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There are no proper Berwald–Einstein manifolds

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Abstract. We prove that a connected Berwald–Einstein manifold is either Riemannian or Ricci-flat.

1. Introduction

On many occasions, S.-S. Chern raised the following question: *Does every smooth manifold admit an Einstein–Finsler metric?* The problem is extremely involved and has been intensely studied. However, it is still remains open, although there are several partial results.

Most of the currently available Einstein–Finsler metrics are either of Randers type or Ricci flat, see e.g., [2], [3], [6], [9], [12]. To attack the problem, it is indeed natural to consider first some special Finsler manifolds. A promising class is given by invariant Einstein–Finsler functions on homogeneous manifolds; for some results on homogeneous Einstein–Finsler functions we refer to [4]. Another important and well-understood class of Finsler manifolds is formed by Berwald manifolds. It turns out, however, that this class does not admit proper Einstein–Finsler functions. In this short note we prove the following

Theorem 1. *A connected Berwald–Einstein manifold is either Riemannian or Ricci-flat.*

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2. Preliminaries

In general we follow the conventions of [1] and [11]. We denote by M an n-dimensional connected smooth manifold. The tangent bundle and the slit tangent bundle of M are $\tau : TM \to M$ and $\mathring{\tau} : \mathring{T}M \to M$, respectively. To avoid subtle technicalities, we shall frequently use local coordinates. Then (u^i) stands for a generic local coordinate system on M, and (x^i, y^i) is the induced local coordinate system on TM. (Here $x^i = u^i \circ \tau$, $y^i(v) = v(u^i)$.)

A *Finsler function* for M is a positive-homogeneous function $F : TM \to [0, \infty[$ such that F is smooth on $\mathring{T}M$ and the matrix

$$(g_{ij}^F) := \left(\frac{\partial^2 \frac{1}{2} F^2}{\partial y^i \partial y^j} \right) \tag{1}$$

is positive definite at every point of its domain. Then the pair (M, F) is called a *Finsler manifold.*

Let (M, F) be a Finsler manifold. The *fundamental tensor* g_F of (M, F) is the Riemannian metric on the pull-back bundle $\mathring{\tau}^* TM$ whose components with respect to the local frame $\big(\widehat{\frac{\partial}{\partial u^i}}\big)_{i=1}^n$ induced by $\big(\frac{\partial}{\partial u^i}\big)_{i=1}^n$ are given by (1). The Finsler function F induces a *canonical spray* on TM, given locally by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where

$$G^i = \frac{1}{4} (g^F)^{il} \left(\frac{\partial^2 (F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2)}{\partial x^l} \right), \quad ((g^F)^{ij}) := (g_{ij}^F)^{-1}.$$

In spite of the coordinate formulation, $\mathbf{G}$ is a globally defined C^1 vector field on TM, smooth on $\mathring{T}M$. For an intrinsic description of the canonical spray, see, e.g., [11, section 9.2.2]. The *Jacobi endomorphism* of (M, F) (called the *Riemann curvature* in [8]) is a type $(1, 1)$ tensor field on $\mathring{\tau}^* TM$, given locally by

$$\mathbf{K} = K_k^i \widehat{\frac{\partial}{\partial u^i}} \otimes \widehat{du^k},$$

where $(\widehat{du^i})_{i=1}^n$ is the dual frame of $\big(\widehat{\frac{\partial}{\partial u^i}}\big)_{i=1}^n$ and

$$K_k^i = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i \partial G^j}{\partial y^j \partial y^k}.$$

Again, a coordinate-free definition may be found in [11]. The trace $\mathrm{Ric}_F = \mathrm{tr}(\mathbf{K})$ of $\mathbf{K}$ is a smooth function on $\mathring{T}M$, called the *Ricci curvature* of (M, F).

Definition 2. A Finsler manifold (M, F) is called a *Einstein–Finsler manifold* if its Ricci curvature is related to the Finsler function by

$$\mathrm{Ric}_F = (\lambda \circ \tau) F^2 =: \lambda^{\mathsf{v}} F^2, \tag{2}$$

where $\lambda \in C^\infty(M)$.

Note that if the manifold has dimension ≥ 3 and the Finsler function is Riemannian, then the function λ in Definition 2 is constant by Schur's lemma [5, Lemma 3]. This assertion is also true in the Randers case; see [7]. However, it is still an open problem in Finsler geometry whether the above assertion holds for a general Finsler manifold of dimension ≥ 3, and this is the key point in our proof of Theorem 1.

Now we recall that a Finsler manifold is called a *Berwald manifold* if its Berwald curvature vanishes. Locally, this means that

$$G^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = 0$$

for all $i, j, k, l \in \{1, \dots, n\}$, therefore the Christoffel symbols G^i_{jk} of the Finslerian Berwald derivative 'depend only on position'. Thus there exists a family (Γ^i_{jk}) of (locally defined) smooth functions such that $G^i_{jk} = \Gamma^i_{jk} \circ \tau$. The so obtained family (Γ^i_{jk}) is the family of Christoffel symbols of a torsion-free covariant derivative on M, called the *base covariant derivative* of (M, F). For details, we refer to [11, section 9.8].

Theorem 3. *If (M, F) is a Berwald manifold then there exists a Riemannian metric g on M whose Levi–Civita derivative is the base covariant derivative of (M, F).*

We note that this is just a reformulation of Z. I. Szabó's clever observation [10, Theorem 1]; see also [11, Theorem 9.8.6]. If a Riemannian metric g satisfies the condition in Theorem 3 we say that (M, F) and (M, g) (or F and g) are *affine equivalent.*

3. Proof of Theorem 1

If $\dim M = 2$, then (M, F) is automatically a locally Minkowski Finsler manifold or a Riemannian–Finsler manifold. This can be concluded from Szabó's proof of his famous structure theorem on Berwald manifolds [10, Theorem 3]. There are also direct proofs; see [1, section 10.6], or [11, section 9.9.4].

In order to prove Theorem 1 in the case $\dim M \geq 3$, we first deduce the following

Lemma 4. *If a Berwald–Einstein manifold* (M, F) *is affine equivalent to a Riemannian manifold* (M, g)*, then*

$$\lambda^{\mathsf{v}} F^2(u) = \mathrm{Ric}_g(u,u) \quad \text{for all } u \in \mathring{T}M, \tag{3}$$

where Ric_g *is the Ricci tensor of* (M, g).

Proof. In view of (2), we have only to show that $\mathrm{Ric}_F(u) = \mathrm{Ric}_g(u,u)$. We can apply Lemma 7.15.3 and formula (8.2.4) in [11] to find that the curvature tensor R of (M, g) and the Jacobi endomorphism of (M, F) are related by

$$\mathbf{K}_u(v) = R_p(v,u)u; \quad u, v \in T_pM, \ u \neq 0.$$

The Ricci tensor of (M, g) is given by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y); \quad X, Y, Z \in \mathfrak{X}(M).$$

Thus we have $\mathrm{Ric}_F(u) = \mathrm{tr}(v \mapsto \mathbf{K}_u(v)) = \mathrm{tr}(v \mapsto R_p(v,u)u) = \mathrm{Ric}_g(u,u)$. □

Proof of Theorem 1. Suppose that (M, F) is a Berwald–Einstein manifold. If the function λ in (2) is everywhere zero then (M, F) is Ricci-flat. Now assume that the set $U := \{p \in M \mid \lambda(p) \neq 0\} \subset M$ is nonempty, and let A be one of its connected components. By Theorem 3, there exists a Riemannian metric g on M which is affine equivalent to F. Then Ric_g is a symmetric bilinear form, and from Lemma 4 it follows that (A, F) is Riemannian. The fundamental tensor g^F of (A, F) reduces to a Riemannian metric on A, denoted for simplicity by the same symbol. Then we have $F^2(u) = g^F(u,u)$ for all $u \in TA$. Since (A, F) and (A, g^F) are obviously affine equivalent, we get

$$\lambda g^F(u,u) = \lambda^{\mathsf{v}} F^2(u) \overset{\text{Lemma 4}}{=} \mathrm{Ric}_{g^F}(u,u), \quad u \in TA.$$

So (A, g^F) is an Einstein manifold and λ is constant on A.

We obtained that for any component A of U, the function $\lambda \restriction A$ is constant and (A, F) is a Riemannian manifold. Since U has countably many components, the image $\lambda(M) \subset \lambda(U) \cup \{0\} \subset \mathbb{R}$ of λ must be countable. However, M is connected, so $\lambda(M)$ is an interval, hence, by its countability, it consists of only a single point. Therefore λ is constant, $U = M$, and (M, F) is Riemannian. □

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