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Fixed points and exponential stability of stochastic functional partial differential equations driven by fractional Brownian motion

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Abstract. In this paper, the fixed point theory is used to investigate the stability for stochastic functional partial differential equations driven by fractional Brownian motion

$$dX(t) = [AX(t) + f(t, X_t)]dt + g(t)dB_Q^H(t),$$

where $H \in (1/2, 1)$. The obtained results improve the results due to CARABALLO, GARRIDO-ATIENZA, TANIGUCHI [3].

1. Introduction

In 1892, LYAPUNOV established a method to investigate the stability problems. From then on, the Lyapunov's direct method was widely used to investigate the stability properties of ordinary, functional, partial differential and integro-differential equations. However, there exist a number of serious obstacles if the delays are unbounded or if the equations has unbounded terms [2]. In recent years, several authors made a attempt to use the fixed point theory to investigate the stability, which shows that some of those difficulties vanish or might be overcome when applying fixed point theory (see [1], [2], [4], [5]). The fixed point theory does not only solve the stability problem but also has other remarkable advantage

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over Lypunov's. The conditions of the former are usually averages but those of the latter are often pointwise (see [2]).

In this paper, we consider the following stochastic functional partial differential equation

$$\begin{cases} dX(t) = [AX(t) + f(t, X_t)]dt + g(t)dB_Q^H(t), \quad t \geq 0, \\ X(s) = \varphi(s), \quad -r \leq s \leq 0, \ r \geq 0, \end{cases} \tag{1.1}$$

under suitable conditions on the operator A, the coefficient functions f, g and the initial value φ. Here $B_Q^H(t)$ denotes a fractional Brownian motion (fBm) with $H \in (1/2, 1)$ (see [3]).

The purpose of this paper is to investigate the exponential stability in mean square of mild solution of stochastic functional partial differential equations driven by fractional Brownian motion by means of the fixed point theory.

This paper is organized as follows. In Section 2 some necessary preliminaries on the stochastic integration equations with respect to fractional Browanian motion are established. In Section 3 the exponential stability in mean square of mild solution of stochastic functional partial differential equations driven by fractional Brownian motion is proved, the results in [3] are improved.

2. Preliminaries

Let $(U, |\cdot|_U, (\cdot,\cdot)_U)$ and $(K, |\cdot|_K, (\cdot,\cdot)_K)$ be two real separable Hilbert spaces. Let $L(K, U)$ denote the space of all bounded linear operators from K to U. Let Q be a non-negative self-adjoint nuclear operator on K. Let $L_Q^0(K, U)$ denote the space of all $\eta \in L(K, U)$ such that $\eta Q^{\frac{1}{2}}$ is a Hibert-Schmidt operator. The norm is given by

$$|\eta|^2_{L^0(K,U)} = \left|\eta Q^{\frac{1}{2}}\right|^2_{HS} = \mathrm{tr}(\eta Q \eta^*).$$

Then η is called a Q-Hilbert–Schmidt operator from K to U.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $\{\beta_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in K. Then we consider a K-valued stochastic process $B_Q^H(t)$ given formally by the following series:

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{1/2} e_n, \quad t \geq 0.$$

If Q is a non-negative self-adjoint nuclear operator, then this series converges in the space K, that is, it holds that $B_Q^H(t) \in L^2(\Omega, K)$. Then, we can say that the above $B_Q^H(t)$ is a K-valued Q-cylindrical fractional Brownian motion with covariance operator Q.

Lemma 2.1 (Caraballo [3]). *Let $\varphi : [0,T] \longmapsto L_Q^0(K,U)$ such that*

$$\sum_{n=1}^{\infty} \|(\varphi Q^{1/2} e_n)\|_{L^{1/H}([0,T];U)} < \infty \tag{2.1}$$

holds, and for any α, $\beta \in [0,T]$ with $\alpha > \beta$,

$$\mathbb{E}\left|\int_\beta^\alpha \varphi(s) dB_Q^H(s)\right|_U^2 \le cH(2H-1)(\alpha-\beta)^{2H-1} \sum_{n=1}^{\infty} \int_\beta^\alpha |\varphi(s) Q^{1/2} e_n|_U^2 ds,$$

where $c = c(H)$. If, in addition,

$$\sum_{n=1}^{\infty} |\varphi(t) Q^{1/2} e_n|_U \text{ is uniformly convergent for } t \in [0,T],$$

then

$$\mathbb{E}\left|\int_\beta^\alpha \varphi(s) dB_Q^H(s)\right|_U^2 \le cH(2H-1)(\alpha-\beta)^{2H-1} \int_\beta^\alpha |\varphi(s)|^2_{L_Q^0(K,U)} ds. \tag{2.2}$$

Consider $(\Omega, \mathcal{F}, \mathbb{P})$, the complete probability space which was previously introduced. Denote $\mathcal{F}_t = \mathcal{F}_0$ for all $t \le 0$.

We denote by $C(a,b;L^2(\Omega;U)) = C(a,b;L^2(\Omega,\mathcal{F},\mathbb{P};U))$ the Banach space of all continuous functions from $[a,b]$ into $L^2(\Omega;U)$ endowed with the supremum.

Consider two fixed real numbers $r \ge 0$ and $T > 0$. If $X \in C(-r,T;L^2(\Omega;U))$ for each $t \in [0,T]$ we denote by $X_t \in C(-r,0;L^2(\Omega;U))$, the function defined by $X_t(s) = X(t+s)$, for $s \in [-r,0]$.

We consider the exponential stability of mild solution to the following stochastic functional partial differential equation:

$$\begin{cases} dX(t) = [AX(t) + f(t,X_t)]dt + g(t) dB_Q^H(t), & t \in [0,T], \\ X(t) = \varphi(t), & t \in [-r,0], \end{cases} \tag{2.3}$$

where $B_Q^H(t)$ is the fractional Brownian motion which was previously introduced, the initial value $\varphi \in C(-r,0;L^2(\Omega;U))$ and $A : \mathrm{Dom}(A) \subset U \to U$ is the infinitesimal generator of a strongly continuous semigroup $S(\cdot)$ on U, that is, for $t \ge 0$, it holds

$$|S(t)|_U \le Me^{\rho t}, \; M \ge 1, \quad \rho \in \mathbb{R}.$$

$f : [0,T] \times C(-r,0;U) \to U$ a family of nonlinear operators defined for almost every t (a.e.t) which satisfies

(f.1) *The mapping* $t \in (0,T) \to f(t,\xi) \in U$ *is Lebesgue measurable, for a.e.t and for all* $\xi \in C(-r,0;U)$.

(f.2) *There exists a constant* $C_f > 0$ *such that for any* $X, Y \in C(-r,T;U)$ *and* $t \in [0,T]$,

$$\int_0^t |f(s,X_s) - f(s,Y_s)|_U^2 ds \le C_f \int_{-r}^t |X(s) - Y(s)|_U^2 ds. \tag{f.3}$$

$$\int_0^T |f(s,0)|_U^2 ds < \infty.$$

Moreover, for $g : [0,T] \to L_Q^0(K,U)$ *we assume the following conditions: for the complete orthonormal basis* $\{e_n\}_{n\in\mathbb{N}}$ *in* K, *we have*

$$\sum_{n=1}^{\infty} \|gQ^{1/2}e_n\|_{L^2([0,T];U)} < \infty. \tag{g.1}$$

$$\sum_{n=1}^{\infty} |g(t)Q^{1/2}e_n|_U \text{ is uniformly convergent for } t \in [0,T]. \tag{g.2}$$

Definition 2.1. A U-valued process $X(t)$ is called a mild solution of (2.3) if $X \in C(-r,T;L^2(\Omega;U))$, $X(t) = \varphi(t)$ for $t \in [-r,0]$, and, for $t \in [0,T]$, satisfies

$$X(t) = S(t)\varphi(0) + \int_0^t S(t-s)f(s,X_s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s) \quad \mathbb{P}\text{-a.s.} \tag{2.4}$$

Notice that, because of (g.1) and $H \in (1/2,1)$, (2.1) holds, which implies that the stochastic integral in (2.4) is well-defined since $S(\cdot)$ is a strongly continuous semigroup. Moreover, (g.1) together with (g.2) immediately imply that, for $\forall t \in [0,T]$,

$$\int_0^T |g(s)|^2_{L_Q^0(K,U)} ds < \infty.$$

Theorem 2.1 (Caraballo [3])**.** *Under the assumptions on* A *and condition* (f.1)–(f.3) *and* (g.1)–(g.2), *for every* $\varphi \in C(-r,0;L^2(\Omega;U))$ *there exists a unique mild solution* X *to* (2.3).

Definition 2.2. Equation (2.3) is said to be exponentially stable in mean square, if for any initial value φ, there exists a pair of constants $\gamma > 0$ and $C > 0$ such that

$$\mathbb{E}|X(t)|^2 \le Ce^{-\gamma t}, \quad t \ge 0. \tag{2.5}$$

In order to set the stability problem, we assume that the following hold, which are also imposed in [3]:

$$|S(t)|_U \leq Me^{-\lambda t}, \quad \forall t \geq 0, \text{ where } M \geq 1 \text{ and } \lambda > 0. \tag{2.6}$$

There exists a constant $C_f \geq 0$, such that, for any $X, Y \in C(-r, T; U)$, and for all $t \geq 0$

$$\int_0^t e^{ms}|f(t, X_s) - f(t, Y_s)|_U^2 ds \leq C_f \int_{-r}^t e^{ms}|X(s) - Y(s)|_U^2 ds, \\ \text{for all } 0 \leq m \leq \lambda, \tag{2.7}$$

and

$$\int_0^\infty e^{\lambda s}|f(s, 0)|_U^2 ds < \infty. \tag{2.8}$$

In addition to assumptions (g.1) and (g.2), assume

$$\int_0^\infty e^{\lambda s}|g(s)|^2_{L_Q^0(K,U)} ds < \infty. \tag{2.9}$$

3. Exponential stability in mean square

In this section, we will consider the exponential stability in mean square of mild solution of (2.3) by means of the fixed point theory.

Theorem 3.1. *Suppose that conditions* (2.6)–(2.9) *hold. Then equation* (2.3) *is exponentially stable in mean square if*

$$\lambda^2 > C_f M^2. \tag{3.1}$$

PROOF. Denote by $\mathcal{S}$ the Banach space of all $\mathcal{F}$-adapted processes $\phi(t, w)$: $[-r, \infty) \times \Omega \longrightarrow \mathbb{R}$, which is almost surely continuous in t for fixed $\omega \in \Omega$. Moreover, $\phi(s, w) = \varphi(s)$ for $s \in [-r, 0]$ and $e^{\alpha t}\mathbb{E}|\phi(t, w)|_U^2 \longrightarrow 0$ as $t \longrightarrow \infty$, where α is a positive constant such that $0 < \alpha < \lambda$.

Define an operator $\pi : \mathcal{S} \longrightarrow \mathcal{S}$ by $(\pi X)(t) = \psi(t)$ for $t \in [-r, 0]$ and for $t \geq 0$,

$$(\pi X)(t) = S(t)\varphi(0) + \int_0^t S(t-s)f(s, X_s)ds \\ + \int_0^t S(t-s)g(s)dB_Q^H(s) := \sum_{i=1}^3 I_i(t). \tag{3.2}$$

We first verify the continuity in mean square of π on $[0,\infty)$. Let $X \in \mathcal{S}$, $t_1 \geq 0$, and r be positive and sufficiently enough, then

$$\mathbb{E}|(\pi X)(t_1+r)-(\pi X)(t_1)|_U^2 \leq 3\sum_{i=1}^{3}\mathbb{E}|I_i(t_1+r)-I_i(t)|_U^2.$$

Obviously

$$\mathbb{E}|I_i(t_1+r)-I_i(t)|_U^2 \longrightarrow 0, \quad i=1,2, \text{ as } r\longrightarrow 0.$$

Further, by using Cauchy–Schwarz inequality, we get

$$\mathbb{E}|I_3(t_1+r)-I_3(t_1)|_U^2 \leq 2\mathbb{E}\left|\int_0^{t_1}(S(t_1+r-s)-S(t_1-s))g(s)dB_Q^H(s)\right|_U^2$$
$$+2\mathbb{E}\left|\int_{t_1}^{t_1+r}S(t_1+r-s)g(s)dB_Q^H(s)\right|_U^2 := J_1+J_2.$$

Firstly, applying inequality (2.2) and condition (2.6) to J_1

$$\begin{aligned}J_1 &= 2\mathbb{E}\left|\int_0^{t_1}(S(t_1+r-s)-S(t_1-s))g(s)dB_Q^H(s)\right|_U^2\\ &\leq 2cH(2H-1)t_1^{2H-1}\int_0^{t_1}|S(t_1-s)(S(r)-Id)g(s)|^2_{L_Q^0(K,U)}ds\\ &\leq 2cH(2H-1)t_1^{2H-1}M^2\int_0^{t_1}|(S(r)-Id)g(s)|^2_{L_Q^0(K,U)}ds \longrightarrow 0\end{aligned}$$

when $r\longrightarrow 0$ since for every s fixed

$$S(r)g(s)\longrightarrow g(s), \qquad |S(r)g(s)|_{L_Q^0(K,U)} \leq M|g(s)|_{L_Q^0(K,U)}.$$

Applying inequality (2.2) and condition (2.6) to J_2, we can obtain

$$J_2 \leq 2cH(2H-1)r^{2H-1}M^2\int_{t_1}^{t_1+r}|g(s)|^2_{L_Q^0(K,U)}ds \longrightarrow 0 \quad \text{as } r\longrightarrow 0.$$

Thus, π is indeed continuous in mean square on $[0,\infty)$. Next, we show that $\pi(\mathcal{S})\subset\mathcal{S}$. It follows from (3.3) that

$$\begin{aligned}e^{\alpha t}\mathbb{E}|(\pi X)(t)|_U^2 \leq 3e^{\alpha t}\mathbb{E}|S(t)\varphi(0)|_U^2 + 3e^{\alpha t}\mathbb{E}\left|\int_0^t S(t-s)f(s,X_s)ds\right|_U^2\\ + 3e^{\alpha t}\mathbb{E}\left|\int_0^t S(t-s)g(s)dB_Q^H(s)\right|_U^2. \quad (3.3)\end{aligned}$$

Now we estimate the terms on the right-hand side of (3.4). Firstly, by the condition (2.6), we can obtain

$$3e^{\alpha t}\mathbb{E}|S(t)\varphi(0)|_U^2 \le 3M^2e^{-2\lambda t}e^{\alpha t}|\varphi(0)|_U^2 \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.4}$$

Secondly, Hölder's inequality and (2.7), (2.8) yield

$$\begin{aligned}
&3e^{\alpha t}\mathbb{E}\left|\int_0^t S(t-s)f(s,X_s)ds\right|_U^2 \le 3e^{\alpha t}\mathbb{E}\left[\int_0^t |S(t-s)f(s,X_s)|_U ds\right]^2\\
&\le 3e^{\alpha t}\mathbb{E}\left[\int_0^t Me^{-\lambda(t-s)}|f(s,X_s)|_U ds\right]^2 \le 3M^2e^{\alpha t}\lambda^{-1}\mathbb{E}\left[\int_0^t e^{-\lambda(t-s)}|f(s,X_s)|_U^2 ds\right]\\
&= 3M^2e^{\alpha t}\lambda^{-1}\mathbb{E}\left[\int_0^t e^{-\lambda(t-s)}|f(s,X_s)-f(s,0)+f(s,0)|_U^2 ds\right]\\
&\le 6M^2C_fe^{\alpha t}\lambda^{-1}\int_{-r}^t e^{-\lambda(t-s)}\mathbb{E}|X(s)|_U^2 ds + 6M^2e^{\alpha t}\lambda^{-1}\int_0^t e^{-\lambda(t-s)}|f(s,0)|_U^2 ds\\
&\le 6M^2C_fe^{(\alpha-\lambda)t}\lambda^{-1}\int_{-r}^t e^{(\lambda-\alpha)s}e^{\alpha s}\mathbb{E}|X(s)|_U^2 ds\\
&\quad + 6M^2e^{(\alpha-\lambda)t}\lambda^{-1}\int_0^t e^{\lambda s}|f(s,0)|_U^2 ds := k_1(t)+k_2(t).
\end{aligned}$$

For any $X(t)\in\mathcal{S}$ and any $\varepsilon>0$ there exists a $t_1>0$, such that $e^{\alpha s}\mathbb{E}|X(s)|_U^2 < \frac{\lambda(\lambda-\alpha)\varepsilon}{6M^2C_f}$ for $t>t_1$ we can get

$$6M^2C_fe^{(\alpha-\lambda)t}\lambda^{-1}\int_{t_1}^t e^{(\lambda-\alpha)s}e^{\alpha s}\mathbb{E}|X(s)|_U^2 ds < \varepsilon,$$

As $e^{(\alpha-\lambda)t}\to 0$ as $t\to\infty$, there exists $t_2>t_1$ such that for any $t\ge t_2$, we have

$$6M^2C_fe^{(\alpha-\lambda)t}\lambda^{-1}\int_{-r}^{t_1} e^{(\lambda-\alpha)s}e^{\alpha s}\mathbb{E}|X(s)|_U^2 ds < \varepsilon,$$

so for any $t>t_2$, we can obtain

$$\begin{aligned}
k_1(t) &\le 6M^2C_fe^{(\alpha-\lambda)t}\lambda^{-1}\int_{t_1}^t e^{(\lambda-\alpha)s}e^{\alpha s}\mathbb{E}|X(s)|_U^2 ds\\
&\qquad + 6M^2C_fe^{(\alpha-\lambda)t}\lambda^{-1}\int_{-r}^{t_1} e^{(\lambda-\alpha)s}e^{\alpha s}\mathbb{E}|X(s)|_U^2 ds < 2\varepsilon.
\end{aligned}$$

So from the above, we conclude that $k_1(t)\longrightarrow 0$ as $t\longrightarrow\infty$.

As $e^{(\alpha-\lambda)t} \to 0$ as $t \to \infty$ and condition (2.8), we can obtain $k_2(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

That is to say

$$3e^{\alpha t}\mathbb{E}\left|\int_0^t S(t-s)f(s,X_s)ds\right|_U^2 \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.5}$$

As for the third term on the right-hand side of (3.4), for any $X(t) \in \mathcal{S}$, $t \in [-r,\infty)$ we have

$$\begin{aligned} &3e^{\alpha t}\mathbb{E}\left|\int_0^t S(t-s)g(s)dB_Q^H(s)\right|_U^2 \\ &\quad \le 3M^2cH(2H-1)t^{2H-1}e^{\alpha t}\int_0^t e^{-2\lambda(t-s)}|g(s)|^2_{L_Q^0(K,U)}ds \\ &\quad \le 3M^2cH(2H-1)t^{2H-1}e^{(\alpha-\lambda)t}\int_0^t e^{\lambda s}|g(s)|^2_{L_Q^0(K,U)}ds \longrightarrow 0. \end{aligned} \tag{3.6}$$

Thus, from (3.4)–(3.7), we know that $e^{\alpha t}\mathbb{E}|(\pi X)(t)|_U^2 \longrightarrow 0$ as $t \longrightarrow 0$. So we conclude that $\pi(\mathcal{S}) \subset \mathcal{S}$.

Thirdly, we will show that π is contractive. For $X, Y \in \mathcal{S}$, as proceeding as we did previously, we can obtain

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,T]}|(\pi X)(t)-(\pi Y)(t)|^2 &= \mathbb{E}\sup_{s\in[0,T]}\left|\int_0^t S(t-s)(f(s,X_s)-f(s,Y_s))ds\right|_U^2 \\ &\le \sup_{s\in[0,T]}\mathbb{E}|X(s)-Y(s)|_U^2 \times C_f M^2\lambda^{-2}. \end{aligned} \tag{3.7}$$

Thus by (3.1) we know that π is a contraction mapping.

Hence by the Contraction Mapping Theorem, π has a unique fixed point $X(t)$ in $\mathcal{S}$, which is a solution of (2.3) with $X(s) = \varphi(s)$ on $[-r, 0]$ and $e^{\alpha t}\mathbb{E}|X(t)|_U^2$ as $t \longrightarrow \infty$ This completes the proof. □

Remark 3.1. Our results improve the results in [3]. Our condition is

$$\lambda^2 > C_f M^2. \tag{3.8}$$

However, the corresponding condition in [3] is

$$\lambda^2 > 6C_f M^2. \tag{3.9}$$

In this sense, this paper improves the results in [3].

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