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Units in FD_{2p}

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Abstract. In this paper, we present the structure of the group of $*$-unitary units in the group algebra FD_{2p}, where F is a finite field of characteristic $p > 2$, D_{2p} is the dihedral group of order $2p$, and $*$ is the canonical involution of the group algebra FD_{2p}. We also provide the structure of the maximal p-subgroup of the unit group $\mathscr{U}(FD_{2p})$ and compute a basis of its center.

1. Introduction

Let FG be the group algebra of a group G over a field F. For a normal subgroup H in G, the natural homomorphism $G \to G/H$ can be extended to an algebra homomorphism from FG to $F[G/H]$, defined by $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g gH$. The kernel of this homomorphism, denoted by $\Gamma(H)$, is the ideal generated by $\{h-1 \mid h \in H\}$. Therefore, $FG/\Gamma(H) \cong F[G/H]$. In particular, for $H = G$, $\Gamma(G)$ is known as the augmentation ideal of the group algebra FG. Since $FG/\Gamma(G)\cong F$, it follows that the Jacobson radical $J(FG)$ is contained in $\Gamma(G)$. The equality occurs if G is a finite p-group and F is a field of characteristic p and hence $1+\Gamma(G)$ is same as the normalized unit group $V(FG)$ of the group algebra FG. Therefore, $\mathscr{U}(FG) = V(FG)\times F^*$, where F^* is the cyclic group of all nonzero elements of F.

If $x = \sum_{g\in G} x_g g$ is an element of FG, then the element $x^* = \sum_{g\in G} x_g g^{-1}$ is called the conjugate of x. The map $x \mapsto x^*$ is an anti-automorphism of FG of order 2, which is known as the canonical involution of the group algebra FG. An element $x \in \mathscr{U}(FG)$ is called unitary if $x^* = x^{-1}$. The unitary units of the

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unit group $\mathscr{U}(FG)$ form a subgroup $\mathscr{U}_*(FG)$, and is called unitary subgroup of $\mathscr{U}(FG)$.

Sandling in[6] described the structure of $V(FG)$ for an arbitrary finite abelian p-group G and a finite field F of p elements. In [7] Sandling provided generators and relators of $V(FG)$ for each 2-group G of order dividing 16 over a finite field F with 2 elements. Creedon and Gildea in [10] described the structure of $V(FD_8)$ for the dihedral group D_8 of order 8 and a finite field F of characteristic 2. For dihedral groups of orders 6 and 10, the structure of unit group is presented in [2] and [1]. In [13] Kaur and Khan described the structure of the unit group $\mathcal{U}(FD_{2p})$ over a finite field F with two elements. Gildea in [8] studied some properties of the center of the maximal p-subgroup of the unit group $\mathscr{U}(FD_{2p})$ over a finite field F of characteristic p. However, basis of its center is not known.

The set of all unitary units in the normalized unit group $V(FG)$ forms a subgroup of $\mathscr{U}_*(FG)$. We denote it by $V_*(FG)$. The unitary subgroup $\mathscr{U}_*(FG)$ coincides with $V_*(FG)$ if F is a finite field of characteristic 2. Otherwise, it coincides with $V_*(FG) \times \langle -1 \rangle$. Bovdi and Sakacs in [14] described the structure of $V_*(FG)$, where G is a finite abelian group and F is a finite field of characteristic p. Bovdi and Erdei in [4] provided the structure of the unitary subgroup $V_*(F_2G)$, where G is a nonabelian group of order 8 and 16. V. Bovdi and Rosa in [3] computed the order of the unitary subgroup of the group of units, when G is either an extraspecial 2-group or the central product of such a group with a cyclic group of order 4 and F is a finite field of characteristic 2. They also computed the order of the unitary subgroup $V_*(FG)$, where G is a 2-group with a finite abelian subgroup A of index 2 and an element b such that b inverts every element in A and the order of b is 2 or 4. V. Bovdi and Rozgonyi in [5] described the structure of $V_*(F_2G)$ if order of b is 4. If G is a nonabelian group of order 8 and F is a finite field of characteristic 2, then the structure of $V_*(FG)$ is described in [12] and [11].

Here, we obtain generators of the unitary subgroup $\mathscr{U}_*(FD_{2p})$ for dihedral group D_{2p} of order $2p$ over a finite field F of characteristic p. We also obtain a basis of the center of the maximal p-subgroup of the unit group $\mathscr{U}(FD_{2p})$. Finally, we establish that the maximal p-subgroup of $\mathscr{U}(FG)$ is a general product of the unitary subgroup with a metabelian group.

Let $D_{2p} = \langle a, b \mid a^p = 1 = b^2, b^{-1}ab = a^{-1} \rangle$. The distinct conjugacy classes of D_{2p} are $C_0 = \{1\}$, $C_i = \{a^i, a^{-i}\}$, for $1 \leq i \leq l$, where $l = \frac{p-1}{2}$ and $C = \{b, ab, a^2b, \cdots, a^{p-1}b\}$. For a set H, if $\hat{H}$ denotes the sum of all the elements of H, then $\{\hat{C}_0, \hat{C}_1, \hat{C}_2, \cdots, \hat{C}_l, \hat{C}\}$ forms a F-basis for the center $Z(FD_{2p})$ of the group algebra FD_{2p}.

2. Unit group of FD_{2p}

If A denotes the normal subgroup of D_{2p} generated by element a in D_{2p}, then $FD_{2p}/\Gamma(A) \cong FC_2$. Since $\Gamma(A)$ is a nilpotent ideal of FD_{2p}, we have $\mathscr{U}(FD_{2p})/(1+\Gamma(A)) \cong \mathscr{U}(FC_2)$, which is isomorphic to $F^* \times F^*$. Let $\theta : \mathscr{U}(FD_{2p}) \to \mathscr{U}(FC_2)$ be a group epimorphism defined by

$$\sum_{i=0}^{p-1} \alpha_i a_i + \sum_{j=0}^{p-1} \beta_j a^j b \mapsto \sum_{i=0}^{p-1} \alpha_i + \sum_{j=0}^{p-1} \beta_j x,$$

where $C_2 = \langle x \rangle$. We can define a group homomorphism $\psi : \mathcal{U}(FC_2) \to \mathcal{U}(FD_{2p})$ by $a_0 + a_1 x \mapsto a_0 + a_1 b$. Since $\theta \circ \psi = 1$, we have $\mathscr{U}(FD_{2p}) \cong (1+\Gamma(A)) \rtimes F^* \times F^*$. If order of F is p^n, then $1+\Gamma(A)$ is a nilpotant group of order p^{4nl} with exponent p, where $l = \frac{p-1}{2}$.

We first establish the structure of the unitary subgroup $\mathscr{U}_*(FD_{2p})$.

3. Structure of the unitary subgroup $\mathscr{U}_*(FD_{2p})$

Theorem 1. *If A is the normal subgroup of D_{2p}, then the unitary subgroup $\mathscr{U}_*(FD_{2p})$ of the group algebra FD_{2p} is the semidirect product of the normal subgroup $V_*(FA)$ with an elementary abelian 2-group.*

We need following lemmas:

Lemma 2. *The group of unitary units in $1+\Gamma(A)$ is $V_*(FA)$.*

Proof. Assume that $v = 1 + x_1 + x_2 b$, where $x_i \in \omega(FA)$ is an arbitrary element of $1+\Gamma(A)$. Since $\omega(FA)$ is a nilpotent ideal, $1+x_1$ is an invertible element and thus v can be written as $v = u(1+xb)$, where $u \in V(FA)$ and $x \in \omega(FA)$. In particular, if v is a unitary unit, then from the equation $v^*v = 1$, we obtain $u^*u + bx^*u^*uxb = 1$ and $bx^*u^*u + u^*uxb = 0$. Moreover, since $by = y^*b$ for any $y \in FA$, we have $u \in V_*(FA)$. Further, note that $u^{-1}v = 1 + xb$ is a symmetric unit as well as a unitary unit. Since the exponent of $1+\Gamma(A)$ is p, it implies that $1 + xb = 1$ and hence $v = u$. □

Since A is a normal subgroup of D_{2p}, $FD_{2p}/\Gamma(A)$ is an algebra over F. Also note that $\Gamma(A)$ is $*$-stable nil ideal, and therefore the set of all unitary units in $FD_{2p}/\Gamma(A)$ form a subgroup of $\mathscr{U}(FD_{2p}/\Gamma(A))$ and is denoted by $\mathscr{U}_*(FD_{2p}/\Gamma(A))$.

Lemma 3. *The unitary subgroup* $\mathcal{U}_*(FD_{2p}/\Gamma(A))$ *of* $\mathscr{U}(FD_{2p}/\Gamma(A))$ *is the group generated by* $\{-1+\Gamma(A), b+\Gamma(A)\}$.

Proof. If $u+\Gamma(A)$ is a unitary unit of $\mathcal{U}(FD_{2p}/\Gamma(A))$, then $(u+\Gamma(A))^*(u+\Gamma(A)) = 1+\Gamma(A)$ and hence u^*u is a symmetric unit in $1+\Gamma(A)$ because $\Gamma(A)$ is a $*$-stable nil ideal. Further, if $u = x_0+x_1b$, where $x_0, x_1 \in FA$, then $uu^* = x_0x_0^* + x_1x_1^* + 2x_0x_1b$. For $x = \sum_{i=0}^{p-1}(\alpha_i a^i + \beta_i a^i b)$, we can define $\chi(x) = \sum_{i=0}^{p-1}(\alpha_i+\beta_i)$. Since uu^* is an element of $1+\Gamma(A)$, it follows that $\chi(x_0x_0^* + x_1x_1^*) = 1$ and $\chi(x_0x_1) = 0$. In particular, if $x_0 = \sum_{i=0}^{p-1}\alpha_i a^i$ and $x_1 = \sum_{i=0}^{p-1}\beta_i a^i$, then we obtain

$$\sum_{i=0}^{p-1}\alpha_i^2 + \sum_{i=0}^{p-1}\beta_i^2 + 2\sum_{\substack{i,j=0\\ i<j}}^{p-1}\alpha_i\alpha_j + 2\sum_{\substack{i,j=0\\ i<j}}^{p-1}\beta_i\beta_j = 1 \tag{1}$$

$$\left(\sum_{i=0}^{p-1}\alpha_i\right)\left(\sum_{j=0}^{p-1}\beta_j\right) = 0. \tag{2}$$

Now if $(\sum_{i=0}^{p-1}\alpha_i) = 0$, then from equation (1) we obtain $(\sum_{i=0}^{p-1}\beta_i)^2 = 1$ and hence $\sum_{i=0}^{p-1}\beta_i = \pm 1$. Thus, either x_1 or $-x_1$ is an element of $V(FA)$ and so the unitary units in $FD_{2p}/\Gamma(A)$ are $\pm b+\Gamma(A)$. Further, if $(\sum_{i=0}^{p-1}\beta_i) = 0$, then, in a similar way, one can show that the unitary units in $FD_{2p}/\Gamma(A)$ are $\pm 1+\Gamma(A)$. □

Proof of the theorem. Note that $V_*(FA)$ is a normal subgroup of $\mathcal{U}_*(FG)$ and $\langle V_*(FA), b, -1\rangle \subseteq \mathcal{U}_*(FG)$. Now suppose that $u \in \mathcal{U}(FG) \setminus V_*(FA)$ is a unitary unit, then $u+\Gamma(A)$ is a unitary unit in $\mathcal{U}(FG/\Gamma(A))$. In particular, if $u+\Gamma(A) = b+\Gamma(A)$, then $u = bx$ for some $x \in V_*(FA)$. Hence, $\mathcal{U}_*(FG) = V_*(FA) \rtimes (\langle b\rangle \times \langle -1\rangle)$. □

4. The structure of center $Z(1+\Gamma(A))$

To find a basis of center $Z(1+\Gamma(A))$ of the maximal p-subgroup $1+\Gamma(A)$ of the unit group $\mathcal{U}(FD_{2p})$, we need following lemma:

Lemma 4. *If* $\omega_i = (a^i - a^{-i})(1+b)$ *and* $\omega_i' = (a^i - a^{-i})(1-b)$ *for*$1 \le i \le l$, *then the set* $\{\omega_i, \omega_i', \omega_i\omega_i', \omega_i'\omega_i \mid 1 \le i \le l\}$ *is a free* F*- basis of* $\Gamma(A)$ *as a free* F*-module.*

PROOF. It is known that the set $\{(a^i-1),(a^i-1)b \mid 1\le i\le 2l\}$ is a free F-basis of $\Gamma(A)$ as a free F-module. Observe that $\omega_i\omega_j=0$, $\omega_i'\omega_j'=0$ for $1\le i,j,\le l$. Also note that $\omega_i\omega_i'=2(a^{2i}+a^{-2i}-2)(1-b)$ and $\omega_i'\omega_i=2(a^{2i}+a^{-2i}-2)(1+b)$. Thus, if $t=2i$, then

$$\left.\begin{aligned}(a^t-1)&=\frac{1}{4}(\omega_{2i}+\omega_{2i}')+\frac{1}{8}(\omega_i\omega_i'+\omega_i'\omega_i)\\(a^t-1)b&=\frac{1}{4}(\omega_{2i}-\omega_{2i}')-\frac{1}{8}(\omega_i\omega_i'-\omega_i'\omega_i)\end{aligned}\right\}0<2i\le l$$

$$\left.\begin{aligned}(a^t-1)&=-\frac{1}{4}(\omega_{p-2i}+\omega_{p-2i}')+\frac{1}{8}(\omega_i\omega_i'+\omega_i'\omega_i)\\(a^t-1)b&=-\frac{1}{4}(\omega_{p-2i}-\omega_{p-2i}')-\frac{1}{8}(\omega_i\omega_i'-\omega_i'\omega_i)\end{aligned}\right\}l<2i\le(p-1)$$

If $t=p-2i$, then

$$\left.\begin{aligned}(a^t-1)&=-\frac{1}{4}(\omega_{2i}+\omega_{2i}')+\frac{1}{8}\omega_i\omega_i'+\omega_i'\omega_i)\\(a^t-1)b&=-\frac{1}{4}(\omega_{2i}-\omega_{2i}')-\frac{1}{8}(\omega_i\omega_i'-\omega_i'\omega_i)\end{aligned}\right\}0<2i\le l$$

$$\left.\begin{aligned}(a^t-1)&=\frac{1}{4}(\omega_{p-2i}+\omega_{p-2i}')+\frac{1}{8}(\omega_i\omega_i'+\omega_i'\omega_i)\\(a^t-1)b&=\frac{1}{4}(\omega_{p-2i}-\omega_{p-2i}')-\frac{1}{8}(\omega_i\omega_i'-\omega_i'\omega_i)\end{aligned}\right\}l<2i\le(p-1)$$

Therefore, $\Gamma(A)=\operatorname{span}\{\omega_i,\omega_i',\omega_i\omega_i',\omega_i'\omega_i \mid 1\le i\le l\}$. Since the dimension of $\Gamma(A)$ over F is $4l$, the result follows. $\square$

Assume that F is a finite field of order p^n. Let $f(x)$ be a monic irreducible polynomial of degree n over F_p such that $F\cong F_p[x]/\langle f(x)\rangle$ and α denote the residue class of x modulo $\langle f(x)\rangle$.

Theorem 5. *If $u_{i,k}=1+\alpha^i(a-1)^k$, then the center $Z(1+\Gamma(A))$ is an elementary abelian p group of order $p^{n(l+1)}$ with the set $\{u_{i,k}^*u_{i,k},1+\alpha^i\hat{A}b \mid 0\le i\le(n-1),\ 1\le k\le(p-1)$ and k is even$\}$ as a basis.*

PROOF. First we prove that $Z(1+\Gamma(A))=Z(FD_{2p})\cap(1+\Gamma(A))$. Let $x\in Z(FD_{2p})\cap(1+\Gamma(A))$. It is clear that x is of the form

$$x=1+\sum_{i=1}^{l}\alpha_i(\hat{C}_i-2)+\beta\hat{A}b,$$

where $\alpha_i,\beta\in F$. Since $Z(FD_{2p})\cap 1+\Gamma(A)\subseteq Z(1+\Gamma(A))$, it implies that

$|Z(1+\Gamma(A))| \geq p^{n(l+1)}$. To show the equality, we compute the dimension of $Z(\Gamma(A))$ over F. Take $x \in Z(\Gamma(A))$ such that

$$x = \sum_{i=1}^{l} \alpha_i \omega_i + \sum_{i=1}^{l} \alpha'_i \omega'_i + \sum_{i=1}^{l} \alpha_{ii} \omega_i \omega'_i + \sum_{i=1}^{l} \alpha'_{ii} \omega'_i \omega_i.$$

First, note that if $i = 2k+1$, where $1 \leq i \leq l$ then

$$\omega_1 \omega'_i = \omega_{k+1}\omega'_{k+1} - \omega_k \omega'_k \quad \text{and} \quad \omega'_i \omega_1 = \omega'_{k+1}\omega_{k+1} - \omega'_k \omega_k.$$

For $i = 2k$, where $1 \leq i \leq l$

$$\omega_1 \omega'_i = \omega_{l-k}\omega'_{l-k} - \omega_{l-(k-1)}\omega'_{l-(k-1)} \quad \text{and} \quad \omega'_i \omega_1 = \omega'_{l-k}\omega_{l-k} - \omega'_{l-(k-1)}\omega_{l-(k-1)}.$$

Next, observe that if l is odd, then

$$\omega_1 \omega'_j \omega_j = 4\omega_{2j+1} - 4\omega_{2j-1} - 8\omega_1, \quad \text{for } 1 \leq j \leq \frac{l-1}{2}$$

$$\omega_1 \omega'_j \omega_j = -4\omega_{2l-2j} + 4\omega_{2l-2j+2} - 8\omega_1, \text{ for } \frac{l+3}{2} \leq j \leq l-1$$

$$\omega_1 \omega'_{\frac{l+1}{2}} \omega_{\frac{l+1}{2}} = -4\omega_{l-1} - 4\omega_l - 8\omega_1$$

$$\omega_1 \omega'_l \omega_l = 4\omega_2 - 8\omega_1$$

and if l is even, then

$$\omega_1 \omega'_j \omega_j = 4\omega_{2j+1} - 4\omega_{2j-1} - 8\omega_1, \quad \text{for } 1 \leq j \leq \frac{l-2}{2}$$

$$\omega_1 \omega'_j \omega_j = -4\omega_{2l-2j} + 4\omega_{2l-2j+2} - 8\omega_1, \quad \text{for } \frac{l+2}{2} \leq j \leq l-1$$

$$\omega_1 \omega'_{\frac{l}{2}} \omega_{\frac{l}{2}} = -4\omega_l - 4\omega_{l-1} - 8\omega_1$$

$$\omega_1 \omega'_l \omega_l = 4\omega_2 - 8\omega_1$$

Now, after substituting these values in the equations $\omega_1 x = x\omega_1$ and $\omega'_1 x = x\omega'_1$, we obtain $\alpha_i = \alpha'_i = 0 \ \forall \ 1 \leq i \leq l$ and the following set of l equations in $2l$ variables.

$$\alpha'_{ii} - \alpha'_{i+1i+1} - \alpha_{ii} + \alpha_{i+1i+1} = 0$$

$$-3(\alpha'_{11} - \alpha_{11}) - 2\sum_{i=2}^{l} (\alpha'_{ii} - \alpha_{ii}) = 0$$

Further, observe that the last equation can be written as

$$-\sum_{k=1}^{l-1}(2k+1)(\alpha'_{kk}-\alpha'_{k+1k+1}-\alpha_{kk}+\alpha_{k+1k+1})=0$$

Therefore, the dimension of the solution space of the above system of linear equation is $l+1$ and hence $|Z(1+\Gamma(A))| \le p^{n(l+1)}$. Thus, $|Z(1+\Gamma(A))| = p^{n(l+1)}$ and

$$Z(1+\Gamma(A)) = \left\{1+\sum_{i=1}^{l}\alpha_i(\hat{C}_i-2)+\beta\hat{A}b \mid \alpha_i,\beta\in F\right\} = S_*(FA)\times(1+F\hat{A}b).$$

Further, note that

$$\mathscr{B} = \{u^*_{i,k}u_{i,k} \mid 0\le i\le (n-1),\ 1\le k\le (p-1) \text{ and } k \text{ is even}\}$$

forms a basis of $S_*(FA)$ and hence the result follows. □

5. Structure of $1+\Gamma(A)$

In this section, we obtain the structure of $1+\Gamma(A)$. We shall use the following result:

Theorem 6 (Pavesic [9]). *Let $(e_1,e_2,\dots e_n)$ be an ordered n-tuple of orthogonal idempotents in a unital ring R such that $1=e_1+e_2+\cdots+e_n$ and each e_i strongly preserves a circle subgroup M of the circle group of R, i.e. $e_iM\subseteq M$ and $Me_i\subseteq M$, then $(M,\circ)=(L,\circ)\circ(D,\circ)\circ(U,\circ)$, where $L=\{m\in M \mid (\forall\ i)\ e_im=e_im\overline{e_i}\}$, $D=\{m\in M \mid (\forall\ i)\ e_im=e_ime_i\}$ and $U=\{m\in M \mid (\forall\ i)\ e_im=e_im\underline{e_i}\}$, where $\overline{e_i}=e_{i+1}+e_{i+2}+\cdots+e_n$, $\underline{e_i}=e_1+e_2+\cdots+e_{i-1}$, $\underline{e_1}=0$ and $\overline{e_n}=0$.*

Theorem 7. *If A is the normal subgroup of D_{2p}, then $1+\Gamma(A)$ is the general product of unitary subgroup with a metabelian group.*

Proof. Note that $\Gamma(A)$ is a circle group and $\{e_1=\frac{1+b}{2}, e_2=\frac{1-b}{2}\}$ is a complete set of orthogonal idempotents that strongly preserve $\Gamma(A)$. Therefore, $(\Gamma(A),o)=(L,o)o(D,o)o(U,o)$, where $(L,o)=\oplus_{i=1}^{l}F\omega'_i$, $(U,o)=\oplus_{i=1}^{l}F\omega_i$ and $(D,o)=\oplus_{i=1}^{l}F\omega_i\omega'_i\oplus_{i=1}^{l}F\omega'_i\omega_i$. Hence, $1+\Gamma(A)$ is a general product of elementary abelain p-groups $(1+L),(1+D)$, and $1+U$. Moreover, $1+D$ normalize $1+L$; thus the group $W=(1+L)\rtimes(1+D)$ is a metabelian group.

If $C_{1+\Gamma(A)}(a)$ is a centralizer of a in $1+\Gamma(A)$, then it is clear that it is an elementary abelian p-group of order p^{np}. Since $V_*(FA)$ and $Z(1+\Gamma(A))$ are in $C_{1+\Gamma(A)}(a)$ and their intersection is identity, it follows that $C_{1+\Gamma(A)}(a) = V_*(FA) \times Z(1+\Gamma(A))$. Also note that $W \cap C_{1+\Gamma(A)}(a) = Z(1+\Gamma(A))$. Thus we obtain that $1+\Gamma(A) = WC_{1+\Gamma(A)}(a)$ is a general product of W and $V_*(FA)$. $\square$

If α is the residue class of $x \mod \langle f(x)\rangle$, then the set $\{1+\alpha^i\omega'_j \mid 0 \le i \le n-1,\ 1 \le j \le l\}$ generates $(1+L)$ and $\{1+\alpha^i\omega_j \mid 0 \le i \le n-1,\ 1 \le j \le l\}$ generates $(1+U)$. The following lemma provides the structure of $1+D$.

Lemma 8. *The set* $\{1+\alpha^i(a-a^{-1})^{2k}(1-b), 1+\alpha^i(a-a^{-1})^{2k}(1+b) \mid 1 \le k \le l$ *and*$0 \le i \le (n-1)\}$ *forms a basis of* $1+D$.

Proof. First, we show that $\sum_{i=1}^{l} F\omega_i\omega'_i = \text{span}\{(a-a^{-1})^{2k}(1-b) \mid 1 \le k \le l\}$. For this, we prove $\omega_k\omega'_k \in \text{span}\{(a-a^{-1})^{2k}(1-b) \mid 1 \le k \le l\}$ by induction over k. The result is trivial for $k=1$, as $\omega_1\omega'_1 = 2(a-a^{-1})^2(1-b)$. Assume the result for $k-1$. Consider $\omega_k\omega'_k$. Notice that

$$\omega_k\omega'_k = 2(a^k - a^{-k})^2(1-b) = 2(a^{2k} + a^{-2k} - 2)(1-b).$$

Now,

$$\begin{aligned}(a-a^{-1})^{2k} &= \sum_{j=0}^{k-1}(-1)^j\binom{2k}{j}(a^{2k-2j} + a^{-(2k-2j)}) + (-1)^k\binom{2k}{k} \\ &= \sum_{j=0}^{k-1}(-1)^j\binom{2k}{j}(a^{2k-2j} + a^{-(2k-2j)} - 2) \\ (a-a^{-1})^{2k}(1-b) &= \sum_{j=0}^{k-1}(-1)^j\frac{1}{2}\binom{2k}{j}\omega_{k-j}\omega'_{k-j}\end{aligned}$$

Therefore, $\frac{1}{2}\omega_k\omega'_k = (a-a^{-1})^{2k}(1-b) - (\sum_{j=1}^{k-1}(-1)^j\frac{1}{2}\binom{2k}{j}\omega_{k-j}\omega'_{k-j})$. Hence, by the induction hypothesis, we obtain

$$\omega_k\omega'_k \in \text{span}\{(a-a^{-1})^{2k}(1-b) \mid 1 \le k \le l\}.$$

Therefore, $\sum_{i=1}^{l} F\omega_i\omega'_i = \sum_{k=1}^{l} F(a-a^{-1})^{2k}(1-b)$. It implies that

$$1+\sum_{i=1}^{l} F\omega_i\omega'_i = 1+\sum_{k=1}^{l} F(a-a^{-1})^{2k}(1-b) = \prod_{k=1}^{l}\prod_{i=0}^{n-1}(1+\alpha^i(a-a^{-1})^{2k})(1-b)$$

Similarly, we can show that

$$1+\sum_{i=1}^{l} F\omega'_i\omega_i = \prod_{k=1}^{l}\prod_{i=0}^{n-1}(1+\alpha^i(a-a^{-1})^{2k})(1+b)$$

Since $1+D = (1+\sum_{i=1}^{l} F\omega_i\omega'_i) \times (1+\sum_{i=1}^{l} F\omega'_i\omega_i)$, the result follows. $\square$

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