-zero derivation acting as a homomorphism or anti-homomorphism on L, unless when $L \subseteq Z(R)$.

Later, Rehman (in [27]) and Albaş and Argaç (in [1]) study the case when the derivation d is replaced by a generalized derivation G associated to a derivation d. In both papers it is proved that if $0 \neq G$ acts as a homomorphism or anti-homomorphism on I, a non-zero ideal of the prime ring R, then either $d = 0$ or R is commutative. In particular, if assume that G acts as a homomorphism on I, then either R is commutative or G is the identity map on R. On the other hand, if assume that G acts as an anti-homomorphism on I, then R is commutative.

Many researchers develop the previous mentioned results, by studying derivations and other kinds of additive mappings acting on Lie ideals, two-sided ideals and one-sided ideals of prime and semiprime rings. Most of the obtained results are concerned with homomorphisms, anti-homomorphisms and derivations, see for instance [8], [30].

More recently, in [9] Dhara *et al.* prove the following result

Theorem. *Let R be a prime ring, U the right Utumi quotient ring of R, C its extended centroid, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C, F a non-zero generalized derivation of R, I a non-zero right ideal of R, $m \geq 2$ a fixed integer. Denote $f(I)$ the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in I. If $F(u^m) = F(u)^m$, for any $u \in f(I)$, then one of the following holds:*

(1) *$IC = eRC$ for some idempotent element $e \in \mathrm{soc}(RC)$ and $f(x_1, \dots, x_n)$ is central valued on $eRCe$;*

(2) *there exist $a, b \in U$ and $\alpha, \beta \in C$ such that $F(x) = ax + xb$, $(a - \alpha)I = (b - \beta)I = (0)$, with $(\alpha + \beta)^{m-1} = 1$;*

(3) *there exists $a \in U$ such that $F(x) = ax$ and $aI = (0)$.*

Following this line of investigation, in this paper we will continue the study of generalized derivations of R acting on the elements of a suitable subset S of R.

The main results of this paper are the following:

Theorem 1. *Let R be a prime ring, U the right Utumi quotient ring of R, C its extended centroid, $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C, F, G two generalized derivations of R, $m \geq 1$ a fixed integer. Denote $f(R)$ the set of all evaluations of the polynomial $f(x_1, \dots, x_n)$ in R. If $F(u^m) = G(u)^m$, for any $u \in f(R)$, then one of the following holds:*

1. $\mathrm{char}(R) = 2$ *and $R \subseteq M_2(C)$, the ring of 2×2 matrices over C;*
2. *there exists $\lambda \in C$ such that $F(x) = \lambda^m x$, $G(x) = \lambda x$, for all $x \in R$;*
3. *$f(x_1, \dots, x_n)^m$ is central valued on R and there exist $\lambda \in C$ and $q \in U$, such that $F(x) = \lambda^m x + [q, x]$, $G(x) = \lambda x$, for all $x \in R$.*

Theorem 2. *Let R be a prime ring, U the right Utumi quotient ring of R, C its extended centroid, I a non-zero right ideal of R, $f(x_1, \dots, x_n)$ a multilinear polynomial over C, F, G two generalized derivations of R, $m \geq 1$ a fixed integer. Denote $f(I)$ the set of all evaluations of the polynomial $f(x_1, \dots, x_n)$ in I. If $F(u^m) = G(u)^m$, for any $u \in f(I)$, then one of the following holds:*

1. *There exist $a', c', q' \in U$ and $\beta, \lambda \in C$ such that $F(x) = a'x + \lambda^{m-1}[q', x]$, $G(x) = c'x + [q', x]$, for all $x \in R$, with $(c' - \lambda)I = 0$, $(a' - \lambda^m)I = 0$ and $(q - \beta)I = 0$;*
2. *$IC = eRC$ for some idempotent element $e \in \mathrm{soc}(RC)$ and $f(x_1, \dots, x_n)$ is central valued on $eRCe$;*
3. *$IC = eRC$ for some idempotent element $e \in \mathrm{soc}(RC)$, $\mathrm{char}(R) = 2$ and $eRCe$ satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree 4;*
4. *$IC = eRC$ for some idempotent element $e \in \mathrm{soc}(RC)$, $f(x_1, \dots, x_n)^m$ is central valued on $eRCe$ and there exist $a', b', c' \in U$ and $\lambda \in C$, such that $F(x) = a'x + [b', x]$, $G(x) = c'x$, for all $x \in R$, with $b'e = eb'e$, $(c' - \lambda)e = 0$ and $(a' - \lambda^m)e = 0$;*
5. *$IC = eRC$ for some idempotent element $e \in \mathrm{soc}(RC)$, $f(x_1, \dots, x_n)^m$ is central valued on $eRCe$ and there exist $a', b', c', q' \in U$ and $\beta \in C$, such that*

$F(x) = a'x + [b', x]$, $G(x) = c'x + [q', x]$, for all $x \in R$, with $b'e = eb'e$, $c'e = 0$, $a'e = 0$ and $(q' - \beta)e = 0$.

We also remark that:

Fact 1. Every generalized derivation Δ on a dense right ideal of a semiprime ring R can be extended to U and assumes the form $\Delta(x) = px + h(x)$, for some $p \in U$ and h derivation on U (Theorem 3 in [20]).

Throughout this paper, unless specially stated, R is always a prime ring with center $Z(R)$, right Utumi quotient ring U and extended centroid C, I a non-zero right ideal of R. The definition, axiomatic formulations and properties of this quotient ring can be found in [4] (Chapter 2). Moreover let $f(x_1, \ldots, x_n)$ be a non-central polynomial over C. We will use the following notation:

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$$

for some $\alpha_\sigma \in K$ and S_n the symmetric group of degree n. Moreover we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n), \quad \forall r_1, r_2, \ldots, r_n \in R.$$

We remark that, by Fact 1, we assume that $F(x) = ax + d(x)$ and $G(x) = cx + g(x)$, for some $a, c \in U$ and d, g derivations on U.

Hence we have that R satisfies the generalized differential identity

$$a f(x_1, \ldots, x_n)^m + \sum_{j=0}^{m-1} f(x_1, \ldots, x_n)^j d(f(x_1, \ldots, x_n)) f(x_1, \ldots, x_n)^{m-j-1} - \big(c f(x_1, \ldots, x_n) + g(f(x_1, \ldots, x_n))\big)^m \quad (1)$$

that is

$$a f(x_1, \ldots, x_n)^m + \sum_{j=0}^{m-1} f(x_1, \ldots, x_n)^j \Big(f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n) \Big) f(x_1, \ldots, x_n)^{m-j-1} - \Big(c f(x_1, \ldots, x_n) + f^g(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, g(x_i), \ldots, x_n) \Big)^m . \quad (2)$$

We introduce some preliminary results which will be useful in the sequel.

Proposition 1. *Let $a, c \in U$ and $d, g : R \to R$ derivations of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C and denote*

$$\Phi(x_1, \ldots, x_n) = \big(af(x_1, \ldots, x_n)^m + d(f(x_1, \ldots, x_n)^m)\big) - \big(cf(x_1, \ldots, x_n) + g(f(x_1, \ldots, x_n))\big)^m. \quad (3)$$

If $\Phi(x_1, \ldots, x_n)$ is a generalized polynomial identity for I, then either R is a GPI-ring or $d(I)I = (0)$, $g(I)I = (0)$ and there exists $\lambda \in C$ such that $(c - \lambda)I = (0)$ and $(a - \lambda^m)I = (0)$.

Proof. Assume that R does not satisfy any non-trivial generalized polynomial identity.
Let $u \in I$, then R satisfies

$$\Phi(ux_1, \ldots, ux_n) = \big(af(ux_1, \ldots, ux_n)^m + d(f(ux_1, \ldots, ux_n)^m)\big) - \big(cf(ux_1, \ldots, ux_n) + g(f(ux_1, \ldots, ux_n))\big)^m. \quad (4)$$

In case both d and g are inner derivations, induced respectively by two elements $b, q \in U$, then $F(x) = ax + [b, x] = a'x - xb$ and $G(x) = cx + [q, x] = c'x - xq$, with $a' = a + b$ and $c' = c + q$. Therefore, by (4), R satisfies

$$\big(a'f(ux_1, \ldots, ux_n)^m - f(ux_1, \ldots, ux_n)^m b\big) - \big(c'f(ux_1, \ldots, ux_n) - f(ux_1, \ldots, ux_n)q\big)^m. \quad (5)$$

If $\{1, b, q\}$ are linearly C-independent, by [7], since R does not satisfy any non-trivial generalized polynomial identity and by (5), it follows that R satisfies $f(ux_1, \ldots, ux_n)^m b$, that is $b = 0$, a contradiction. Analogously, if $\{u, a'u, c'u\}$ are linearly C-independent, we get $a'u = 0$, a contradiction.
Hence, we assume that there exist $\alpha, \beta, \gamma, \lambda \in C$, such that

$$q = \alpha + \beta b, \quad c'u = \gamma u + \lambda a'u. \quad (6)$$

Now R satisfies

$$a'f(ux_1, \ldots, ux_n)^m - f(ux_1, \ldots, ux_n)^m b - \big((\gamma + \lambda a')f(ux_1, \ldots, ux_n) - f(ux_1, \ldots, ux_n)(\alpha + \beta b)\big)^m. \quad (7)$$

Assume $\{a'u, u\}$ C-linearly independent. Thus, by (7) R satisfies

$$a'f(ux_1, \ldots, ux_n)\big(f(ux_1, \ldots, ux_n)^{m-1} - \lambda(\lambda a'f(ux_1, \ldots, ux_n) - f(ux_1, \ldots, ux_n)(\alpha + \beta b - \gamma))^{m-1}\big)$$

in particular R satisfies

$$f(ux_1,\dots,ux_n)^{m-1} - \lambda\big(\lambda a' f(ux_1,\dots,ux_n) - f(ux_1,\dots,ux_n)(\alpha+\beta b-\gamma)\big)^{m-1}. \quad (8)$$

Again since R is not a GPI-ring, then (8) is a trivial generalized polynomial identity for R, thus

$$\lambda^2 a' f(ux_1,\dots,ux_n)\big(\lambda a' f(ux_1,\dots,ux_n) - f(ux_1,\dots,ux_n)(\alpha+\beta b-\gamma)\big)^{m-2}$$

is also a trivial generalized polynomial identity for R. Continuing this process we obtain that $\lambda^m a' f(ux_1,\dots,ux_n)$ is satisfied by R, which is a contradiction, unless when $\lambda = 0$. In this last case, by (8) we also have the contradiction $f(ux_1,\dots,ux_n)^{m-1} = 0$.

Hence, there exists $\lambda' \in C$ such that $a'u = \lambda' u$ and (7) reduces to

$$f(ux_1,\dots,ux_n)\big(f(ux_1,\dots,ux_n)^{m-1}(\lambda' - b) - \big((\gamma+\lambda\lambda'-\alpha-\beta b)f(ux_1,\dots,ux_n)\big)^{m-1}(\gamma+\lambda\lambda'-\alpha-\beta b)\big).$$

Since R does not satisfy any non-trivial generalized polynomial identity, then R satisfies

$$f(ux_1,\dots,ux_n)^{m-1}(\lambda' - b) - \big((\gamma+\lambda\lambda'-\alpha-\beta b)f(ux_1,\dots,ux_n)\big)^{m-1}(\gamma+\lambda\lambda'-\alpha-\beta b). \quad (9)$$

Since R is not a GPI-ring, then by (9), $\{(\gamma+\lambda\lambda'-\alpha-\beta b)u, u\}$ are not linearly C-independent, that is there exists $\lambda'' \in C$ such that $(\gamma+\lambda\lambda'-\alpha-\beta b)u = \lambda'' u$.

If $\beta \neq 0$, then $\{bu, u\}$ are C linearly dependent as well as $\{qu, u\}$. In this case, the previous argument shows that for any $u \in I$ any of $\{a'u, u\}$, $\{bu, u\}$, $\{c'u, u\}$, $\{qu, u\}$ are linearly C-dependent, and in this case it is well known that there exist $\mu', \mu'', \nu', \nu'' \in C$ such that $(a' - \mu')I = (0)$, $(b - \mu'')I = (0)$, $(c' - \nu')I = (0)$ and $(q - \nu'')I = (0)$. In particular this means that $d(I)I = (0)$ and $g(I)I = 0$, and also both $d(f(x_1,\dots,x_n))$ and $g(f(x_1,\dots,x_n))$ are differential identities for I.

On the other hand, if $\beta = 0$, then $q = \alpha \in C$ and (9) reduces to

$$f(ux_1,\dots,ux_n)^{m-1}(\lambda' - b) - f(ux_1,\dots,ux_n)^{m-1}\lambda''(\gamma+\lambda\lambda'-\alpha) \quad (10)$$

which implies $b \in C$. Thus, $d = g = 0$, and as above $(a' - \mu')I = (0)$ and $(c' - \nu')I = (0)$.

In any case, I satisfies

$$\big(\mu' f(x_1,\dots,x_n)^m - \nu'^m f(x_1,\dots,x_n))^m\big)$$

that is $\mu' - \nu'^m = 0$, as required.

Thus we suppose in all that follows that at least one of d and g must be not U-inner. In this situation we prove that a number of contradictions occurs.

If $g = 0$, we may assume that d is not U-inner; then, for any $u \in I$, by (4) and by KHARCHENKO's theorem (see [13]) R satisfies

$$\begin{aligned} af(ux_1,\dots,ux_n)^m + \sum_{j=0}^{m-1} f(ux_1,\dots,ux_n)^j \Big(f^d(ux_1,\dots,ux_n) \\ + \sum_i f(ux_1,\dots,ut_i + d(u)x_i,\dots,ux_n)\Big) f(ux_1,\dots,ux_n)^{m-j-1} \\ - \big(cf(ux_1,\dots,ux_n)\big)^m. \end{aligned} \tag{11}$$

In particular,

$$\sum_{j=0}^{m-1} f(ux_1,\dots,ux_n)^j \sum_i f(ux_1,\dots,ut_i,\dots,ux_n) f(ux_1,\dots,ux_n)^{m-j-1}$$

is a non-trivial generalized polynomial identity for R, a contradiction.

Consider now the case $d = 0$. Thus we may assume that g is not U-inner; then, by (4) and KHARCHENKO's theorem (see [13]), for any $u \in I$, R satisfies

$$\begin{aligned} af(ux_1,\dots,ux_n)^m \\ -\Big(cf(ux_1,\dots,ux_n) + f^g(ux_1,\dots,ux_n) + \sum_i f(ux_1,\dots,uz_i{+}g(u)x_i,\dots,ux_n)\Big)^m \end{aligned}$$

and in particular $f(uz_1, ux_2,\dots,ux_n)^m$ is a non trivial generalized polynomial identity for R, which is again a contradiction.

In all that follows we assume both $d \neq 0$ and $g \neq 0$. Now we have the following cases:

Case 1: d and g are C-linear independent modulo X-inner derivations. In this case, by (4) and for any $u \in I$, applying again Kharchenko's theorem, it follows that R satisfies:

$$\begin{aligned} af(ux_1,\dots,ux_n)^m + \sum_{j=0}^{m-1} f(ux_1,\dots,ux_n)^j \Big(f^d(ux_1,\dots,ux_n) + \sum_i f(ux_1,\dots,ut_i \\ + d(u)x_i,\dots,ux_n)\Big) f(ux_1,\dots,ux_n)^{m-j-1} - \Big(cf(ux_1,\dots,ux_n) \end{aligned}$$

$$+ f^g(ux_1,\ldots,ux_n) + \sum_i f(ux_1,\ldots,uz_i+g(u)x_i,\ldots,ux_n)\Big)^m. \qquad (12)$$

As above, R satisfies the blended component $f(uz_1,ux_2,\ldots,ux_n)^m$, and this is a contradiction.

Case 2: d and g are C-linear dependent modulo X-inner derivations.
In this case there exist a non-central element $q\in U$ and $\alpha,\beta\in C$ such that $\alpha d+\beta g=ad(q)$, the inner derivation induced by q.

- If $\alpha=0$, then $g(x)=[x,\beta^{-1}q]$, for all $x\in R$ and d is not an inner derivation. For any $u\in I$, by Kharchenko's theorem (4) reduces to

$$af(ux_1,\ldots,ux_n)^m+\sum_{j=0}^{m-1} f(ux_1,\ldots,ux_n)^j\Big(f^d(ux_1,\ldots,ux_n) + \sum_i f(ux_1,\ldots,ut_i+d(u)x_i,\ldots,ux_n)\Big)f(ux_1,\ldots,ux_n)^{m-j-1} - \Big(cf(ux_1,\ldots,ux_n)+[f(ux_1,\ldots,ux_n),\beta^{-1}q]\Big)^m. \qquad (13)$$

  In particular R satisfies

$$\sum_{j=0}^{m-1} f(ux_1,\ldots,ux_n)^j\Big(\sum_i f(ux_1,\ldots,ut_i,\ldots,ux_n)\Big)f(ux_1,\ldots,ux_n)^{m-j-1} \qquad (14)$$

  which is again a non-trivial generalized polynomial identity, a contradiction.

- If $\beta=0$, then $d(x)=[x,\alpha^{-1}q]$, for all $x\in R$ and g is not an inner derivation. For $u\in I$, by Kharchenko's theorem (4) reduces to

$$af(ux_1,\ldots,ux_n)^m+\big[f(ux_1,\ldots,ux_n)^m,\alpha^{-1}q\big]-\Big(cf(ux_1,\ldots,ux_n) + f^g(ux_1,\ldots,ux_n)+\sum_i f(ux_1,\ldots,uz_i+g(u)x_i,\ldots,ux_n)\Big)^m \qquad (15)$$

  and in particular R satisfies $f(uz_1,ux_2,\ldots,ux_n)^m$. As above, this leads to a contradiction.

- Finally, we analyze the case both $\alpha\neq 0$ and $\beta\neq 0$, hence $g(x)=\gamma d(x)+[x,q']$, for all $x\in U$, where $\gamma=-\alpha\beta^{-1}$ and $q'=\beta^{-1}q$.

  In this case we may assume that d is not an inner derivation. Therefore, by (4) and $u\in I$, R satisfies

$$af(ux_1,\ldots,ux_n)^m+\sum_{j=0}^{m-1} f(ux_1,\ldots,ux_n)^j\Big(f^d(ux_1,\ldots,ux_n)$$

$$+\sum_i f(ux_1,\ldots,d(u)x_i+ud(x_i),\ldots,ux_n)\Big)f(ux_1,\ldots,ux_n)^{m-j-1}$$
$$-\Big(cf(ux_1,\ldots,ux_n)+\gamma f^d(ux_1,\ldots,ux_n)+\sum_i \gamma f(ux_1,\ldots,d(u)x_i$$
$$+ud(x_i),\ldots,ux_n)+[f(ux_1,\ldots,ux_n),q']\Big)^m. \quad (16)$$

Since d is not inner, by Kharchenko's result and (16), it follows that R satisfies

$$af(ux_1,\ldots,ux_n)^m+\sum_{j=0}^{m-1} f(ux_1,\ldots,ux_n)^j\Big(f^d(ux_1,\ldots,ux_n)+$$
$$+\sum_i f(ux_1,\ldots,ut_i,\ldots,ux_n)\Big)f(ux_1,\ldots,ux_n)^{m-j-1}$$
$$-\Big(cf(ux_1,\ldots,ux_n)+\gamma f^d(ux_1,\ldots,ux_n)+\sum_i \gamma f(ux_1,\ldots,ut_i,\ldots,ux_n)$$
$$+[f(x_1,\ldots,x_n),q']\Big)^m$$

and, in particular, for $x_1=0$, $\gamma^m f(ut_1,ux_2\ldots,ux_n)^m$ is a generalized polynomial identity for R, a contradiction again. □

Proposition 2. *Let R be a prime ring, I a non-zero two-sided ideal of R and $f(x_1,\ldots,x_n)$ a non-central valued polynomial over C, the extended centroid of R. If $F:R\to R$ is a generalized derivation associated with a derivation $d:R\to R$ such that $F(f(r_1,\ldots,r_n))\in C$ for all $r_1,\ldots,r_n\in I$, then either* $\mathrm{char}(R)=2$ *and $R\subseteq M_2(C)$, the ring of 2×2 matrices over C, or $F(x)=0$, for all $x\in R$.*

Proof. In light of Fact 1, we have that there exists $a\in U$, the Utumi quotient ring of R, such that $F(x)=ax+d(x)$, for all $x\in R$. Thus I satisfies the generalized differential identity

$$\big[af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n)),x_{n+1}\big].$$

Since by [21], I and R satisfy the same differential identities, then we have that R satisfies $F(f(x_1,\ldots,x_n))\in C$. Let S the additive subgroup generated by the subset

$$\{f(r_1,\ldots,r_n)\mid r_1,\ldots,r_n\in R\}.$$

S is a Lie ideal of R, indeed for any $r\in R$, $y_1,\ldots,y_n\in R$ one has

$$[r,f(y_1,\ldots,y_n)]=\sum_i f(y_1,\ldots,[r,y_i],\ldots,y_n)\in S.$$

If S is non-commutative then, by [15] (page 4-5), either $\mathrm{char}(R) = 2$ and R satisfies the standard identity $s_4(x_1, \ldots, x_4)$ or there exists a non-zero two-sided ideal J of R such that $0 \neq [J, R] \subseteq S$. In the first case, it is well known that $R \subseteq U = M_2(C)$, the 2×2 ring of matrices over C.

In the latter case it is easy to see that $F([r_1, r_2]) \in C$, for all $r_1, r_2 \in J$. In particular $[F(u), u] = 0$ for all $u \in [J, J]$. Since $[J, J]$ is a non-central Lie ideal of R, it follows easily that F must be zero (see for example Theorem 3.3 in [12]).

Hence we may consider the only case when S is commutative.

Thus $[f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)]$ is an identity in R. This means that there exist a field K and a positive integer m such that $[f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)]$ is also an identity in $M_m(K)$. If $m = 1$, R is commutative, thus we suppose $m \geq 2$. Since $f(x_1, \ldots, x_n)$ is not central valued on R, there exist $r_1, \ldots, r_n \in M_m(K)$ such that $f(r_1, \ldots, r_n) = a \notin Z(R)$, so that $[a, f(y_1, \ldots, y_n)]$ is also a generalized identity in $M_m(K)$. By a result of Lee (see [22, Lemma 5 and Theorem 6]), we have the contradiction that $f(x_1, \ldots, x_n)$ is central valued on R. □

As easy consequences we get the following:

Fact 2. Let R be a prime ring, I a non-zero two-sided ideal of R and $f(x_1, \ldots, x_n)$ a non-central polynomial over C, the extended centroid of R. Let $a, b \in U$ be such that $af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b \in C$ for all $r_1, \ldots, r_n \in I$, then either $\mathrm{char}(R) = 2$ and $R \subseteq M_2(C)$ or $a = -b \in C$.

Fact 3. Let R be a prime ring, I a non-zero two-sided ideal of R and $f(x_1, \ldots, x_n)$ a non-central polynomial over C, the extended centroid of R. Let $a \in U$ be such that $[a, f(r_1, \ldots, r_n)] \in C$ for all $r_1, \ldots, r_n \in I$, then either $\mathrm{char}(R) = 2$ and $R \subseteq M_2(C)$ or $a \in C$.

Fact 4. Let R be a prime ring, I a non-zero two-sided ideal of R and $f(x_1, \ldots, x_n)$ a polynomial over C, the extended centroid of R. Assume that $f(x_1, \ldots, x_n)$ is not a polynomial identity for R and let $0 \neq a \in U$ be such that $af(r_1, \ldots, r_n) \in C$ for all $r_1, \ldots, r_n \in I$. Then either $\mathrm{char}(R) = 2$ and $R \subseteq M_2(C)$ or $a \in C$ and $f(x_1, \ldots, x_n)$ is central valued on R.

Proof. If $f(x_1, \ldots, x_n)$ is not central valued on R and since $a \neq 0$, then, by Proposition 2, it follows $\mathrm{char}(R) = 2$ and $R \subseteq M_2(C)$. On the other hand, if $f(x_1, \ldots, x_n)$ is central valued on R, then, for any $x, r_1, \ldots, r_n \in R$, we get

$$0 = [af(r_1, \ldots, r_n), x] = [a, x]f(r_1, \ldots, r_n).$$

Hence, by [6] and since $f(x_1, \ldots, x_n)$ is not an identity for R, it follows $[a, x] = 0$, for all $x \in R$, and we are done. □

2. The case of inner generalized derivations

In this section we study the case when the generalized derivations F, G are inner, induced by the elements $a, b, c, q \in U$, that is, for all $x \in R$, $F(x) = ax + xb$ and $G(x) = cx + xq$. Hence R satisfies the following generalized polynomial identity

$$P(x_1, \ldots, x_n) = (af(x_1, \ldots, x_n)^m + f(x_1, \ldots, x_n)^m b) - \big(cf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)q\big)^m. \quad (17)$$

As a reduction of Proposition 1, the following holds:

Proposition 3. *Assume $P(x_1, \ldots, x_n)$ is generalized polynomial identity for R. Then either R is a GPI-ring or $a, b, c, q \in C$ and $(c+q)^m = a + b$.*

Lemma 1. *Let $R = M_t(C)$, the ring of $t \times t$ matrices over the field C, with $t > 1$, a, b, c, q elements of R such that R satisfies the relation (17). Then $c, q \in Z(R)$ and one of the following holds:*

(1) $\mathrm{char}(R) = 2$ *and R satisfies s_4;*

(2) $f(x_1, \ldots, x_n)^m$ *is central valued on R and $(c+q)^m - (a+b) = 0$;*

(3) $b = (c+q)^m - a \in Z(R)$.

Proof. Since $f(x_1, \ldots, x_n)$ is not central, by Lemma 5 in [19] and Lemma 2 in [24], there exist $u_1, \ldots, u_n \in M_t(C)$ and $\gamma \in C - \{0\}$, such that $f(u_1, \ldots, u_n) = \gamma e_{kl}$, with $k \neq l$. Here e_{kl} denotes the usual matrix unit with 1 in (k, l)-entry and zero elsewhere. Moreover, since the set $\{f(v_1, \ldots, v_n) : v_1, \ldots, v_n \in M_t(C)\}$ is invariant under the action of all C-automorphisms of $M_t(C)$, for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_t(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$.

Say $c = \sum_{rs} c_{rs} e_{rs}$ and $q = \sum_{rs} q_{rs} e_{rs}$, where $c_{rs}, q_{rs} \in C$. In (17) assume that $f(x_1, \ldots, x_n) = e_{ij}$, then

$$\big(ce_{ij} + e_{ij}q\big)^m = 0$$

which implies both c and q are diagonal matrices in R.

Let φ be the inner automorphism on $M_t(C)$ defined as follows:

$$\varphi(x) = (1 + e_{ij})x(1 - e_{ij}) = x + e_{ij}x - xe_{ij} - e_{ij}xe_{ij}, \quad i \neq j.$$

Since the set $\{f(r_1, \ldots, r_n) \mid r_i \in R\}$ is invariant under the action of φ, the elements $\varphi(a)$, $\varphi(b)$, $\varphi(c)$ and $\varphi(q)$ must satisfy the same conditions which are satisfied by a, b, c and q. Thus, if denote $\varphi(c) = \sum c'_{rs} e_{rs}$ and $\varphi(q) = \sum q'_{rs} e_{rs}$, with $c'_{rs}, q'_{rs} \in C$, we have that both $c'_{lm} = 0$ and $q'_{lm} = 0$, for all $l \neq m$. Easy computations show that $c_{ll} = c_{mm}$ and $q_{ll} = q_{mm}$, for all $l \neq m$, that is both $c \in Z(R)$ and $q \in Z(R)$. If denote $\lambda = c + q$, then R satisfies

$$(a - \lambda^m) f(x_1, \ldots, x_n)^m + f(x_1, \ldots, x_n)^m b.$$

By Fact 2, it follows that one of the following holds:

(1) $\mathrm{char}(R) = 2$ and R satisfies s_4;

(2) $f(x_1, \ldots, x_n)^m$ is central valued on R and $\lambda^m = a + b$;

(3) $b = \lambda^m - a \in Z(R)$. □

Proposition 4. *Let R be a prime ring, a, b, c, q elements of R such that R satisfies the relation* (17). *Then $c, q \in C$ and one of the following holds:*

(1) $\mathrm{char}(R) = 2$ *and R satisfies s_4;*

(2) $f(x_1, \ldots, x_n)^m$ *is central valued on R and* $(c + q)^m = a + b$;

(3) $b = (c + q)^m - a \in C$.

Proof. By Proposition 3, $P(x_1, \ldots, x_n)$ is a non-trivial generalized polynomial identity for R. Moreover U and $U \otimes_C \overline{C}$ are both centrally closed algebras ([10], Theorems 2.5 and 3.5) and, in case C is infinite, they satisfy the same generalized polynomial identities.

Hence, replacing R by U or $U \otimes_C \overline{C}$, as well as C is finite or infinite, we may assume, without loss of generality, $C = Z(R)$ and R is a C-algebra centrally closed. By Martindale's theorem in [26], R is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over C.

Consider the case $\dim_C(V) = t$, with t finite positive integer ≥ 2. In this condition R is a simple ring which satisfies a non-trivial generalized polynomial identity, moreover $M_t(C)$ satisfies the same generalized identity of R and we get the conclusion by Lemma 1.

Let now $\dim_C V = \infty$. Since the set $\{f(r_1, \ldots, r_n) \mid r_1, \ldots, r_n \in R\}$ is dense in R, from (17) we have

$$(aX^m + X^m b) - (cX + Xq)^m \tag{18}$$

for all $X \in R$ (see Lemma 2 in [29]). Moreover, eHe is a simple central algebra finite dimensional over C, for any minimal idempotent element $e \in H = \mathrm{soc}(R)$. We may assume H non-commutative, otherwise also R must be commutative. Notice that H satisfies relation (18) (see for example [17, proof of Theorem 1]).

Since H is a simple ring then one of the following holds: either H does not contain any non-trivial idempotent element or H is generated by its idempotents.

In this last case, let $e^2 = e \in H$ and replace in (18) X with $ex(1-e)$. Thus $(cex(1-e)+ex(1-e)q)^m = 0$. Right multiplying by e, it follows $(ex(1-e)q)^m e = 0$, and a fortiori $(x(1-e)qe)^{m+1} = 0$, for all $x \in R$. By [11], we get $(1-e)qe = 0$, for any idempotent $e \in H$.

This implies that, for any idempotent element of rank 1, $qe = eqe$. In a similar way we may prove $eq = eqe$. Hence $[q, e] = 0$, for any idempotent element of rank 1, and $[q, H] = 0$, since H is generated by these idempotent elements. This argument gives that $q \in C$. Analogously, left multiplying $(cex(1-e)+ex(1-e)q)^m = 0$ by $(1-e)$, we have $(1-e)(cex(1-e))^m = 0$ and so $((1-e)cex)^{m+1} = 0$. As above $(1-e)ce = 0$ for any idempotent $e \in H$, then $c \in C$.

Denote $c + q = \lambda \in C$, then H satisfies $(a - \lambda^m)X^m + X^m b$. By Fact 2, and since H does not satisfy any polynomial identity, it follows $b = \lambda^m - a \in C$.

Assume now H does not contain idempotent elements, then H is a finite dimensional division algebra over C. If C is finite then H is a finite division ring, that is H is a commutative field and so R is commutative too.

If C is infinite then $H \otimes_C K \cong M_r(K)$, where K is a splitting field of H. In this case, a Vandermonde determinant argument shows that (18) is still an identiy for $M_r(K)$. As above one can see that if $r \geq 2$ then $c, q \in C$ and $b = \lambda^m - a \in C$. □

3. Generalized derivations in prime rings and semiprime rings

We recall that, by [21, Theorem 3], U satisfies the differential identity (2). We first analyze the special cases when at least one of F and G is zero:

Fact 5. If $G = 0$ then one of the following holds:

(1) $F = 0$;

(2) $\mathrm{char}(R) = 2$ and $R \subseteq M_2(C)$, the ring of 2×2 matrices over C;

(3) $f(x_1, \ldots, x_n)^m$ is central valued on R and there exists $q \in U$, such that $F(x) = [q, x]$, for all $x \in R$.

Proof. By Proposition 2, either $F = 0$, or $\mathrm{char}(R) = 2$ and $R \subseteq M_2(C)$, or $f(x_1, \ldots, x_n)^m$ is central valued on R. In this last case, U is a central simple algebra, finite dimensional over its center, and there exists $t \geq 1$ such that $U = M_t(C)$, the ring of $t \times t$ matrices over C. Moreover, since U satisfies $F(f(x_1, \ldots, x_n)^m)$, then $af(x_1, \ldots, x_n)^m + d(f(x_1, \ldots, x_n)^m)$ is a differential identity for U. Therefore, for all $r_1, \ldots, r_n \in U$,

$$af(r_1, \ldots, r_n)^m = -d(f(r_1, \ldots, r_n)^m) \in C \tag{19}$$

which implies $a \in C$, since $f(x_1, \dots, x_n)^m$ is central valued on R.

In case d is an inner derivation of U, then there exists an element $q \in U$ such that $d(x) = [q, x]$, for all $x \in R$. Therefore $d(f(r_1, \dots, r_n)^m) = 0$, for all $r_1, \dots, r_n \in U$, since $f(x_1, \dots, x_n)^m$ is central valued on U.

Thus $af(r_1, \dots, r_n)^m = -d(f(r_1, \dots, r_n)^m) = 0$, for all $r_1, \dots, r_n \in U$. Since $f(x_1, \dots, x_n)^m$ is central valued and $f(x_1, \dots, x_n)$ is not an identity for U, it follows that $a = 0$ and $F(x) = d(x) = [q, x]$, for all $x \in R$.

We now assume that d is not inner. Since U satisfies

$$af(x_1, \dots, x_n)^m + \sum_{j=0}^{m-1} f(x_1, \dots, x_n)^j \big(f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n)\big) f(x_1, \dots, x_n)^{m-j-1} \quad (20)$$

then, by KHARCHENKO's theorem ([13]), it follows that U satisfies

$$af(x_1, \dots, x_n)^m + \sum_{j=0}^{m-1} f(x_1, \dots, x_n)^j \Big(f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, t_i, \dots, x_n)\Big) f(x_1, \dots, x_n)^{m-j-1}. \quad (21)$$

Therefore the blended component

$$\sum_{j=0}^{m-1} f(x_1, \dots, x_n)^j \Big(\sum_i f(x_1, \dots, t_i, \dots, x_n)\Big) f(x_1, \dots, x_n)^{m-j-1} \quad (22)$$

is a polynomial identity for U. In particular, for $t_1 = x_1$ and any $t_i = 0$ (for all $i \neq 1$) in relation (22), we have that U satisfies $m \cdot f(x_1, \dots, x_n)^m$. Since $f(x_1, \dots, x_n)$ is not an identity for U, it follows that $\mathrm{char}(U) = m$. Hence, application of Theorem 10 in [24] implies the contradiction that $f(x_1, \dots, x_n)$ is central valued on U, unless when $\mathrm{char}(U) = 2$ and $t = 2$, that is $\mathrm{char}(R) = 2$ and $R \subseteq M_2(C)$, as required. □

Fact 6. If $F = 0$ then $G = 0$.

PROOF. By our assumption, $G(f(x_1, \dots, x_n))^m$ is a generalized differential polynomial identity for R. In this case, by [25, Thoerem 3] and since $f(x_1, \dots, x_n)$ is not central valued on R, we get the required conclusion. □

Remark 1. We would like to point out that the results obtained in Facts 5 and 6 are implicitely contained in the conclusions of Theorem 1.

3.1. The proof of Theorem 1. In case both d and g are inner derivations, induced respectively by two elements $b, q \in U$, then we have $F(x) = ax + [b, x] = (a+b)x - xb$ and $G(x) = cx + [q, x] = (c+q)x - xq$. Therefore the conclusion follows by main theorem of previous section.

Thus we suppose in all that follows that at least one of d and g must be not U-inner. In this situation we prove that a number of contradictions occurs.

If $g = 0$, we may assume that d is not U-inner; then, by (2) and Kharchenko's theorem ([13]) U satisfies

$$af(x_1,\dots,x_n)^m + \sum_{j=0}^{m-1} f(x_1,\dots,x_n)^j \Big(f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,t_i,\dots,x_n)\Big) f(x_1,\dots,x_n)^{m-j-1} - \big(cf(x_1,\dots,x_n)\big)^m \tag{23}$$

In particular, U satisfies the blended component

$$\sum_{j=0}^{m-1} f(x_1,\dots,x_n)^j \sum_i f(x_1,\dots,t_i,\dots,x_n) f(x_1,\dots,x_n)^{m-j-1}$$

Consider a non-central element $b \in U$ and replace t_i by $[b, x_i]$ for any $i = 1,\dots,n$. Then, for all $x_1,\dots,x_n \in U$

$$\sum_{j=0}^{m-1} f(x_1,\dots,x_n)^j [b, f(x_1,\dots,x_n)] f(x_1,\dots,x_n)^{m-j-1} = 0$$

that is $[b, f(x_1,\dots,x_n)^m] = 0$, for all $x_1,\dots,x_n \in U$. By Fact 3 and since $b \notin C$, we have that either $\mathrm{char}(U) = 2$ and $U = M_2(C)$, or $f(x_1,\dots,x_n)^m$ is central valued on U. In this last case, U is a central simple algebra, finite dimensional over its center, and there exists $t \geq 1$ such that $U = M_t(C)$, the ring of $t \times t$ matrices over C.

Since $f(x_1,\dots,x_n)$ is not central, by Lemma 5 in [19] and Lemma 2 in [24], for any $k \neq l$, there exist $u_1,\dots,u_n \in M_t(C)$ and $\gamma \in C - \{0\}$, such that $f(u_1,\dots,u_n) = \gamma e_{kl}$. Thus, for $g = 0$ and $f(u_1,\dots,u_n) = \gamma e_{kl}$ in relation (1), we have $(ce_{kl})^m = 0$, and a standard argument shows that $c \in C$. Hence, again by (1), since both $f(x_1,\dots,x_n)^m$ and $d(f(x_1,\dots,x_n)^m)$ are central matrices, we get $(c^m - a)f(x_1,\dots,x_n)^m \in C$. By Fact 4 it follows $c^m - a = \lambda \in C$. Thus $a \in C$ and $M_t(C)$ satisfies

$$\lambda f(x_1,\dots,x_n)^m = d\big(f(x_1,\dots,x_n)^m\big). \tag{24}$$

By using the same argument in Fact 5 (see relation (19)), one may prove that $\lambda = 0$ and d is an inner derivation of $M_t(C)$. Hence $a, c \in C$ and $c^m = a$ and $d(x) = [q, x]$, for all $x \in R$ and suitable $q \in U$, as required.

Consider now the case $d = 0$. Thus we may assume that g is not U-inner; then, by (2) and Kharchenko's theorem (see [13]) U satisfies

$$af(x_1,\dots,x_n)^m - \Big(cf(x_1,\dots,x_n) + f^g(x_1,\dots,x_n) + \sum_i f(x_1,\dots,z_i,\dots,x_n)\Big)^m$$

and in particular $f(z_1, x_2, \dots, x_n)^m$ is a polynomial identity for U, which leads to the contradiction that $f(x_1, \dots, x_n)$ is a polynomial identity for U (see Main Theorem in [6]).

In all that follows we assume both $d \neq 0$ and $g \neq 0$. Now we have the following cases:

Case 1: d and g are C-linear independent modulo X-inner derivations.
In this case, by (2) and applying again Kharchenko's theorem, it follows that U satisfies:

$$\begin{aligned} af(x_1,\dots,x_n)^m &+ \sum_{j=0}^{m-1} f(x_1,\dots,x_n)^j \Big(f^d(x_1,\dots,x_n) \\ &+ \sum_i f(x_1,\dots,t_i,\dots,x_n)\Big) f(x_1,\dots,x_n)^{m-j-1} \\ &- \Big(cf(x_1,\dots,x_n) + f^g(x_1,\dots,x_n) + \sum_i f(x_1,\dots,z_i,\dots,x_n)\Big)^m. \end{aligned} \tag{25}$$

As above, U satisfies the blended component $f(z_1, x_2, \dots, x_n)^m$, which is a contradiction again.

Case 2: d and g are C-linear dependent modulo X-inner derivations.
In this case there exist a non-central element $q \in U$ and $\alpha, \beta \in C$ such that $\alpha d + \beta g = ad(q)$, the inner derivation induced by q.

- If $\alpha = 0$, then $g(x) = [x, \beta^{-1} q]$, for all $x \in U$ and d is not an inner derivation. By Kharchenko's theorem (2) reduces to

$$\begin{aligned} af(x_1,\dots,x_n)^m &+ \sum_{j=0}^{m-1} f(x_1,\dots,x_n)^j \Big(f^d(x_1,\dots,x_n) \\ &+ \sum_i f(x_1,\dots,t_i,\dots,x_n)\Big) f(x_1,\dots,x_n)^{m-j-1} \\ &- \big(cf(x_1,\dots,x_n) + [f(x_1,\dots,x_n), \beta^{-1} q]\big)^m. \end{aligned} \tag{26}$$

In particular U satisfies

$$\sum_{j=0}^{m-1} f(x_1,\dots,x_n)^j \left(\sum_i f(x_1,\dots,t_i,\dots,x_n)\right) f(x_1,\dots,x_n)^{m-j-1}. \tag{27}$$

Consider a non-central element $b \in U$, then in (27) replace any t_i with $[b, x_i]$. Hence U satisfies $[b, f(x_1,\dots,x_n)^m]$. Since $b \notin C$, by Fact 3 it follows that either $\mathrm{char}(U) = 2$ and $U = M_2(C)$ or $f(x_1,\dots,x_n)^m$ is central valued on U. In this last case, U is a central simple algebra, finite dimensional over its center, and there exists $t \geq 1$ such that $U = M_t(C)$, the ring of $t \times t$ matrices over C.

Since $f(x_1,\dots,x_n)$ is not central, by Lemma 5 in [19] and Lemma 2 in [24], for any $i \neq j$, there exist $u_1,\dots,u_n \in M_t(C)$, such that $f(u_1,\dots,u_n) = e_{ij}$. Hence by (26) we have

$$\left(ce_{ij} + [e_{ij}, \beta^{-1}q]\right)^m = 0. \tag{28}$$

In particular, right multiplying by e_{ij}, it follows $(e_{ij}(\beta^{-1}q))^m e_{ij} = 0$ for all $i \neq j$, which implies that q is a diagonal matrix. As above, a standard argument shows that $q \in C$, that is $g = 0$, a contradiction.

- If $\beta = 0$, then $d(x) = [x, \alpha^{-1}q]$, for all $x \in U$ and g is not an inner derivation. By Kharchenko's theorem (2) reduces to

$$af(x_1,\dots,x_n)^m + \left[f(x_1,\dots,x_n)^m, \alpha^{-1}q\right] - \left(cf(x_1,\dots,x_n) + f^g(x_1,\dots,x_n) + \sum_i f(x_1,\dots,z_i,\dots,x_n)\right)^m \tag{29}$$

  and in particular U satisfies $f(z_1,x_2,\dots,x_n)^m$. As above, this leads to the contradiction that $f(x_1,\dots,x_n)$ is a polynomial identity for$U$.

- Finally, we analyze the case both $\alpha \neq 0$ and $\beta \neq 0$, hence $g(x) = \gamma d(x) + [x, q']$, for all $x \in U$, where $\gamma = \alpha^{-1}\beta$ and $q' = \alpha^{-1}q$.

  In this case we may assume that d is not an inner derivation. Therefore, by (2), U satisfies

$$af(x_1,\dots,x_n)^m + \sum_{j=0}^{m-1} f(x_1,\dots,x_n)^j \Bigg(f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,d(x_i),\dots,x_n)\Bigg) f(x_1,\dots,x_n)^{m-j-1} - \Bigg(cf(x_1,\dots,x_n) + \gamma f^d(x_1,\dots,x_n) + \sum_i \gamma f(x_1,\dots,d(x_i),\dots,x_n)$$

$$+ [f(x_1,\ldots,x_n),q']\Big)^m. \tag{30}$$

Since d is not inner, by Kharchenko's result and (30), it follows that U satisfies

$$af(x_1,\ldots,x_n)^m + \sum_{j=0}^{m-1} f(x_1,\ldots,x_n)^j \Big(f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,t_i,\ldots,x_n)\Big) f(x_1,\ldots,x_n)^{m-j-1} - \Big(cf(x_1,\ldots,x_n) + \gamma f^d(x_1,\ldots,x_n) + \sum_i \gamma f(x_1,\ldots,t_i,\ldots,x_n) + [f(x_1,\ldots,x_n),q']\Big)^m$$

and, in particular, for $x_1 = 0$, $\gamma^m f(t_1, x_2 \ldots, x_n)^m$ is a polynomial identity for U. Since $\gamma \neq 0$, as above we get a contradiction.

By using standard arguments, one may obtain the following results as easy consequence of Theorem 1:

Theorem 3. *Let R be a prime ring, U the right Utumi quotient ring of R, C its extended centroid, L a non-central Lie ideal of R, F and G two generalized derivations of R, $m \geq 1$ a fixed integer. If $F(u^m) = G(u)^m$, for any $u \in L$, then one of the following holds:*

(1) *$F = G = 0$;*

(2) $\mathrm{char}(R) = 2$ *and $R \subseteq M_2(C)$, the ring of 2×2 matrices over C;*

(3) *there exists $\lambda \in C$ such that $F(x) = \lambda^m x$, $G(x) = \lambda x$, for all $x \in R$;*

(4) *$R \subseteq M_2(C)$ and there exist $\lambda \in C$ and a suitable derivation $d : R \to R$, such that $F(x) = \lambda^m x + d(x)$, $G(x) = \lambda x$, for all $x \in R$.*

Theorem 4. *Let R be a prime ring, U the right Utumi quotient ring of R, C its extended centroid, I a non-central ideal of R, F and G two generalized derivations of R, $m \geq 1$ a fixed integer. If $F(u^m) = G(u)^m$, for any $u \in I$, then either $F = G = 0$ or there exists $\lambda \in C$ such that $F(x) = \lambda^m x$, $G(x) = \lambda x$, for all $x \in R$.*

Theorem 5. *Let R be a prime ring, U the right Utumi quotient ring of R, C its extended centroid, $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C, F a non-zero generalized derivation of R, $m \geq 1$ a fixed integer. Denote $f(R)$ the* *set of all evaluations of the polynomial* $f(x_1,\dots,x_n)$ *in* R. *If* $F(u^m)=F(u)^m$, *for any* $u\in f(R)$, *then one of the following holds:*

(1) $\mathrm{char}(R)=2$ *and* $R\subseteq M_2(C)$, *the ring of* 2×2 *matrices over* C;

(2) *there exists* $\lambda\in C$ *such that* $F(x)=\lambda^m x$, *for all* $x\in R$, *with* $\lambda^{m-1}=1$.

Theorem 6. *Let* R *be a prime ring,* U *the right Utumi quotient ring of* R, C *its extended centroid,* $f(x_1,\dots,x_n)$ *a non-central multilinear polynomial over* C, F *a non-zero generalized derivation of* R, $m\geq 1$ *a fixed integer. Denote* $f(R)$ *the set of all evaluations of the polynomial* $f(x_1,\dots,x_n)$ *in* R. *If* F *acts as a Jordan homomorphism on* $f(R)$ *then* F *is the identity map on* R, *unless when* $\mathrm{char}(R)=2$ *and* $R\subseteq M_2(C)$.

We would like to conclude this section with the following generalization to semiprime rings. We first recall the following:

Remark 2. Let R be a prime ring, d a derivation of R and $F(x)=ax+d(x)$ a generalized derivation of R. If $F(x)=0$, for all $x\in R$, then $a=0$ and $d(R)=0$.

Theorem 7. *Let* R *be a semiprime ring,* U *the right Utumi quotient ring of* R, C *its extended centroid,* F *and* G *two non-zero generalized derivations of* R, $m\geq 1$ *a fixed integer. If* $F(u^m)=G(u)^m$, *for any* $u\in R$, *then either* R *contains a non-zero central ideal or there exists* $\lambda\in C$ *such that* $F(x)=\lambda^m x$, $G(x)=\lambda x$, *for all* $x\in R$.

PROOF. By Fact 1, we assume that $F(x)=ax+d(x)$ and $G(x)=cx+g(x)$, for some $a,c\in U$ and d,g derivations on U.

Hence we have that R satisfies the generalized differential identity

$$ax^m+d(x^m)-\big(cx+g(x)\big)^m. \tag{31}$$

By [21, Theorem 3], U satisfies the differential identity (31). Now let B the boolean algebra of central idempotents of U. Let M any maximal ideal of B, then MU is a prime ideal of U, which is invariant under d and $\bigcap_M MU=0$ (see [2, Lemma 1, Theorem 1]). Let $\overline{d_M}$ and $\overline{g_M}$ be the derivations induced by d and g on U/MU.

So one has

$$\overline{a}\overline{r^m}+\overline{d_M}(\overline{r^m})-\big(\overline{cr}+\overline{g_M}(\overline{r})\big)^m=0$$

for all $\overline{r}\in U/MU$. Therefore, by Theorem 4 and Remark 2, one of the following holds:

(1) either both $\overline{d_M}$ and $\overline{g_M}$ are zero derivations on U/MU, that is $d(U)\subseteq MU$ and $g(U)\subseteq MU$. Moreover $\overline{a}\in Z(\overline{U})$ and $\overline{b}\in Z(\overline{U})$, that is $[a,U]\subseteq MU$ and $[b,U]\subseteq MU$;

(2) or U/MU is commutative, that is $[U,U] \subseteq MU$.

In any case we have $[d(U),U] \subseteq \bigcap MU = 0$, $[g(U),U] \subseteq \bigcap MU = 0$, $[a,U] \subseteq \bigcap MU = 0$ and $[b,U] \subseteq \bigcap MU = 0$.

If either $d \neq 0$ or $g \neq 0$, then by [18], R contains a non-zero central ideal.

Assume both $d = 0$ and $g = 0$. Thus U satisfies $ax^m - (bx)^m$, and since $a, b \in C$, one has that U satisfies $(a - b^m)x^m$. In particular, by replacing x with $y(a - b^m)$, it follows that U satisfies $((a - b^m)x)^{m+1}$. Therefore $(a - b^m)U$ is a nil right ideal of U. If $(a - b^m)U \neq 0$, then by [16, Lemma 2.1.1], R has a non-zero nilpotent right ideal, which is a contradiction because of the semiprimeness of U. Thus $(a - b^m)U = 0$, and we conclude that $a = b^m$. $\square$

4. Generalized derivations on right ideals

Here we assume that the right ideal I of R satisfies

$$\big(af(x_1,\dots,x_n)^m + d(f(x_1,\dots,x_n)^m)\big) - \big(cf(x_1,\dots,x_n) + g(f(x_1,\dots,x_n))\big)^m. \quad (32)$$

Remark 3. In all that follows we write the polynomial $f(x_1,\dots,x_n)$ by using the following notation:

$$f(x_1,\dots,x_n) = \sum_i g_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)x_i$$

where any g_i is a multilinear polynomial of degree $n-1$ and x_i never appears in any monomial of g_i. Note that if there exists an idempotent $e \in H = \mathrm{Soc}(U)$ such that any g_i is a polynomial identity for eHe, then we get the conclusion that $f(x_1,\dots,x_n)$ is a polynomial identity for eHe.

Thus, if one assumes that $f(x_1,\dots,x_n)$ is not a polynomial identity for eHe, then there exists an index i and $r_1,\dots,r_{n-1} \in eHe$ such that $g_i(r_1,\dots,r_{n-1}) \neq 0$. Now let $f(x_1,\dots,x_n) = g_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)x_i + h(x_1,\dots,x_n)$ where g_i and h are multilinear polynomials, x_i never appears in any monomials of g_i and x_i never appears as last variable in any monomials of h. Without loss of generality we assume $i = n$, say $g_n(x_1,\dots,x_{n-1}) = t(x_1,\dots,x_{n-1})$ and so $f(x_1,\dots,x_n) = t(x_1,\dots,x_{n-1})x_n + h(x_1,\dots,x_n)$ where $t(eHe) \neq 0$.

We also would like to recall a well know result in literature:

Fact 7. Let R be a prime ring, I a non-zero right ideal of R, d a derivation of R such that $d(I)I = 0$. Then there exists $q \in U$ and $\alpha \in C$ such that $d(x) = [q, x]$ for all $x \in R$ and $(q - \alpha)I = 0$.

Proof. Let d be an inner derivation induced by the element $q \in Q$ and $x, y \in I$, $r \in R$. Then $[q, xr]y = 0$, whence $qxry - xrqy = 0$, that is $qx = \beta_x x$, with $\beta_x \in C$, and analogously $qy = \beta_y y$, $q(x + y) = \beta_{x+y}(x + y)$. From this, it is easy to see that β_x is independent from the choice of $x \in I$, therefore there exists $\beta \in C$ such that $(q - \beta)I = 0$.

Let now d an outer derivation. Then, for any $0 \neq c \in I$, R satisfies $d(cx)cy = d(c)xcy + cd(x)cy$. By using Kharchenko's result in [13], it follows that R satisfies $d(c)x_1cx_2 + cx_3cx_2$ and in particular R satisfies the blended component cx_3cx_2. This means, since R is prime, that $c = 0$, a contradiction. □

4.1. The proof of Theorem 2. Of course, in case $f(x_1, \dots, x_n)x_{n+1}$ is an identity for I, we are done. Thus, in all that follows, we assume that I does not satisfy $f(x_1, \dots, x_n)x_{n+1}$.

We also suppose that the following hold simultaneously:

(1) either $d(I)I \neq (0)$, or $g(I)I \neq (0)$, or $[c, I]I \neq 0$, or $(a - c^m)I \neq 0$.

(2) $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ is not an identity for I;

(3) if $\mathrm{char}(R) = 2$, $s_4(x_1, \dots, x_4)x_5$ is not an identity for I;

(4) either $[f(x_1, \dots, x_n)^m, x_{n+1}]x_{n+2}$ is not an identity for I, or $g(I)I \neq 0$, or $[c, I]I \neq 0$, or $(a - c^m)I \neq 0$;

We proceed to derive a contradiction. By Lemma 1, we may assume that R is a GPI ring, so is also U (see [3] and [7]). By [26] U is a primitive ring with $H = \mathrm{Soc}(U) \neq 0$, moreover we may assume that $f(x_1, \dots, x_n)x_{n+1}$ is not an identity for IH, otherwise, by [3] and [7], it should be an identity also for IU, which is a contradiction. Let $a_1, \dots, a_{n+2} \in IH$ such that $[f(a_1, \dots, a_n), a_{n+1}]a_{n+2} \neq 0$.

Analogously, we assume that there exist $b_1, \dots, b_5$; $c_1, \dots, c_{n+7}$; $h_1, \dots, h_7 \in IH$ such that

- either $d(h_1)h_2 \neq 0$, or $g(h_3)h_4 \neq 0$, or $[c, h_5]h_6 \neq 0$, or $(a - c^m)h_7 \neq 0$.
- $s_4(b_1, \dots, b_4)b_5 \neq 0$ if $\mathrm{char}(H) = 2$;
- either $[f(c_1, \dots, c_n)^m, c_{n+1}]c_{n+2} \neq 0$, or $g(c_{n+3})c_{n+4} \neq 0$, or $[c, c_{n+5}]c_{n+6} \neq 0$, or $(a - c^m)c_{n+7} \neq 0$;

Since H is a regular ring, exists $e^2 = e \in H$ such that

$$eH = \sum_{i=1}^{n+2} a_i H + \sum_{j=1}^{5} b_j H + \sum_{k=1}^{n+7} c_k H + \sum_{l=1}^{7} h_l H$$

where $a_i = ea_i$, $b_j = eb_j$, $c_k = ec_k$, $h_l = eh_l$, for all $i = 1, \ldots, n+2$, $j = 1, \ldots, 5$, $k = 1, \ldots, n+7$ and $l = 1, \ldots, 7$.

By our assumption and by [21, Theorem 2] we also assume that (32) is an identity for IU. In particular (32) is an identity for eH. It follows that, for all $r_1, \ldots, r_n \in H$,

$$\big(af(er_1, \ldots, er_n)^m + d(f(er_1, \ldots, er_n)^m)\big) \\ - \big(cf(er_1, \ldots, er_n) + g(f(er_1, \ldots, er_n))\big)^m. \quad (33)$$

As we said above, write $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_{n-1})x_n + h(x_1, \ldots, x_n)$, where x_n never appears as last variable in any monomials of h.

Let $r \in H$ and pick $r_n = er(1-e)$. Hence we have $f(er_1, \ldots, er_n(1-e)) = t(er_1, \ldots, er_{n-1})er_n(1-e)$, and by (33) it follows

$$\big(ct(er_1, \ldots, er_{n-1})er_n(1-e) + g(t(er_1, \ldots, er_{n-1})er_n(1-e))\big)^m \quad (34)$$

and left multiplying by $(1-e)$, one has

$$(1-e)(ce + g(e))e\big(t(er_1, \ldots, er_{n-1})er_n\big)^{m+1} = 0$$

that is H satisfies

$$(1-e)(ce + g(e))e\big(t(ex_1, \ldots, ex_{n-1})eH\big)^{m+1}.$$

By [11], H satisfies $(1-e)(ce+g(e))t(ex_1, \ldots, ex_{n-1})eH$, that is $(1-e)(ce+g(e))t(er_1, \ldots, er_{n-1})e = 0$, for all $r_1, \ldots, r_n \in H$. Since eHe is a simple artinian ring and $t(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [6, Lemma 2], $(1-e)(ce+g(e)) = 0$ and so $G(e) = ce + g(e) = ece + eg(e) \in eH$. Thus $G(eH) \subseteq eH$. Therefore the generalized derivation G induces another one $\overline{G}$, which is defined in the prime ring $\overline{IH} = \frac{IH}{IH \cap l_H(IH)}$, where $l_H(IH)$ is the left annihilator in H of IH, and $\overline{G}(\overline{x}) = \overline{G(x)}$, for all $x \in IH$.

On the other hand, left multiplying (33) by $(1-e)$, and since $(1-e)(ce+g(e)) = 0$, we also have that H satisfies

$$(1-e)\big(aef(ex_1, \ldots, ex_n)^m + d(e)f(ex_1, \ldots, ex_n)^m)\big)$$

that is $(1-e)(ae+d(e))ef(r_1e, \ldots, r_ne)^m = 0$, for all $r_1, \ldots, r_n \in H$. Again by [6], and since $f(x_1, \ldots, x_n)x_{x+1}$ is not an identity for eH, we get $(1-e)(ae+d(e))e = 0$. By using the same above argument, it follows $F(eH) \subseteq eH$.

Moreover we obviously have that (32) is a differential identity for $\overline{eH}$. So, the application of Theorem 1 to the prime ring $\overline{eH}$ implies that one of the following holds:

(1) $d(eH)eH = (0)$ and $g(eH)eH = (0)$, $[c, eH]eH = 0$ and $(a-c^m)eH = 0$. This is a contradiction since either $d(h_1)h_2 \neq 0$, or $g(h_3)h_4 \neq 0$, or $[c, h_5]h_6 \neq 0$, or $(a - c^m)h_7 \neq 0$;

(2) $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ is an identity for eH. This contradicts with $[f(a_1, \dots, a_n), a_{n+1}]a_{n+2} \neq 0$.

(3) $\mathrm{char}(H) = 2$ and eH satisfies the standard identiy $s_4(x_1, \dots, x_4)x_5$, which contradicts with $s_4(b_1, \dots, b_4)b_5 \neq 0$;

(4) eH satisfies $[f(x_1, \dots, x_n)^m, x_{n+1}]x_{n+2}$, $g(eH)eH = 0$, $[c, eH]eH = 0$ and $(a - c^m)eH = 0$. Also in this case we get a contradiction, since either $[f(c_1, \dots, c_n)^m, c_{n+1}]c_{n+2} \neq 0$, or $g(c_{n+3})c_{n+4} \neq 0$, or $[c, c_{n+5}]c_{n+6} \neq 0$, or $(a - c^m)c_{n+7} \neq 0$.

The previous contradictions imply that one of the following holds:

(1): $d(I)I = (0)$, $g(I)I = (0)$, $[c, I]I = 0$, $(a - c^m)I = 0$. In particular by Fact 7, there exist $b, q \in U$ and $\alpha, \beta, \lambda \in C$ such that $d(x) = [b, x]$, $g(x) = [q, x]$ for all $x \in R$ and $(b - \alpha)I = 0$, $(q - \beta)I = 0$, $(c - \lambda)I = 0$ and $(a - \lambda^m)I = 0$. Moreover, in this case, relation (32) reduces to

$$f(x_1, \dots, x_n)^m \big(\lambda^m + \alpha - b) - \lambda^{m-1}(\lambda + \beta - q)\big).$$

Since $f(x_1, \dots, x_n)x_{n+1}$ is not an identity for I and by [6], we get $b = \alpha + \lambda^{m-1}(q - \beta)$, that is $F(x) = ax + \lambda^{m-1}[q, x]$ (this is the conclusion (1) of Theorem 2).

(2): $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ is an identity for I; then, by [23] (see the proof of Theorem 6, page 17, rows 3–8) it follows that there exists an idempotent element $e \in \mathrm{Soc}(U)$ such that $CI = eRC$ and $f(x_1, \dots, x_n)$ is central valued on $eRCe$ (this is the conclusion (2) of Theorem 2).

(3): $\mathrm{char}(R) = 2$ and $s_4(x_1, \dots, x_4)x_5$ is an identity for I; also in this case there exists an idempotent element $e \in \mathrm{Soc}(U)$ such that $CI = eRC$ and $s_4(x_1, \dots, x_4)$ is an identity for $eRCe$ (this is the conclusion (3) of Theorem 2).

(4): $[f(x_1, \dots, x_n)^m, x_{n+1}]x_{n+2}$ is an identity for I, $g(I)I = 0$, $[c, I]I = 0$ and $(a - c^m)I = 0$. In partucular by Fact 7, there exist $q \in U$ and $\alpha, \lambda \in C$ such that $g(x) = [q, x]$ for all $x \in R$ and $(q - \alpha)I = 0$, $(c - \lambda)I = 0$ and $(a - \lambda^m)I = 0$. Also in this case, by [23], we have that $CI = eRC$ and $f(x_1, \dots, x_n)^m$ is central valued on $eRCe$.

Then by (32), it follows that eRC satisfies

$$(\lambda^m + b)f(x_1, \dots, x_n)^m - f(x_1, \dots, x_n)^m b - \big((\lambda + \alpha)f(x_1, \dots, x_n) - f(x_1, \dots, x_n)q\big).$$

Left multiplying by $(1-e)$, we have that $(1-e)bf(x_1,\dots,x_n)^m$ is a generalized polynomial identity for eRC. Thus by [6] and since $f(x_1,\dots,x_n)x_{n+1}$ is not an identity for eRC, we get $(1-e)be=0$, that is $be\in eRCe$. Hence $[b,f(r_1,\dots,r_n)^m]=0$ for all $r_1,\dots,r_n\in eRCe$ and (32) reduces to

$$\lambda^{m-1}f(x_1,\dots,x_n)^m(\alpha-q).$$

Once again by [6] and since $f(x_1,\dots,x_n)x_{n+1}$ is not an identity for eRC, one has that either $\lambda=0$ or $q=\alpha\in C$. In the first case $ae=ce=0$ (and we get the conclusion (5) of Theorem 2); in the latter one, it follows $q\in C$ (and we get the conclusion (4) of Theorem 2).

Remark 4. We conclude this paper by considering what happens when either $F(I)=0$ or $G(I)=0$. We observe that, in both cases and by using the same argument in the proof of Theorem 2, one has:

(1) if $F(I)=0$, then either $G(I)=0$ or one of the following holds:
   1. either there exists an idempotent element $e\in \mathrm{Soc}(U)$ such that $CI=eRC$ and $f(x_1,\dots,x_n)$ is central valued on $eRCe$;
   2. or there exist $c,q\in U$ and $\lambda\in C$ such that $G(x)=cx+[q,x]$ for all $x\in R$, with $cI=0$ and $(q-\lambda)I=0$.

(2) if $G(I)=0$, then either $F(I)=0$ or there exists an idempotent element $e\in \mathrm{Soc}(U)$ such that $CI=eRC$ and one of the following holds
   1. either $f(x_1,\dots,x_n)$ is central valued on $eRCe$;
   2. or $\mathrm{char}(R)=2$ and $eRCe$ satisfies $s_4(x_1,\dots,x_4)$;
   3. or $f(x_1,\dots,x_n)^m$ is central valued on $eRCe$ and there exist $a,q\in U$ and $\alpha\in C$ such that $F(x)=ax+[q,x]$ for all $x\in R$, with $ae=0$ and $qe=eqe$.

Notice that the previous results are special cases of the conclusions of Theorem 2.

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